Stitching Direct

```
MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].MatrixExp[η2 ρ[CU@y]].
  MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\gamma(\alpha_1+\alpha_2)}\left(1+\gamma\epsilon\eta_2\xi_1\right) & e^{\gamma\alpha_1}\gamma\left(e^{\gamma\alpha_2}\xi_2+\xi_1\left(1+e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2\right)\right) \\ e^{\gamma\alpha_2}\epsilon\left(\eta_2+e^{\gamma\alpha_1}\eta_1\left(1+\gamma\epsilon\eta_2\xi_1\right)\right) & 1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1+e^{\gamma\alpha_2}\gamma\epsilon\left(\eta_2+e^{\gamma\alpha_1}\eta_1\left(1+\gamma\epsilon\eta_2\xi_1\right)\right)\xi_2 \end{pmatrix}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
  MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
 e^(τ0 ϵ γ) MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\{\{e^{\gamma\alpha_2}(e^{\gamma\alpha_1}+e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1),\ e^{\gamma\alpha_1}\gamma\xi_1+e^{\gamma\alpha_2}\gamma(e^{\gamma\alpha_1}+e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1)\xi_2\},$$
$$\{e^{\gamma\alpha_2}(e^{\gamma\alpha_1}\epsilon\eta_1+\epsilon\eta_2(1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1)),$$
$$1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1+e^{\gamma\alpha_2}\gamma(e^{\gamma\alpha_1}\epsilon\eta_1+\epsilon\eta_2(1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1))\xi_2\}\} ==$$
$$\{\{e^{\alpha 0\gamma+\gamma\epsilon\tau 0},\ e^{\alpha 0\gamma+\gamma\epsilon\tau 0}\gamma\xi 0\},\ \{e^{\alpha 0\gamma+\gamma\epsilon\tau 0}\epsilon\eta 0,\ e^{\gamma\epsilon\tau 0}(1+e^{\alpha 0\gamma}\gamma\epsilon\eta 0\xi 0)\}\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]][[1]]
```

... Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is
Log[e^(γ (α0+ϵ τ0))] − Log[e^(γ α2) (e^(γ Subscript[≪2≫]) + e^(Times[≪2≫]) γ ϵ η2 ξ1)] == 0.

... Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

... Solve: Equations may not give solutions for all "solve" variables.

$$\Bigg\{\tau 0\to\frac{-\mathrm{Log}\left[e^{\alpha 0\gamma}\right]+\mathrm{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2}+e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_2\xi_1\right]}{\gamma\epsilon},\ \eta 0\to\frac{1}{\gamma\epsilon\left(\xi_1+e^{\gamma\alpha_2}\xi_2+e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_1\xi_2\right)}$$
$$e^{-\gamma\alpha_1}\Bigg(\frac{1}{2}+\frac{1}{2}e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1+\frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2+\frac{1}{2}e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2+\frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2-$$
$$\frac{1}{2}\sqrt{\Big(\left(-1-e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1-e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2-e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2-e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2\right)^2+}$$
$$4e^{-\alpha 0\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\left(-e^{\gamma\alpha_1}\eta_1\xi_1-\eta_2\xi_1-e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1^2-e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2-e^{\gamma\alpha_2}\eta_2\xi_2-\right.$$
$$\left.\left.2e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2-e^{\gamma\alpha_2}\gamma\epsilon\eta_2^2\xi_1\xi_2-e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2^2\xi_1^2\xi_2\right)\Big)\Bigg),$$
$$\xi 0\to\frac{1}{e^{\gamma\alpha_1}\eta_1+\eta_2+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1}e^{-\gamma\alpha_2}\Bigg(\frac{1}{2\gamma\epsilon}+\frac{1}{2}e^{\gamma\alpha_1}\eta_1\xi_1+\frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2+$$
$$\frac{1}{2}e^{\gamma\alpha_2}\eta_2\xi_2+\frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2-$$
$$\frac{1}{2\gamma\epsilon}\Bigg(\sqrt{\Big(\left(-1-e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1-e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2-e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2-e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2\right)^2+}$$
$$4e^{-\alpha 0\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\left(-e^{\gamma\alpha_1}\eta_1\xi_1-\eta_2\xi_1-e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1^2-e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2-e^{\gamma\alpha_2}\eta_2\xi_2-\right.$$
$$\left.\left.2e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2-e^{\gamma\alpha_2}\gamma\epsilon\eta_2^2\xi_1\xi_2-e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2^2\xi_1^2\xi_2\right)\Big)\Bigg)\Bigg)\Bigg\}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
    MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
  T0 MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\{\{e^{\gamma\alpha_2}(e^{\gamma\alpha_1}+e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1),\ e^{\gamma\alpha_1}\gamma\xi_1+e^{\gamma\alpha_2}\gamma(e^{\gamma\alpha_1}+e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1)\xi_2\},\ \{e^{\gamma\alpha_2}(e^{\gamma\alpha_1}\epsilon\eta_1+\epsilon\eta_2(1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1)),\ 1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1+e^{\gamma\alpha_2}\gamma(e^{\gamma\alpha_1}\epsilon\eta_1+\epsilon\eta_2(1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1))\xi_2\}\} == \{\{e^{\alpha 0\gamma}T0,\ e^{\alpha 0\gamma}T0\,\gamma\,\xi 0\},\ \{e^{\alpha 0\gamma}T0\,\epsilon\,\eta 0,\ T0(1+e^{\alpha 0\gamma}\gamma\epsilon\eta 0\,\xi 0)\}\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {T0, η0, α0, ξ0}]][[1]]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{T0\to\frac{1}{1+\gamma\epsilon\eta_2\xi_1},\ \eta 0\to\frac{\eta_1+e^{-\gamma\alpha_1}\eta_2+\gamma\epsilon\eta_1\eta_2\xi_1}{1+\gamma\epsilon\eta_2\xi_1},\ \alpha 0\to\frac{\mathrm{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2}(1+\gamma\epsilon\eta_2\xi_1)^2\right]}{\gamma},\ \xi 0\to\frac{e^{-\gamma\alpha_2}\xi_1+\xi_2+\gamma\epsilon\eta_2\xi_1\xi_2}{1+\gamma\epsilon\eta_2\xi_1}\right\}$$

```
SSϵ[ℰ_] := Block[{ϵ}, Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, Together]];
(* Shielded ϵ-Series *)
```

Logos

```
CΛ[t1_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
  {eqn, d, b, c, sol, λ, q, ν, ξ, η},
  eqn = ρ[e^(ξ x_CU)].ρ[e^(η y_CU)] == ρ[e^(d y_CU)].ρ[e^(c (t 1_CU - 2 ϵ a_CU))].ρ[e^(b x_CU)];
  sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0;
  λ = e^(-η y - ξ x + η ξ t) Normal@Series[e^(c t + d y - 2 ϵ c a + b x) /. sol, {ϵ, 0, $k}];
  q = e^(ν (-t ξ η + η y + ξ x + δ y x));
  Collect[ν q^-1 DP_{ξ→D_x, η→D_y}[λ][q] /. ν → (1 + t δ)^-1, ϵ, Simplify] /.
   {t → t1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1}];
```

Logos

```
QΛ[T1_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
  {G, F, fs, f, bs, e, b, es, λ, q, ν, ξ, η, t},
  G = Simp[
    Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[x_QU ** # - # ** x_QU] &, y_QU, $k + 1]];
  fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, l}, {j, 0, l}, {k, 0, l}];
  F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ e^l QU@{y^i, a^j, x^k});
  es = Flatten[
    Table[Coefficient[e, b] == 0, {e, {F - 1_QU /. η → 0, F ** G - y_QU ** F - ∂_η F}}, {b, bs}]];
  {λ} = F /. DSolve[es, fs, η] /. {e- → 1, QU → Times};
  q = e^(ν (-t ξ η + η y + ξ x + δ y x));
  Collect[ν q^-1 DP_{ξ→D_x, η→D_y}[λ][q] /. ν → (1 + t δ)^-1 /. t → (T - 1) / ℏ, ϵ, Simplify] /.
   {T → T1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1}];
```

Logos

```
wc[CU] = t; wc[QU] = (T - 1) / ℏ;
Λ[U_] := Λ[U] = Module[{Q, w}, Q = (-w ξ η + η y + ξ x + δ y x) / (1 + w δ);
    Collect[(1 + w δ)^-1 𝕖^-Q DP_{ξ→D_x, η→D_y}[λ[U]][𝕖^Q] /. w → wc[U], ϵ, Simplify]];
Λ[U_, t1_, T1_, y1_, a1_, x1_, ξ1_, η1_, δ1_] :=
  Λ[U] /. {t → t1, T → T1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1, δ → δ1};
```

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) ⧴ c D[λ, {x, m}, {y, n}]]
```

Logos

```
Λ_U_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, uu, ww, d},
    {yax, q, p} = List @@ Λ_U[{υ, ω}, {u, w}];
    Œ_U[yax, Q = (υ uu + ω ww + δ uu ww + d υ ω) / (1 - d δ), Expand[(1 - d δ)^-1 𝕖^-Q
        SP_{υ→uu, ω→ww}[p 𝕖^Q]]] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1, uu → u, ww → w}];
```

```
Rord_{u_i_, w_j_ → k_}[Œ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, δ1 = ∂_{u_i, w_j} Q},
    {yax, q, p} = List @@ If[δ1 === 0, Λ_U[{υ, ω}, {u, w}], Λ_U[{υ, ω, δ}, {u, w}]] /.
      {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s};
    Œ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), 𝕖^-q DP_{u_i→D_υ, w_j→D_ω}[P][p 𝕖^q]] /.
     {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
(*SimpPQ[PQ_] := Expand@Collect[PQ, {x_,y_,a_},CC]  /. 𝕖^δ_ ⧴ 𝕖^Expand@Together[δ] /.
   CC→ExpandDenominator@*ExpandNumerator@*Together;*)
SimpPQ[PQ_] := Simplify[PQ //. 𝕖^δ_ ⧴ 𝕖^Simplify[δ]];
```

Syax

Next task: $\mathrm{Exp}_U : U \to Œ$ ...

Next next task: Define $S\Lambda_{U,k}[\eta, \alpha, \xi, \delta]$, whose value is an $Œ_U[\{y_1, a_1, x_1\}_1,\ Q,\ P + 0_k]$ such that

```
U@SΛ_{U,k}[η, α, ξ, δ] == S_1@U@Œ_U[{y_1, a_1, x_1}_1, η y_1 + α a_1 + ξ x_1 + δ x_1 y_1, 0_k].
```

```
SΛ_{U_,k_}[η_, α_, ξ_, δ_]
```

In[∘]:=
```
Block[{$p = 4, $k = 4}, TableForm[StringSplit[
  "y|a|x|CΘ@y_CU|CΘ@a_CU|CΘ@x_CU|QΘ@y_QU|QΘ@a_QU|QΘ@x_QU|A𝔻@y_QU|A𝔻@a_QU|A𝔻@x_QU|S𝔻@y_QU|S𝔻@a_QU|S𝔻@x_QU|S@y_CU|S@a_CU|S@x_CU|S@y_QU|S@a_QU|S@x_QU|Δ@y_CU|Δ@a_CU|Δ@x_CU|Δ@y_QU-y_1|Δ@a_QU|Δ@x_QU-x_1", "|"] /. s_String ↦
  {s, Normal@Simplify@Log@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[∘]//TableForm=

y	$\mathrm{Log}[y]$
a	$\mathrm{Log}[a]$
x	$\mathrm{Log}[x]$
$C\Theta@y_{CU}$	$\mathrm{Log}[-x]$
$C\Theta@a_{CU}$	$\mathrm{Log}[-a]$
$C\Theta@x_{CU}$	$\mathrm{Log}[-y]$
$Q\Theta@y_{QU}$	$a\,\epsilon\,\hbar + \mathrm{Log}\left[-\frac{x}{\sqrt{T}}\right]$
$Q\Theta@a_{QU}$	$\mathrm{Log}[-a]$
$Q\Theta@x_{QU}$	$(a-\gamma)\,\epsilon\,\hbar + \mathrm{Log}\left[-\frac{y}{\sqrt{T}}\right]$
$A\mathbb{D}@y_{QU}$	$\frac{\epsilon\,\hbar\,(x\,y\,\gamma\,\hbar\,(20+10\,t\,\hbar+3\,t^2\,\hbar^2)-4\,a\,(60+40\,t\,\hbar+15\,t^2\,\hbar^2+4\,t^3\,\hbar^3))}{2\,(120+60\,t\,\hbar+20\,t^2\,\hbar^2+5\,t^3\,\hbar^3+t^4\,\hbar^4)} + \frac{\epsilon^2\,\hbar^2\,(24\,a\,t\,x\,y\,\gamma\,\hbar^2\,(80+320\,t\,\hbar+180\,t^2\,\hbar^2+45\,t^3\,\hbar^3+6\,t^4\,\hbar^4}{}$
$A\mathbb{D}@a_{QU}$	$\mathrm{Log}[a]$
$A\mathbb{D}@x_{QU}$	$\mathrm{Log}[x]$
$S\mathbb{D}@y_{QU}$	$-\frac{4\,(2\,a\,t-x\,y\,\gamma)\,\epsilon\,\hbar^2\,(240+t^2\,\hbar^2)}{23040+480\,t^2\,\hbar^2+t^4\,\hbar^4} + \frac{4\,\epsilon^2\,(4\,a\,t\,x\,y\,\gamma\,\hbar^4\,(92160+480\,t^2\,\hbar^2+t^4\,\hbar^4)+x\,y\,\gamma^2\,\hbar^4\,(-69120\,t-46080\,x\,y-1440\,t^3\,\hbar^2+480}{(23040+480\,t^2\,\hbar^2+t}$
$S\mathbb{D}@a_{QU}$	$\mathrm{Log}[a]$
$S\mathbb{D}@x_{QU}$	$\frac{4\,\epsilon\,\hbar\,(x\,y\,\gamma\,\hbar\,(240+120\,t\,\hbar+31\,t^2\,\hbar^2)-2\,a\,(2880+1680\,t\,\hbar+540\,t^2\,\hbar^2+121\,t^3\,\hbar^3))}{23040+11520\,t\,\hbar+3360\,t^2\,\hbar^2+720\,t^3\,\hbar^3+121\,t^4\,\hbar^4} + \frac{4\,\epsilon^2\,\hbar^2\,((23040+11520\,t\,\hbar+3360\,t^2\,\hbar^2+720\,t^3}{}$
$S@y_{CU}$	$\mathrm{Log}[-y]$
$S@a_{CU}$	$\mathrm{Log}[-a]$
$S@x_{CU}$	$\mathrm{Log}[-x]$
$S@y_{QU}$	$(a-\gamma)\,\epsilon\,\hbar + \mathrm{Log}\left[-\frac{y}{T}\right]$
$S@a_{QU}$	$\mathrm{Log}[-a]$
$S@x_{QU}$	$a\,\epsilon\,\hbar + \mathrm{Log}[-x]$
$\Delta@y_{CU}$	$\mathrm{Log}[y_1 + T_1\,y_2]$
$\Delta@a_{CU}$	$\mathrm{Log}[a_1 + a_2]$
$\Delta@x_{CU}$	$\mathrm{Log}[x_1 + x_2]$
$\Delta@y_{QU}-y_1$	$\mathrm{Log}[T_1\,y_2] - \epsilon\,\hbar\,a_1$
$\Delta@a_{QU}$	$\mathrm{Log}[a_1 + a_2]$
$\Delta@x_{QU}-x_1$	$\mathrm{Log}[x_2] - \epsilon\,\hbar\,a_1$

Exp

Task. Define $\mathrm{Exp}_{U,k}[\xi, X, P]$ which computes $e^{\xi X\cdot \mathbb{O}(P)}$ to ϵ^k in the algebra U, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent docile perturbation, giving the answer in Œ-form. Should satisfy $U@\mathrm{Exp}_{U,k}[\xi,\ X,\ P] == \mathbb{S}_U[e^{\xi x},\ x \to X\cdot\mathbb{O}(P)]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi X\cdot\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi X\cdot P_0} F(\xi)\right)$, then $F(\xi=0)=1$ and we have:

$$\mathbb{O}\left(e^{\xi X\cdot P_0}(X P_0 F(\xi) + \partial_\xi F\right) = .$$

$$\mathbb{O}\left(\partial_\xi e^{\xi X\cdot P_0} F(\xi)\right) = \partial_\xi \mathbb{O}\left(e^{\xi X\cdot P_0} F(\xi)\right) = \partial_\xi e^{\xi X\,\mathbb{O}(P)} = e^{\xi X\,\mathbb{O}(P)}\, X\,\mathbb{O}(P) = \mathbb{O}\left(e^{\xi X\cdot P_0} F(\xi)\right) X\,\mathbb{O}(P)$$

This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k\,\varphi$ we find that $F_0 = 1$ and solve for φ.

```
In[•]:= Exp_{U_,0}[ξ_, (X : (x | y))_{i_}, P_] := 𝔼_U[{y_i, a_i, x_i}_i, Normal@P /. ϵ → 0, 1 + 0_0];
Exp_{U_,k_}[ξ_, (X : (x | y))_{i_}, P_] := Module[{P0, f, φ, φs, F, rhs, at0, atξ},
  P0 = Normal@P /. ϵ → 0;
  f = Normal@Last@Exp_{U,k-1}[ξ, X_i, P];
  φs = Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, 2 k}, {j1, 0, 2 k - j2}, {j3, 0, 2 k - j2 - j1}];
  F = f + ϵ^k φs.(φs /. φ_{j1_,j2_,j3_}[ξ] :→ y_i^{j1} a_i^{j2} x_i^{j3});
  rhs = Normal@Last@m_{i,b,c→i}[
      𝔼_U[{y_i, a_i, x_i}_i, ξ X_i P0, F + 0_k] 𝔼_U[{X_b}_b, 0, X_b + 0_k] m_{i→c}@𝔼_U[{y_i, a_i, x_i}_i, 0, P + 0_k]];
  at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}];
  atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 X_i F - rhs, {y_i, a_i, x_i}];
  𝔼_U[{y_i, a_i, x_i}_i, ξ X_i P0, F + 0_k] /. DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]]
 ]
```

The antipode on exponentials in QU.

Computing $S(e^{\xi x})$: If $S(e^{\xi x}) = \mathbb{O}(ax : F e^{-\xi x})$,

then $F_{\xi=0} = 1$ and $\mathbb{O}(ax : (\partial_\xi F - x F) e^{-\xi x}) = \partial_\xi S(e^{\xi x}) = S(x e^{\xi x}) =$

$S(e^{\xi x}) S(x) = \mathbb{O}(ax : F e^{-\xi x})(-e^{\hbar\epsilon a} x) = \mathbb{O}(a x a_2 x_2 : -x_2 F e^{-\xi x + \hbar\epsilon a_2})$, and that's an ODE for F.

SxF

```
In[•]:= SxF[0] = 1;
SxF[k_] := SxF[k] = Module[{fs, bs, F, rhs, at0, atξ},
    fs = Flatten@Table[f_{i,j}[ξ], {i, 0, 2 k}, {j, 0, 2 k - i}];
    F = SxF[k - 1] + ϵ^k fs.(bs = fs /. f_{i_,j_}[ξ] :→ a^i x^j);
    rhs = Normal@Last@Cord[𝔼_QU[{a_1, x_1, a_2, x_2}_1, -ξ x_1, (F /. {a → a_1, x → x_1})
          Series[-x_2 e^{ħ ϵ a_2}, {ϵ, 0, k}]] /. ξ → ħ ξ] /. {ξ → ħ^{-1} ξ, a_1 → a, x_1 → x};
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {a, x}];
    atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) - x F - rhs, {a, x}];
    F /. DSolve[And @@ (at0 ⋃ atξ), fs, ξ][[1]]
   ];
```

```
In[•]:= Timing@Block[{$p = 8, $k = 3}, {
    Collect[SxF[$k], {ϵ, a}],
    HL@Simp[S_1@QU_1@𝕆_QU[{x}, SS[e^{ħ ξ x}]] - QU_1@𝕆_QU[{a, x}, SS[e^{-ħ ξ x} (SxF[$k] /. ξ → ħ ξ)]]]
  }]
```

Out[•]= $\{2.76563, \{1 + \epsilon\left(-a x \xi \hbar - \frac{1}{2} x^2 \gamma \xi^2 \hbar\right) + \epsilon^2\left(\frac{1}{4} x^2 \gamma^2 \xi^2 \hbar^2 - \frac{1}{2} x^3 \gamma^2 \xi^3 \hbar^2 + \frac{1}{8} x^4 \gamma^2 \xi^4 \hbar^2 + a^2\left(-\frac{1}{2} x \xi \hbar^2 + \frac{1}{2} x^2 \xi^2 \hbar^2\right) + a\left(-x^2 \gamma \xi^2 \hbar^2 + \frac{1}{2} x^3 \gamma \xi^3 \hbar^2\right)\right) + \epsilon^3\left(-\frac{1}{12} x^2 \gamma^3 \xi^2 \hbar^3 + \frac{2}{3} x^3 \gamma^3 \xi^3 \hbar^3 - \frac{19}{24} x^4 \gamma^3 \xi^4 \hbar^3 + \frac{1}{4} x^5 \gamma^3 \xi^5 \hbar^3 - \frac{1}{48} x^6 \gamma^3 \xi^6 \hbar^3 + a^3\left(-\frac{1}{6} x \xi \hbar^3 + \frac{1}{2} x^2 \xi^2 \hbar^3 - \frac{1}{6} x^3 \xi^3 \hbar^3\right) + a^2\left(-x^2 \gamma \xi^2 \hbar^3 + \frac{5}{4} x^3 \gamma \xi^3 \hbar^3 - \frac{1}{4} x^4 \gamma \xi^4 \hbar^3\right) + a\left(\frac{1}{2} x^2 \gamma^2 \xi^2 \hbar^3 - \frac{7}{4} x^3 \gamma^2 \xi^3 \hbar^3 + x^4 \gamma^2 \xi^4 \hbar^3 - \frac{1}{8} x^5 \gamma^2 \xi^5 \hbar^3\right)\right), 0\}\}$

Computing $S(e^{\eta y})$: If $S(e^{\eta y}) = \mathbb{O}(ya : F e^{-T^{-1} \eta y})$,

then $F_{\eta=0} = 1$ and $\mathbb{O}\left(ya : \left(\partial_\eta F - T^{-1} y F\right) e^{-T^{-1}\eta y}\right) = \partial_\eta S\left(e^{\eta y}\right) = S\left(y e^{\eta y}\right) = S\left(e^{\eta y}\right) S(y) =$
$\mathbb{O}\left(ya : F e^{-T^{-1}\eta y}\right)\left(-e^{\hbar\epsilon a} T^{-1} y\right) = \mathbb{O}\left(yaa_2 y_2 : -T^{-1} y_2 F e^{-T^{-1}\eta y + \hbar\epsilon a_2}\right)$, and that's an ODE for F.

SyF

In[∘]:=
```
SyF[0] = 1;
SyF[k_] := SyF[k] = Module[{fs, bs, F, rhs, at0, atη},
    fs = Flatten@Table[f_{i,j}[η], {i, 0, 2 k}, {j, 0, 2 k - i}];
    F = SyF[k - 1] + ϵ^k fs.(bs = fs /. f_{i_,j_}[η] :> y^i a^j);
    rhs = Normal@Last@Cord[ℭ_QU[{y1, a1, a2, y2}1, -T^-1 η y1, (F /. {a → a1, y → y1})
            Series[-y2 T^-1 e^(ℏ ϵ a2), {ϵ, 0, k}]] /. η → ℏ η] /. {η → ℏ^-1 η, a1 → a, y1 → y};
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. η → 0, {a, y}];
    atη = (# == 0) & /@ Flatten@CoefficientList[(∂_η F) - T^-1 y F - rhs, {a, y}];
    F /. DSolve[And @@ (at0 ⋃ atη), fs, η][[1]]
  ];
```

In[∘]:=
```
Timing@Block[{$p = 8, $k = 3}, {
   Collect[SyF[$k], {ϵ, a}],
   HL@Simp[S1@QU1@𝕆_QU[{y}, SS[e^(ℏ η y)]] - QU1@𝕆_QU[{y, a}, SS[e^(-ℏ η y/T) (SyF[$k] /. η → ℏ η)]]]
  }]
```

Out[∘]= {10.5469,

$$\left\{1 + \epsilon\left(-\frac{a\,y\,\eta\,\hbar}{T} + \frac{y\,\gamma\,\eta\,\hbar}{T} - \frac{y^2\,\gamma\,\eta^2\,\hbar}{2\,T^2}\right) + \epsilon^2\left(-\frac{y\,\gamma^2\,\eta\,\hbar^2}{2\,T} + \frac{7\,y^2\,\gamma^2\,\eta^2\,\hbar^2}{4\,T^2} - \frac{y^3\,\gamma^2\,\eta^3\,\hbar^2}{T^3} + \frac{y^4\,\gamma^2\,\eta^4\,\hbar^2}{8\,T^4} + a^2\left(-\frac{y\,\eta\,\hbar^2}{2\,T} + \frac{y^2\,\eta^2\,\hbar^2}{2\,T^2}\right) + a\left(\frac{y\,\gamma\,\eta\,\hbar^2}{T} - \frac{2\,y^2\,\gamma\,\eta^2\,\hbar^2}{T^2} + \frac{y^3\,\gamma\,\eta^3\,\hbar^2}{2\,T^3}\right)\right) + \epsilon^3\left(\frac{y\,\gamma^3\,\eta\,\hbar^3}{6\,T} - \frac{25\,y^2\,\gamma^3\,\eta^2\,\hbar^3}{12\,T^2} + \frac{23\,y^3\,\gamma^3\,\eta^3\,\hbar^3}{6\,T^3} - \frac{49\,y^4\,\gamma^3\,\eta^4\,\hbar^3}{24\,T^4} + \frac{3\,y^5\,\gamma^3\,\eta^5\,\hbar^3}{8\,T^5} - \frac{y^6\,\gamma^3\,\eta^6\,\hbar^3}{48\,T^6} + a^3\left(-\frac{y\,\eta\,\hbar^3}{6\,T} + \frac{y^2\,\eta^2\,\hbar^3}{2\,T^2} - \frac{y^3\,\eta^3\,\hbar^3}{6\,T^3}\right) + a^2\left(\frac{y\,\gamma\,\eta\,\hbar^3}{2\,T} - \frac{5\,y^2\,\gamma\,\eta^2\,\hbar^3}{2\,T^2} + \frac{7\,y^3\,\gamma\,\eta^3\,\hbar^3}{4\,T^3} - \frac{y^4\,\gamma\,\eta^4\,\hbar^3}{4\,T^4}\right) + a\left(-\frac{y\,\gamma^2\,\eta\,\hbar^3}{2\,T} + \frac{4\,y^2\,\gamma^2\,\eta^2\,\hbar^3}{T^2} - \frac{19\,y^3\,\gamma^2\,\eta^3\,\hbar^3}{4\,T^3} + \frac{3\,y^4\,\gamma^2\,\eta^4\,\hbar^3}{2\,T^4} - \frac{y^5\,\gamma^2\,\eta^5\,\hbar^3}{8\,T^5}\right)\right), 0\right\}\}$$

Logos

In[∘]:=
```
Λ_U_,0[{ξ1_, η1_}, {x, y}] :=
  Œ_QU[{y, a, x}, ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}), 1 + O_0];
Λ_U_,kk_[{ξ1_, η1_}, {x, y}] := (*Λ_U,kk[{ξ1,η1},{x,y}]=*)
  Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
    fs = Flatten@Table[f_{i,j,k}[η], {i, 0, $k}, {j, 0, $k}, {k, 0, $k}];
    F = O_U[{y, a, x}, Normal@Last@Λ_{U,kk-1}[{ξ, η}, {x, y}]] +
      fs.(bs = fs /. f_{i_,j_,k_}[η] :> ϵ^kk U@{y^i, a^j, x^k});
    es = Flatten[Table[Coefficient[e, b] == 0,
        {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
    F = F /. DSolve[es, fs, η][[1]];
    Œ_U[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
      Simplify[F + O_$k /. {ⅇ- → 1, U → Times}]
     ] /. {ξ → ξ1, η → η1}]];
```

In[∘]:=
```
Timing@Block[{$p = 3, $k = 1}, {
   sexp = m_{3,2,1→1}[Exp_{QU_1,$k}[η, S_1[QU[y_1]] /. QU → Times] Exp_{QU_2,$k}[α, S_2[QU[a_2]] /. QU → Times]
       Exp_{QU_3,$k}[ξ, S_3[QU[x_3]] /. QU → Times]] /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ},
   HL@SimpT[QU@sexp - S_1@O_QU[{y_1, a_1, x_1}_1, SS[ⅇ^(ℏ (η y_1+α a_1+ξ x_1))]]]
  }]
```

Out[∘]=
$$\left\{9.34375, \left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \frac{1}{\hbar T_1}\left(e^{\alpha\gamma\hbar}\eta\xi\hbar^2 - e^{\alpha\gamma\hbar}\eta\xi\hbar^2 T_1 - \alpha\hbar^2 a_1 T_1 - e^{\alpha\gamma\hbar}\xi\hbar^2 T_1 x_1 - e^{\alpha\gamma\hbar}\eta\hbar^2 y_1\right),\right.\right.$$
$$1 + \frac{1}{4\hbar T_1^2}\left(-3 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 - 4 e^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1 + 4 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1 + \right.$$
$$8 e^{\alpha\gamma\hbar}\eta\xi\hbar^3 a_1 T_1 + 4 e^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1^2 - e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1^2 + 6 e^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1 x_1 -$$
$$2 e^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1^2 x_1 - 4 e^{\alpha\gamma\hbar}\xi\hbar^3 a_1 T_1^2 x_1 - 2 e^{2\alpha\gamma\hbar}\gamma\xi^2\hbar^4 T_1^2 x_1^2 +$$
$$6 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 y_1 + 4 e^{\alpha\gamma\hbar}\gamma\eta\hbar^3 T_1 y_1 - 2 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 T_1 y_1 -$$
$$\left.\left.\left. 4 e^{\alpha\gamma\hbar}\eta\hbar^3 a_1 T_1 y_1 - 4 e^{2\alpha\gamma\hbar}\gamma\eta\xi\hbar^4 T_1 x_1 y_1 - 2 e^{2\alpha\gamma\hbar}\gamma\eta^2\hbar^4 y_1^2\right)\epsilon + O[\epsilon]^2\right], 0\right\}\right\}$$