

Pensieve header: A unified verification notebook for the \$sl_2\$-portfolio project, Uxi version. Continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
In[*]:= HL[ε_] := Style[ε, Background → Yellow];
```

DocileQ

DocileQ

```
In[*]:= DQ[ε_] := (Exponent[Normal@ε /.
  {a → a / ε, a_i → a_i / ε, (u : x | y) ⇒ ε-1/2 u, (u : x | y)_i ⇒ ε-1/2 u_i}, ε, Min] ≥ 0);
```

```
In[*]:= DQ /@ {ε2 x y a2, ε2 x2 y3}
```

```
Out[*]:= {True, False}
```

Initialization / Utilities

It is verification-risky to work with low \$E\$!

TD

```
In[*]:= $p = 2; $k = 1; $U = QU; $E := {$k, $p};
$trim := {hp /; p > $p → 0, ek /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
T2t = {T_i → eh t_i, T → eh t}; q_h = eγ ε h;
t2T = {ec- t_i + b- ⇒ Tc/h eb, ec- t + b- ⇒ Tc/h eb, eε ⇒ eExpand@ε};
SS[ε_, op_] := Collect[
  Normal@Series[If[$p > 0, ε, ε / . T2t], {h, 0, $p}],
  h, op];
SS[ε_] := SS[ε, Together];
SST[ε_, op___] := SS[ε / . T2t, op];
Simp[ε_, op_] := Collect[ε, _CU | _QU, op];
Simp[ε_] := Simp[ε, SS[#, Expand] &];
SimpT[ε_] := Collect[ε, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδi,j := If[i == j, 1, 0];
```

Differential polynomials (DP):

Utils

```
In[*]:= DPα→Dx, β→Dy[P_][λ_] :=
  Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) ⇒ c ∂{x,m},{y,n} λ]
```

$$\text{HL}[\text{DP}_{x \rightarrow D_\epsilon, y \rightarrow D_\eta}[x^2 y^3][e^{\delta \xi \eta}] == 6 e^{\delta \eta \xi} \delta^3 \xi + 6 e^{\delta \eta \xi} \delta^4 \eta \xi^2 + e^{\delta \eta \xi} \delta^5 \eta^2 \xi^3]$$

True

CF

```
In[ ]:= CF[ ] := ExpandDenominator@
ExpandNumerator@Together[Expand[ ] /. e^x - e^y -> e^{x+y} /. e^x -> e^{CF[x]}];
```

SeriesData

```
In[ ]:= Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs_] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

SP

```
In[ ]:= SP_{ } [P_] := P; SP_{\xi \to x, ps} [P_] := Expand[P // SP_{ps}] /. f_ . \xi^{d_} -> \partial_{\{x,d\}} f
```

$$\text{SP}_{\{\xi \to x\}}[(\xi^2 + \xi + 3)(x^5 e^x + 7x) + 99a]$$

$$7 + 99a + 21x + 20 e^x x^3 + 15 e^x x^4 + 5 e^x x^5$$

$$\text{SP}_{\{\xi \to x, \eta \to y\}}[(\xi^2 + \xi + 3 + 2 \xi \eta)(x^5 e^x + 7x) + 99a + e^{\delta x y} \xi \eta]$$

$$7 + 99a + 21x + 20 e^x x^3 + 15 e^x x^4 + 5 e^x x^5 + e^{xy \delta} \delta + e^{xy \delta} x y \delta^2$$

DeclareAlgebra

QLImplementation

```
In[ ]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

In[*]:=

```

DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
  gs = Generators /. {opts},
  cs = Centrals /. {opts} /. Centrals -> {}},
  (#u = U@#) & /@gs;
  gp = Alternatives @@ gs; gp = gp | gp; (* gens *)
  sr = Flatten@Table[{g -> ++k, gi -> {i, k}}, {g, gs}]; (* sorting -> *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[_] := Collect[_U, Expand] /. $trim;
  Ui[_] := # /. {t : cp -> ti, u_U -> (#i &) /@u};
  Ui[NCM[]] = pow[_] = U@{ } = 1_U = U[];
  B[U@(x_)i_, U@(y_)i_] := Ui@B[U@x, U@y];
  B[U@(x_)i_, U@(y_)j_] /; i != j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy];
  U@{c_. * (L : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[L, {n}] ** U@{r}];
  U@{c_. * L : gp, r___} := CE[c U[L] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
  U@{L_, r___} := U@{Expand[L], r};
  U[_NonCommutativeMultiply] := U /@ #;
  OU[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, L_List -> L_null, {1}];
    vs = Join@@(First /@ sp);
    us = Join@@(sp /. L_s_ -> (L /. x_i_ -> x_s));
    CE[Total[
      CoefficientRules[poly, vs] /. (p_ -> c_) -> c U@(us^p)
    ] /; x_null -> x];
  pow[_] := pow[_ - 1] ** #;
  SU[_] := CE@Total[
    CoefficientRules[_] /.
      (p_ -> c_) -> c NCM@@MapThread[pow, {Last /@ {ss}, p}]];
  sigma_rs___[c_. * u_U] := (c /. (t : cp)j_ -> tj /. {rs}) U[List@@(u /. v_j_ -> vj /. {rs})];
  m_j_to_k[c_. * u_U] := CE[((c /. (t : cp)j_ -> tk) DeleteCases[u, _j|k]) **
    U@@Cases[u, w_j -> wk] ** U@@Cases[u, _k]];
  U /: c_. * u_U * v_U := CE[c u ** v];
  Si[c_. * u_U] := CE[((c /. Si[U, Centrals]) DeleteCases[u, _i]) **
    Ui[NCM@@Reverse@Cases[u, x_i -> S@U@x]];
  Delta_i_to_j_k[c_. * u_U] := CE[((c /. Delta_i_to_j_k[U, Centrals]) DeleteCases[u, _i]) **
    (NCM@@Cases[u, x_i -> sigma_1_to_j_2_to_k@Delta@U@x] /. NCM[] -> U[])];

```

DeclareMorphism

QLImplementation

```
In[*]:= DeclareMorphism[m_, U_ -> V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
    {(g_ -> img_) -> (m[U[g]] = img), (g_ -> img_) -> (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i]] := V_i[m[U@g]];
  m[U[vs_]] := NCM@@(m/@U/@{vs});
  m[E_] := Simp[E /. oncs /. u_U -> m[u]] /. $trim;
```

Meta-Operations

QLImplementation

```
In[*]:= sigma_RS__[E_Plus] := sigma_RS /@ E;
m_j -> j_ = Identity; m_j -> k_[0] = 0;
m_j -> k_[E_Plus] := Simp[m_j -> k_ /@ E];
m_i_s___, i_, j_ -> k_[E_] := m_j -> k_ @ m_i_s, i -> j @ E;
S_i_[E_Plus] := Simp[S_i_ /@ E];
Delta_i_s___[E_Plus] := Simp[Delta_i_s_ /@ E];
```

Implementing $CU = \mathcal{U}(sl_2^{\mathbb{C}})$

Verify σ and Δ ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\dots}$.

CU

```
In[*]:= DeclareAlgebra[CU, Generators -> {y, a, x}, Centrals -> {t}];
B[a_CU, y_CU] = -y y_CU; B[x_CU, a_CU] = -y x_CU;
B[x_CU, y_CU] = 2 e a_CU - t 1_CU;
(S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU);
S_i_[CU, Centrals] = {t_i -> -t_i};
Delta@y_CU = CU@y_1 + CU@y_2; Delta@a_CU = CU@a_1 + CU@a_2; Delta@x_CU = CU@x_1 + CU@x_2;
Delta_i -> j_, k_[CU, Centrals] = {t_i -> t_j + t_k};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp // HL
}] // Timing
{1.32813,
{(28 t^2 γ^4 + 116 t γ^5 ε) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Implementing QU = $\mathcal{U}_q(\mathfrak{sl}_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T e^{-2ε a ħ}) / ħ, {a, 0, 3}]
```

Aside

$$\frac{1 - T}{\hbar} + 2 T \epsilon a - 2 (T \epsilon^2 \hbar) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

```
In[*]:= HL /@ DQ /@ Series[{{(1 - T e^{-2ε a ħ}) / ħ, e^{ħ ε a}}, {ε, 0, 5}]
```

```
Out[*]:= {True, True}
```

QU

```
In[*]:= DeclareAlgebra[QU, Generators -> {y, a, x}, CentralS -> {t, T}];
B[aQU, yQU] = -γ yQU; B[xQU, aQU] = -γ QU@x;
B[xQU, yQU] := SS[qn - 1] QU@{y, x} + OQU[{a}, SS[(1 - T e^{-2ε a ħ}) / ħ]];
(S@yQU := OQU[{a, y}, SS[-T^{-1} e^{ħ ε a} y]]; S@aQU = -aQU; S@xQU := OQU[{a, x}, SS[-e^{ħ ε a} x]];
Si[QU, CentralS] = {ti -> -ti}, Ti -> Ti^{-1}};
Δ@yQU := OQU[{y1, a1}1, {y2}2, SS[y1 + T1 e^{-ħ ε a1} y2]];
Δ@aQU = QU@a1 + QU@a2; Δ@xQU := OQU[{a1, x1}1, {x2}2, SS[x1 + e^{-ħ ε a1} x2]];
Δi -> j, k[QU, CentralS] = {ti -> tj + tk}, Ti -> Tj Tk}}
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} -> Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}]]
{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> γ QU[y],
{QU[y], QU[x]} -> ((-1 + T) QU[] / ħ) - 2 T ε QU[a] - γ ε ħ QU[y, x]},
{{QU[a], QU[y]} -> -γ QU[y], {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> γ QU[x]},
{{QU[x], QU[y]} -> ((1 - T) QU[] / ħ) + 2 T ε QU[a] + γ ε ħ QU[y, x],
{QU[x], QU[a]} -> -γ QU[x], {QU[x], QU[x]} -> 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
{z1, bas}, {z2, bas}, {z3, bas}]]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ \$p=5, \$k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  HL[z1 ** (z2 ** z3) - rhs // Simp]
}] // Timing

{3.78125, {
  (

$$\frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \epsilon - 280 T \ll 1 \gg \epsilon + 198 T^2 \gamma^5 \epsilon}{\hbar}$$

    ) QU[y, y, y, x, x] +
  <<18>> + (1 + 8 \gamma \epsilon \hbar) QU[y, <<11>>, x], 0}}

```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs @ \$p=5, \$k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  Short[lhs = z1 ** (z2 ** z3)],
  Short[rhs = (QU@@z1) ** ((QU@@z2) ** (QU@@z3))],
  Expand[Limit[rhs /. T2t \cup {QU -> CU}, \hbar -> 0] - lhs] // HL
}] // Timing

{10.125, {
  28 t^2 \gamma^4 CU[y, y, y, x, x] +
  116 t \gamma^5 \epsilon CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, a, a, a, a, x, x, x, x],
  2 (

$$\frac{\gamma^4}{\hbar^2} - \frac{2 T \gamma^4}{\hbar^2} + \frac{T^2 \gamma^4}{\hbar^2} + \frac{\gamma^5 \epsilon}{\hbar} - \frac{2 T \gamma^5 \epsilon}{\hbar} + \frac{T^2 \gamma^5 \epsilon}{\hbar}$$

    ) QU[y, y, y, x, x] +
  <<209>> + (1 + 8 \gamma \epsilon \hbar) QU[y, y, y, <<7>>, x, x, x], 0}}

```

Verifying $\sigma, m, S,$ and Δ .

Verifying $\sigma_{i \rightarrow j, k \rightarrow l}$:

```
In[*]:= CU@x1 + CU@x2 // \sigma_{1 \rightarrow 3, 2 \rightarrow 4}
Out[*]:= CU[x3] + CU[x4]
```

Verifying relabeling using m :

```
In[*]:= t1 t3 CU[y1, a1, x2] + t1 t1 CU[y1, a2, x2] // m_{1 \rightarrow 3}
Out[*]:= CU[a2, x2, y3] t3^2 + CU[x2, y3, a3] t3^2
```

Verifying the meta-associativity of m :

```
In[*]:= Module[{z, u},
  Table[u = CU[z[[1]]1, z[[2]]2, z[[3]]3]; HL[m_{1,3 \rightarrow 3} @ m_{2,3 \rightarrow 3} @ u == m_{2,3 \rightarrow 3} @ m_{1,2 \rightarrow 2} @ u],
  {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
Out[*]:= {{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True}}
```

Verifying the involutivity of S on CU on products of triples:

```
In[ ]:= With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
Out[ ]:= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying that S is an anti-homomorphism on CU/QU:

```
In[ ]:= With[{bas = U /@ {y1, a1, x1}},
  Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
    {z1, bas}, {z2, bas}, {U, {CU, QU}} ] ]
```

```
Out[ ]:= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}
```

Verifying the co-associativity of Δ :

```
In[ ]:= Block[{bas = U /@ {y1, a1, x1}},
  Table[(z1 ** z2 ** z3 //  $\Delta_{1 \rightarrow 1, 2}$  //  $\Delta_{2 \rightarrow 2, 3}$ ) - (z1 ** z2 ** z3 //  $\Delta_{1 \rightarrow 1, 3}$  //  $\Delta_{1 \rightarrow 1, 2}$ ) // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas}, {U, {CU, QU}} ] ]
```

```
Out[ ]:= {{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
  {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
  {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying S- Δ compatibility:

```
In[ ]:= Block[{bas = U /@ {y1, a1, x1}},
  Table[z1 ** z2 ** z3 //  $\Delta_{1 \rightarrow 1, 2}$  // Si //  $m_{1, 2 \rightarrow 1}$  // Simp // HL,
    {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
Out[ ]:= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying S- Δ compatibility for opposite m , only for CU:

```
In[ ]:= Block[{bas = CU /@ {y1, a1, x1}},
  Table[z1 ** z2 ** z3 //  $\Delta_{1 \rightarrow 1, 2}$  // Si //  $m_{2, 1 \rightarrow 1}$  // Simp // HL,
    {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
Out[ ]:= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying m - Δ compatibility:

```
In[ ]:= Block[{bas1 = U /@ {y1, a1, x1}, bas2 = U /@ {y2, a2, x2}},
  Table[{z1 ** z2 ** z3 ** z4 // m1,2->1 // Δ1->1,2} -
    (z1 ** z2 ** z3 ** z4 // Δ1->3,4 // Δ2->5,6 // m3,5->1 // m4,6->2) // Simp // HL,
    {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2} ] ]
```

```
Out[ ]:= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Implementing θ

theta

```
In[ ]:= DeclareMorphism[Cθ, CU → CU, {y → -xCU, a → -aCU, x → -yCU}, {t → -t, T → T-1}] ;
DeclareMorphism[Qθ, QU → QU, {y := OQU[{a, x}, SS[-T-1/2 eħεa x]],
  a → -aQU, x := OQU[{a, y}, SS[-T-1/2 eħεa y]]}, {t → -t, T → T-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas} ]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas} ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas} ]
{QU[y] → - $\frac{QU[x]}{\sqrt{T}}$  -  $\frac{\epsilon \hbar QU[a, x]}{\sqrt{T}}$  → QU[y], QU[a] → -QU[a] → QU[a],
  QU[x] →  $\left(-\frac{1}{\sqrt{T}} + \frac{\gamma \epsilon \hbar}{\sqrt{T}}\right) QU[y]$  -  $\frac{\epsilon \hbar QU[y, a]}{\sqrt{T}}$  → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:


```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[Simp[QΘ[z1 ** z2] - QΘ[z1] ** QΘ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$AD\$f = \gamma \left(\left(\text{Cosh} \left[\hbar \left(a \epsilon + \frac{\gamma \epsilon}{2} - \frac{t}{2} \right) \right] - \text{Cosh} \left[\hbar \sqrt{\left(\frac{t - \gamma \epsilon}{2} \right)^2 + \epsilon \omega} \right] \right) / \right. \\ \left. \left(\hbar e^{\hbar ((a+\gamma) \epsilon - t/2)} \text{Sinh} \left[\frac{\gamma \epsilon \hbar}{2} \right] (a^2 \epsilon + a \gamma \epsilon - a t - \omega) \right) \right);$$

Docility of AD\$:f:

```
In[ ]:= HL@DQ@Block[{$p = 4}, Collect[SS@AD$f /. ω → a1, ε]]
```

Out[]:= True

Scaling behaviour of AD\$:f:

```
HL@Simplify[AD$f == ((AD$f /. γ → 1) /. {ε → γ ε, a → γ-1 a, ω → γ-1 ω})]
```

True

```
HL@FullSimplify[
  AD$f == ((AD$f /. γ → 1) /. {ħ → γ2 ħ, ε → ε / γ, a → a / γ, t → γ-2 t, ω → γ-3 ω})]
```

True

ADeq

$$AD\$ω = \gamma \text{CU}[y, x] + \epsilon \text{CU}[a, a] - (t - \gamma \epsilon) \text{CU}[a];$$

ADeq

```
In[ ]:= DeclareMorphism[AD, QU → CU,
  {a → aCU, x → CU@x, y := SCU[SS[AD$f], a → aCU, ω → AD$ω] ** yCU}
```

Verifying that the asymmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$\text{In[*]:= SD\$g = \sqrt{\left(\left(2 \gamma \left(\text{Cosh} \left[\frac{\hbar}{2} \sqrt{t^2 + \gamma^2 \epsilon^2 + 4 \epsilon \varpi} \right] - \text{Cosh} \left[\frac{t - \epsilon \gamma - 2 \epsilon a}{2 / \hbar} \right] \right) \right) / \left(\text{Sinh} \left[\frac{\gamma \epsilon \hbar}{2} \right] (t (2 a + \gamma) - 2 a (a + \gamma) \epsilon + 2 \varpi) \hbar \right) \right)};$$

SDeq

$$\text{In[*]:= SD\$f = Simplify} \left[e^{\hbar (t/2 - \epsilon a)} (\text{SD\$g} /. \{a \rightarrow -a, t \rightarrow -t\}) \right];$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

$$\text{In[*]:= } \left\{ \text{SD\$P} = \frac{\text{Cosh} \left[\hbar \left(\frac{\epsilon - t}{2} + \epsilon a \right) \right] - \text{Cosh} \left[\hbar \sqrt{\frac{t^2 + \epsilon^2}{4} + \epsilon \varpi} \right]}{\hbar \text{Sinh} \left[\frac{-\epsilon \hbar}{2} \right] (\varpi - \epsilon a^2 + (t - \epsilon) a + t/2)} \right.,$$

$$\text{Simplify} [\text{SD\$P} /. \{a \rightarrow -a - 1, t \rightarrow -t\}] // \text{HL},$$

$$\text{PowerExpand} @ \text{Simplify} \left[(\text{SD\$P} /. \{\hbar \rightarrow \gamma^2 \hbar, \epsilon \rightarrow \epsilon / \gamma, a \rightarrow a / \gamma, t \rightarrow \gamma^{-2} t, \varpi \rightarrow \gamma^{-3} \varpi\}) == \right.$$

$$\text{SD\$g} (\text{SD\$g} /. \{a \rightarrow -a - \gamma, t \rightarrow -t\}) // \text{HL},$$

$$\text{SD\$Q} = \text{Simplify} [\text{SD\$P} /. \{a \rightarrow c - 1/2\}],$$

$$\text{Simplify} [\text{SD\$Q} == (\text{SD\$Q} /. \{c \rightarrow -c, t \rightarrow -t\})] // \text{HL},$$

$$\text{FullSimplify} [\text{SD\$g} == \text{FullSimplify} [$$

$$\sqrt{\text{SD\$Q}} /. c \rightarrow a + 1/2 /. \{\hbar \rightarrow \gamma^2 \hbar, \epsilon \rightarrow \epsilon / \gamma, a \rightarrow a / \gamma, t \rightarrow \gamma^{-2} t, \varpi \rightarrow \gamma^{-3} \varpi\}] // \text{HL},$$

$$\text{HL@DQ@Block} [\{\$p = 4\}, \text{Collect} [\text{SS@SD\$g} /. \omega \rightarrow a_1, \epsilon]],$$

$$\text{HL@DQ@Block} [\{\$p = 4\}, \text{Collect} [\text{SS@SD\$f} /. \omega \rightarrow a_1, \epsilon]]$$

$$\left. \right\}$$

$$\text{Out[*]:= } \left\{ - \left(\left(\left(\text{Cosh} \left[\left(a \epsilon + \frac{1}{2} (-t + \epsilon) \right) \hbar \right] - \text{Cosh} \left[\sqrt{\frac{1}{4} (t^2 + \epsilon^2) + \epsilon \varpi} \hbar \right] \right) \text{Csch} \left[\frac{\epsilon \hbar}{2} \right] \right) / \right.$$

$$\left. \left(\left(\frac{t}{2} + a (t - \epsilon) - a^2 \epsilon + \varpi \right) \hbar \right) \right), \text{True, True},$$

$$- \left(\left(4 \left(\text{Cosh} \left[\frac{1}{2} (t - 2 c \epsilon) \hbar \right] - \text{Cosh} \left[\frac{1}{2} \sqrt{t^2 + \epsilon^2 + 4 \epsilon \varpi} \hbar \right] \right) \text{Csch} \left[\frac{\epsilon \hbar}{2} \right] \right) / \right.$$

$$\left. \left((4 c t + \epsilon - 4 c^2 \epsilon + 4 \varpi) \hbar \right) \right), \text{True, True, True, True} \left. \right\}$$

SDeq

$$\text{In[*]:= SD\$w = \gamma CU[y, x] + \epsilon CU[a, a] - (t - \gamma \epsilon) CU[a] - t \gamma 1_{cu} / 2;$$

SDeq

```
In[*]:= DeclareMorphism[SID, QU -> CU, {a -> aCU,
  x -> SCU[SS[SID$f], a -> aCU, w -> SID$w] ** XCU,
  y -> SCU[SS[SID$g], a -> aCU, w -> SID$w] ** YCU }]
```

Verifying the θ -symmetry:

```
Table[HL@SimpT[C@SID[z]] == SID[Q@z]], {z, QU /@ {y, a, x}}]
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} -> HL@SimpT[SID[z1 ** z2] - SID[z1] ** SID[z2]], {z1, bas}, {z2, bas}]]
{{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> 0, {QU[y], QU[x]} -> 0},
 {{QU[a], QU[y]} -> 0, {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> 0},
 {{QU[x], QU[y]} -> 0, {QU[x], QU[a]} -> 0, {QU[x], QU[x]} -> 0}}
```

The representation ρ

rho

```
In[*]:= rho@yCU = rho@yQU =  $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$ ; rho@aCU = rho@aQU =  $\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ ;
rho@xCU =  $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ ; rho@xQU =  $\begin{pmatrix} 0 & (1 - e^{-\gamma \epsilon \hbar}) / (\epsilon \hbar) \\ 0 & 0 \end{pmatrix}$ ;
rho[eε] := MatrixExp[rho[ε]];
rho[ε_] := (ε /. T2t /. t -> γ ε /. (U : CU | QU)[u___] => Fold[Dot,  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , rho /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[rho[z1 ** z2] == rho[z1].rho[z2]] /. ek. /; k > $k -> 0],
  {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}}]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[rho[eε Uex].rho[eα Uea] == rho[eα Uea].rho[ee-γ α ε Uex]], {U, {CU, QU}}]
{True, True}
```

\mathbb{C} and the logoi \wedge

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

Multiplying OEs

```
In[*]:= CU[s1_, Q1_, P1_] CU[s2_, Q2_, P2_] ^:= CU[s1, s2, Q1 + Q2, P1 P2];
```

CdsO

```
In[*]:= CU@CU[specs___, Q_, P_] := OCU[specs, SS[e^Q P]];
QU@QU[specs___, Q_, P_] := OQU[specs, SS[e^Q P]];
```

Logos

```
In[*]:= c_Integer k_Integer := c + O[ε]^{k+1};
ΛU, k_[{α_, β_}, {x_, x_}] := CU[{x}, (α + β) x, 1_k];
ΛU, k_[{ξ_, α_}, {x, a}] := CU[{a, x}, α a + e^{-α ξ} x, 1_k];
ΛU, k_[{α_, η_}, {a, y}] := CU[{y, a}, α a + e^{-α η} y, 1_k];
```

Table[

```
{ΛU, 1[{α, β}, {u, u}],
  lhs = U@CU[{u1, u2}, ħ (α u1 + β u2), 1], HL[lhs == U@ΛU, 1[ħ {α, β}, {u, u}]]},
{U, {CU, QU}}, {u, {y, a, x}}]
{{CU[{y}, y (α + β), 1 + O[ε]^2],
  CU[] + (α ħ + β ħ) CU[y] + (α^2 ħ^2 / 2 + α β ħ^2 + β^2 ħ^2 / 2) CU[y, y], True},
{CU[{a}, a (α + β), 1 + O[ε]^2], CU[] + (α ħ + β ħ) CU[a] + (α^2 ħ^2 / 2 + α β ħ^2 + β^2 ħ^2 / 2) CU[a, a],
  True}, {CU[{x}, x (α + β), 1 + O[ε]^2],
  CU[] + (α ħ + β ħ) CU[x] + (α^2 ħ^2 / 2 + α β ħ^2 + β^2 ħ^2 / 2) CU[x, x], True}},
{QU[{y}, y (α + β), 1 + O[ε]^2], QU[] + (α ħ + β ħ) QU[y] + (α^2 ħ^2 / 2 + α β ħ^2 + β^2 ħ^2 / 2) QU[y, y],
  True}, {QU[{a}, a (α + β), 1 + O[ε]^2], QU[] + (α ħ + β ħ) QU[a] +
  (α^2 ħ^2 / 2 + α β ħ^2 + β^2 ħ^2 / 2) QU[a, a], True}, {QU[{x}, x (α + β), 1 + O[ε]^2],
  QU[] + (α ħ + β ħ) QU[x] + (α^2 ħ^2 / 2 + α β ħ^2 + β^2 ħ^2 / 2) QU[x, x], True}}}
```

```
{Λ#, 1[{ξ, α}, {x, a}], lhs = #@CU#[{x, a}, ħ (ξ x + α a), 1],
```

```
HL[lhs == #@Λ#, 1[ħ {ξ, α}, {x, a}]] & /@ {CU, QU}
```

```
{{CU[{a, x}, a α + e^{-α ξ} x, 1 + O[ε]^2],
  CU[] + α ħ CU[a] + (ξ ħ - α γ ξ ħ^2) CU[x] + 1/2 α^2 ħ^2 CU[a, a] + α ξ ħ^2 CU[a, x] + 1/2 ξ^2 ħ^2 CU[x, x],
  True}, {QU[{a, x}, a α + e^{-α ξ} x, 1 + O[ε]^2], QU[] + α ħ QU[a] +
  (ξ ħ - α γ ξ ħ^2) QU[x] + 1/2 α^2 ħ^2 QU[a, a] + α ξ ħ^2 QU[a, x] + 1/2 ξ^2 ħ^2 QU[x, x], True}}
```

```
{Λ_{#,2}[{α, η}, {a, y}], lhs = #@C_{#}[{a, y}, ħ (η y + α a), 1],
  HL[lhs = #@Λ_{#,2}[ħ {α, η}, {a, y}]] & /@ {CU, QU}
{{C_{CU}[{y, a}, a α + e^{-α γ} y η, 1 + 0[ε]^3],
  CU[] + α ħ CU[a] + (η ħ - α γ η ħ^2) CU[y] + 1/2 α^2 ħ^2 CU[a, a] + α η ħ^2 CU[y, a] + 1/2 η^2 ħ^2 CU[y, y],
  True}, {C_{QU}[{y, a}, a α + e^{-α γ} y η, 1 + 0[ε]^3], QU[] + α ħ QU[a] +
  (η ħ - α γ η ħ^2) QU[y] + 1/2 α^2 ħ^2 QU[a, a] + α η ħ^2 QU[y, a] + 1/2 η^2 ħ^2 QU[y, y], True}}
```

Goal. In either U , compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
    G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
    fs = Echo@Flatten@Table[f_{1,i,j,k}[η], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = Echo[fs. (bs = fs /. f_{L_,i_,j_,k_}[η] => e^L U@{y^i, a^j, x^k})];
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η -> 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
    sol = Echo@First[F /. DSolve[es, fs, η]];
    Echo[sol /. {e -> 1, U -> Times}];
    Collect[sol /. {e -> 1, U -> Times}, ε, Simplify]
  ]]]
" -t ξ CU[] + 2 ε ξ CU[a] - γ ε ξ^2 CU[x] + CU[y]
" {f_{0,0,0,0}[η], f_{1,0,0,0}[η], f_{1,0,0,1}[η], f_{1,0,1,0}[η],
  f_{1,0,1,1}[η], f_{1,1,0,0}[η], f_{1,1,0,1}[η], f_{1,1,1,0}[η], f_{1,1,1,1}[η]}
" CU[] f_{0,0,0,0}[η] + ε CU[] f_{1,0,0,0}[η] + ε CU[x] f_{1,0,0,1}[η] + ε CU[a] f_{1,0,1,0}[η] + ε CU[a, x] f_{1,0,1,1}[η] +
  ε CU[y] f_{1,1,0,0}[η] + ε CU[y, x] f_{1,1,0,1}[η] + ε CU[y, a] f_{1,1,1,0}[η] + ε CU[y, a, x] f_{1,1,1,1}[η]
" e^{-t η ξ} CU[] + 1/2 e^{-t η ξ} t γ ε η^2 ξ^2 CU[] + 2 e^{-t η ξ} ε η ξ CU[a] - e^{-t η ξ} γ ε η ξ^2 CU[x] - e^{-t η ξ} γ ε η^2 ξ CU[y]
" 1 + 2 a ε η ξ - y γ ε η^2 ξ - x γ ε η ξ^2 + 1/2 t γ ε η^2 ξ^2
  1 + 1/2 ε η ξ (4 a + γ (-2 y η - 2 x ξ + t η ξ))
```

Logos

In[*]:=

```

 $\Lambda_{U,kk}[\{\xi_1, \eta_1\}, \{x, y\}] := \Lambda_{U,kk}[\{\xi_1, \eta_1\}, \{x, y\}] =$ 
Block[{$k = kk, $p = kk}, Module[{ $\xi, \eta, G, F, fs, f, bs, e, b, es$ },
  G = Simp[Table[ $\xi^k/k!$ , {k, 0, $k+1}].NestList[Simp[B[xU, #]] &, yU, $k+1]];
  fs = Flatten@Table[f1,i,j,k[ $\eta$ ], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
  F = fs.(bs = fs /. fL-,i-,j-,k-[ $\eta$ ]  $\rightarrow e^L U@{y^i, a^j, x^k}$ );
  es = Flatten[
    Table[Coefficient[e, b] == 0, {e, {F - 1U /.  $\eta \rightarrow 0$ , F ** G - yU ** F -  $\partial_\eta F$ }}, {b, bs}]];
  F = F /. DSolve[es, fs,  $\eta$ ] [[1]];
   $\mathbb{C}_U[\{y, a, x$ ,
     $\xi x + \eta y + (U /. \{CU \rightarrow -t \eta \xi, QU \rightarrow \eta \xi (1 - T) / \hbar\})$ ,
     $F + \theta_{\$k} /. \{e \rightarrow 1, U \rightarrow Times\}$ 
  ] /. { $\xi \rightarrow \xi_1, \eta \rightarrow \eta_1$ }]];

```

In[*]:= Timing@ $\Lambda_{QU,2}[\{\xi, \eta\}, \{x, y\}]$

Out[*]:= $\{1.64063, \mathbb{C}_{QU}[\{y, a, x\}, y \eta + x \xi + \frac{(1-T) \eta \xi}{\hbar}, 1 + \frac{1}{4 \hbar}$

$$\eta \xi (\gamma \eta \xi - 4 T \gamma \eta \xi + 3 T^2 \gamma \eta \xi + 8 a T \hbar + 2 y \gamma \eta \hbar - 6 T y \gamma \eta \hbar + 2 x \gamma \xi \hbar - 6 T x \gamma \xi \hbar + 4 x y \gamma \hbar^2) \epsilon +$$

$$\left(-a T y \gamma \eta^2 \xi (-\eta \xi + 3 T \eta \xi - 3 \hbar) - a T x \gamma \eta \xi^2 (-\eta \xi + 3 T \eta \xi - 3 \hbar) + 2 a^2 T \eta \xi (T \eta \xi - \hbar) + \right.$$

$$2 a T x y \gamma \eta^2 \xi^2 \hbar - \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi (-\eta \xi + 3 T \eta \xi - \hbar) \hbar - \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 (-\eta \xi + 3 T \eta \xi - \hbar) \hbar +$$

$$\frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 + \frac{1}{24} y^2 \gamma^2 \eta^3 \xi (3 \eta \xi - 18 T \eta \xi + 27 T^2 \eta \xi + 4 \hbar - 28 T \hbar) + \frac{1}{24} x^2 \gamma^2 \eta \xi^3$$

$$(3 \eta \xi - 18 T \eta \xi + 27 T^2 \eta \xi + 4 \hbar - 28 T \hbar) + \frac{1}{2 \hbar} a T \gamma \eta^2 \xi^2 (\eta \xi - 4 T \eta \xi + 3 T^2 \eta \xi + 4 \hbar - 6 T \hbar) +$$

$$\frac{1}{4} x y \gamma^2 \eta \xi (2 \eta^2 \xi^2 - 10 T \eta^2 \xi^2 + 12 T^2 \eta^2 \xi^2 + 5 \eta \xi \hbar - 21 T \eta \xi \hbar + 2 \hbar^2) - \frac{1}{24 \hbar}$$

$$y \gamma^2 \eta^2 \xi (-3 \eta^2 \xi^2 + 21 T \eta^2 \xi^2 - 45 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 10 \eta \xi \hbar + 68 T \eta \xi \hbar - 82 T^2 \eta \xi \hbar -$$

$$6 \hbar^2 + 30 T \hbar^2) - \frac{1}{24 \hbar} x \gamma^2 \eta \xi^2 (-3 \eta^2 \xi^2 + 21 T \eta^2 \xi^2 - 45 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 10 \eta \xi \hbar +$$

$$68 T \eta \xi \hbar - 82 T^2 \eta \xi \hbar - 6 \hbar^2 + 30 T \hbar^2) + \frac{1}{288 \hbar^2} (-1 + T) \gamma^2 \eta^2 \xi^2 (-9 \eta^2 \xi^2 + 63 T \eta^2 \xi^2 -$$

$$135 T^2 \eta^2 \xi^2 + 81 T^3 \eta^2 \xi^2 - 40 \eta \xi \hbar + 272 T \eta \xi \hbar - 328 T^2 \eta \xi \hbar - 36 \hbar^2 + 180 T \hbar^2) \left. \right) \epsilon^2 + O[\epsilon^3] \}$$

$\{\Lambda_{CU,1}[\{\xi, \eta\}, \{x, y\}], lhs = CU@C_{CU}[\{x, y\}, \hbar (\xi x + \eta y), 1],$
 $HL[lhs = CU@\Lambda_{CU,1}[\hbar \{\xi, \eta\}, \{x, y\}]]\}$

$\{C_{CU}[\{y, a, x\}, y \eta + x \xi - t \eta \xi, 1 + \frac{1}{2} \eta \xi (4 a - 2 y \gamma \eta - 2 x \gamma \xi + t \gamma \eta \xi) \epsilon + O[\epsilon^2],$

$$(1 - t \eta \xi \hbar^2) CU[] + 2 \epsilon \eta \xi \hbar^2 CU[a] + \xi \hbar CU[x] + \eta \hbar CU[y] +$$

$$\frac{1}{2} \xi^2 \hbar^2 CU[x, x] + \eta \xi \hbar^2 CU[y, x] + \frac{1}{2} \eta^2 \hbar^2 CU[y, y], True\}$$

```
In[*]:= {ΔQU,1[{ξ, η}, {x, y}], lhs = QU@CQU[{x, y}, ħ (ξ x + η y), 1],
  HL@SimpT[lhs == QU@ΔQU,1[ħ {ξ, η}, {x, y}]]}
```

```
Out[*]:= {CQU[{y, a, x}, y η + x ξ +  $\frac{(1-T)\eta\xi}{\hbar}$ , 1 +  $\frac{1}{4\hbar}$ 
  η ξ (γ η ξ - 4 T γ η ξ + 3 T2 γ η ξ + 8 a T ħ + 2 y γ η ħ - 6 T y γ η ħ + 2 x γ ξ ħ - 6 T x γ ξ ħ + 4 x y γ ħ2) ε +
  O[ε]2}, (1 + η ξ ħ - T η ξ ħ) QU[] + 2 T ε η ξ ħ2 QU[a] + ξ ħ QU[x] +
  η ħ QU[y] +  $\frac{1}{2}$  ξ2 ħ2 QU[x, x] + η ξ ħ2 QU[y, x] +  $\frac{1}{2}$  η2 ħ2 QU[y, y], True}
```

```
{tt = Last[ΔCU,2[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ħ ξ, η → ħ η, x → ħ x, y → ħ y}, ħ]} // Expand
```

```
{1 + (2 a η ξ - y γ η2 ξ - x γ η ξ2 +  $\frac{1}{2}$  t γ η2 ξ2) ε +
  (2 a2 η2 ξ2 - a γ η2 ξ2 - 2 a y γ η3 ξ2 + y γ2 η3 ξ2 +  $\frac{1}{2}$  y2 γ2 η4 ξ2 - 2 a x γ η2 ξ3 + x γ2 η2 ξ3 + a t γ η3 ξ3 -
   $\frac{1}{3}$  t γ2 η3 ξ3 + x y γ2 η3 ξ3 -  $\frac{1}{2}$  t y γ2 η4 ξ3 +  $\frac{1}{2}$  x2 γ2 η2 ξ4 -  $\frac{1}{2}$  t x γ2 η3 ξ4 +  $\frac{1}{8}$  t2 γ2 η4 ξ4) ε2 + O[ε]3,
  (2 a η ξ - y γ η2 ξ - x γ η ξ2 +  $\frac{1}{2}$  t γ η2 ξ2) ε + (-a γ η2 ξ2 + y γ2 η3 ξ2 + x γ2 η2 ξ3 -  $\frac{1}{3}$  t γ2 η3 ξ3) ε2 +
  O[ε]3, 6}
```

```
{tt = Last[ΔQU,2[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\begin{aligned}
 & \left\{ 1 + \left(2 a T \eta \xi + \frac{1}{2} y \gamma \eta^2 \xi - \frac{3}{2} T y \gamma \eta^2 \xi + \right. \right. \\
 & \quad \left. \frac{1}{2} x \gamma \eta \xi^2 - \frac{3}{2} T x \gamma \eta \xi^2 + \frac{\gamma \eta^2 \xi^2}{4 \hbar} - \frac{T \gamma \eta^2 \xi^2}{\hbar} + \frac{3 T^2 \gamma \eta^2 \xi^2}{4 \hbar} + x y \gamma \eta \xi \hbar \right) \epsilon + \\
 & \left(2 a^2 T^2 \eta^2 \xi^2 + 2 a T \gamma \eta^2 \xi^2 - 3 a T^2 \gamma \eta^2 \xi^2 + \frac{1}{8} \gamma^2 \eta^2 \xi^2 - \frac{3}{4} T \gamma^2 \eta^2 \xi^2 + \frac{5}{8} T^2 \gamma^2 \eta^2 \xi^2 + \right. \\
 & \quad a T y \gamma \eta^3 \xi^2 - 3 a T^2 y \gamma \eta^3 \xi^2 + \frac{5}{12} y \gamma^2 \eta^3 \xi^2 - \frac{17}{6} T y \gamma^2 \eta^3 \xi^2 + \frac{41}{12} T^2 y \gamma^2 \eta^3 \xi^2 + \frac{1}{8} y^2 \gamma^2 \eta^4 \xi^2 - \\
 & \quad \frac{3}{4} T y^2 \gamma^2 \eta^4 \xi^2 + \frac{9}{8} T^2 y^2 \gamma^2 \eta^4 \xi^2 + a T x \gamma \eta^2 \xi^3 - 3 a T^2 x \gamma \eta^2 \xi^3 + \frac{5}{12} x \gamma^2 \eta^2 \xi^3 - \frac{17}{6} T x \gamma^2 \eta^2 \xi^3 + \\
 & \quad \frac{41}{12} T^2 x \gamma^2 \eta^2 \xi^3 + \frac{1}{2} x y \gamma^2 \eta^3 \xi^3 - \frac{5}{2} T x y \gamma^2 \eta^3 \xi^3 + 3 T^2 x y \gamma^2 \eta^3 \xi^3 + \frac{1}{8} x^2 \gamma^2 \eta^2 \xi^4 - \\
 & \quad \frac{3}{4} T x^2 \gamma^2 \eta^2 \xi^4 + \frac{9}{8} T^2 x^2 \gamma^2 \eta^2 \xi^4 + \frac{\gamma^2 \eta^4 \xi^4}{32 \hbar^2} - \frac{T \gamma^2 \eta^4 \xi^4}{4 \hbar^2} + \frac{11 T^2 \gamma^2 \eta^4 \xi^4}{16 \hbar^2} - \frac{3 T^3 \gamma^2 \eta^4 \xi^4}{4 \hbar^2} + \\
 & \quad \frac{9 T^4 \gamma^2 \eta^4 \xi^4}{32 \hbar^2} + \frac{a T \gamma \eta^3 \xi^3}{2 \hbar} - \frac{2 a T^2 \gamma \eta^3 \xi^3}{\hbar} + \frac{3 a T^3 \gamma \eta^3 \xi^3}{2 \hbar} + \frac{5 \gamma^2 \eta^3 \xi^3}{36 \hbar} - \frac{13 T \gamma^2 \eta^3 \xi^3}{12 \hbar} + \\
 & \quad \frac{25 T^2 \gamma^2 \eta^3 \xi^3}{12 \hbar} - \frac{41 T^3 \gamma^2 \eta^3 \xi^3}{36 \hbar} + \frac{y \gamma^2 \eta^4 \xi^3}{8 \hbar} - \frac{7 T y \gamma^2 \eta^4 \xi^3}{8 \hbar} + \frac{15 T^2 y \gamma^2 \eta^4 \xi^3}{8 \hbar} - \frac{9 T^3 y \gamma^2 \eta^4 \xi^3}{8 \hbar} + \\
 & \quad x \gamma^2 \eta^3 \xi^4 - \frac{7 T x \gamma^2 \eta^3 \xi^4}{8 \hbar} + \frac{15 T^2 x \gamma^2 \eta^3 \xi^4}{8 \hbar} - \frac{9 T^3 x \gamma^2 \eta^3 \xi^4}{8 \hbar} - 2 a^2 T \eta \xi \hbar + 3 a T y \gamma \eta^2 \xi \hbar + \\
 & \quad \frac{1}{4} y \gamma^2 \eta^2 \xi \hbar - \frac{5}{4} T y \gamma^2 \eta^2 \xi \hbar + \frac{1}{6} y^2 \gamma^2 \eta^3 \xi \hbar - \frac{7}{6} T y^2 \gamma^2 \eta^3 \xi \hbar + 3 a T x \gamma \eta \xi^2 \hbar + \frac{1}{4} x \gamma^2 \eta \xi^2 \hbar - \\
 & \quad \frac{5}{4} T x \gamma^2 \eta \xi^2 \hbar + 2 a T x y \gamma \eta^2 \xi^2 \hbar + \frac{5}{4} x y \gamma^2 \eta^2 \xi^2 \hbar - \frac{21}{4} T x y \gamma^2 \eta^2 \xi^2 \hbar + \frac{1}{2} x y^2 \gamma^2 \eta^3 \xi^2 \hbar - \\
 & \quad \frac{3}{2} T x y^2 \gamma^2 \eta^3 \xi^2 \hbar + \frac{1}{6} x^2 \gamma^2 \eta \xi^3 \hbar - \frac{7}{6} T x^2 \gamma^2 \eta \xi^3 \hbar + \frac{1}{2} x^2 y \gamma^2 \eta^2 \xi^3 \hbar - \frac{3}{2} T x^2 y \gamma^2 \eta^2 \xi^3 \hbar + \\
 & \quad \left. \frac{1}{2} x y \gamma^2 \eta \xi \hbar^2 + \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi \hbar^2 + \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 \hbar^2 + \frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 \right) \epsilon^2 + 0[\epsilon]^3, \\
 & \left(2 a T \eta \xi + \frac{1}{2} y \gamma \eta^2 \xi - \frac{3}{2} T y \gamma \eta^2 \xi + \frac{1}{2} x \gamma \eta \xi^2 - \frac{3}{2} T x \gamma \eta \xi^2 + \frac{\gamma \eta^2 \xi^2}{4 \hbar} - \right. \\
 & \quad \left. \frac{T \gamma \eta^2 \xi^2}{\hbar} + \frac{3 T^2 \gamma \eta^2 \xi^2}{4 \hbar} + x y \gamma \eta \xi \hbar \right) \epsilon + \\
 & \left(2 a T \gamma \eta^2 \xi^2 - 3 a T^2 \gamma \eta^2 \xi^2 + \frac{1}{8} \gamma^2 \eta^2 \xi^2 - \frac{3}{4} T \gamma^2 \eta^2 \xi^2 + \frac{5}{8} T^2 \gamma^2 \eta^2 \xi^2 + \frac{5}{12} y \gamma^2 \eta^3 \xi^2 - \right. \\
 & \quad \frac{17}{6} T y \gamma^2 \eta^3 \xi^2 + \frac{41}{12} T^2 y \gamma^2 \eta^3 \xi^2 + \frac{5}{12} x \gamma^2 \eta^2 \xi^3 - \frac{17}{6} T x \gamma^2 \eta^2 \xi^3 + \frac{41}{12} T^2 x \gamma^2 \eta^2 \xi^3 + \\
 & \quad \frac{5 \gamma^2 \eta^3 \xi^3}{36 \hbar} - \frac{13 T \gamma^2 \eta^3 \xi^3}{12 \hbar} + \frac{25 T^2 \gamma^2 \eta^3 \xi^3}{12 \hbar} - \frac{41 T^3 \gamma^2 \eta^3 \xi^3}{36 \hbar} - 2 a^2 T \eta \xi \hbar + 3 a T y \gamma \eta^2 \xi \hbar + \\
 & \quad \frac{1}{4} y \gamma^2 \eta^2 \xi \hbar - \frac{5}{4} T y \gamma^2 \eta^2 \xi \hbar + \frac{1}{6} y^2 \gamma^2 \eta^3 \xi \hbar - \frac{7}{6} T y^2 \gamma^2 \eta^3 \xi \hbar + 3 a T x \gamma \eta \xi^2 \hbar + \\
 & \quad \frac{1}{4} x \gamma^2 \eta \xi^2 \hbar - \frac{5}{4} T x \gamma^2 \eta \xi^2 \hbar + \frac{5}{4} x y \gamma^2 \eta^2 \xi^2 \hbar - \frac{21}{4} T x y \gamma^2 \eta^2 \xi^2 \hbar + \frac{1}{6} x^2 \gamma^2 \eta \xi^3 \hbar - \\
 & \quad \left. \frac{7}{6} T x^2 \gamma^2 \eta \xi^3 \hbar + \frac{1}{2} x y \gamma^2 \eta \xi \hbar^2 + \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi \hbar^2 + \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 \hbar^2 + \frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 \right) \epsilon^2 + 0[\epsilon]^3, 6 \}
 \end{aligned}$$

Logos

```
In[*]:= Simp[CU_[specs___, Q_, P_] := CU[specs, CF[Q], CF[P]];
```

Logos

```
In[*]:= ΔU_,k_[{u1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{v, ω, yax, q, p, Q, d},
  {yax, q, p} = List@@ΔU_,k_[{v, ω}, {u, w}];
  CU[yax, Q = (v u + ω w + δ u w + d v ω) / (1 - d δ),
  Expand[(1 - d δ)^-1 e^-Q DP_{v→D_v, ω→D_w}[p][e^Q] + θ_R] /. {d → ∂_{v, ω} Q} /. {v → u1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
  {ΔCU, $k}[ħ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@ΔCU[{x, y}, ħ (ξ x + η y + δ x y), 1_{$k}], 5],
  HL@Simp[lhs - CU@ΔCU, $k][ħ {ξ, η, δ}, {x, y}]]
]
```

$$\left\{ \text{CU} \left[\{y, a, x\}, \frac{xy \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar}, \right. \right.$$

$$\frac{1}{1 + t \delta \hbar} + \left((4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 - \right.$$

$$4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 +$$

$$4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 -$$

$$6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 +$$

$$8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 +$$

$$4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 +$$

$$4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 +$$

$$4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4) \epsilon) /$$

$$(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5) + 0[\epsilon]^2],$$

$$\left(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 + \right.$$

$$2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 -$$

$$\left. 9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4 \right) \text{CU}[\] +$$

$$(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 +$$

$$18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4) \text{CU}[a] +$$

$$\llcorner 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 \text{CU}[y, y, y, y, x, x, x] +$$

$$\frac{1}{24} \delta^4 \hbar^4$$

$$\text{CU}[y, y, y, y, x, x, x], \mathbf{0} \}$$

$\{\Delta_{\text{Qu},2}[\{\xi, \eta, \delta\}, \{x, y\}], \text{lhs} = \text{QU}@\text{CQu}[\{x, y\}, \hbar(\xi x + \eta y + \delta xy), 1],$
 $\text{HL}@\text{SimpT}[\text{lhs} = \text{QU}@\Delta_{\text{Qu},1}[\hbar\{\xi, \eta, \delta\}, \{x, y\}]]\}$

$$\left\{ \text{CQu}[\{y, a, x\}, \frac{\dots 1 \dots}{\dots 1 \dots}, \right.$$

$$\frac{\hbar}{-\delta + T \delta + \hbar} + \left((-8 a T \delta^4 \hbar^2 + 24 a T^2 \delta^4 \hbar^2 - 24 a T^3 \delta^4 \hbar^2 + 8 a T^4 \delta^4 \hbar^2 + \dots 149 \dots + \right.$$

$$4 x^2 y^2 \gamma \delta^2 \hbar^6 + 4 x y^2 \gamma \delta \eta \hbar^6 + 4 x^2 y \gamma \delta \xi \hbar^6 + 4 x y \gamma \eta \xi \hbar^6) \epsilon) /$$

$$(-4 \delta^5 + 20 T \delta^5 - 40 T^2 \delta^5 + 40 T^3 \delta^5 - 20 T^4 \delta^5 + 4 T^5 \delta^5 + \dots 12 \dots + 40 T^3 \delta^3 \hbar^2 +$$

$$40 \delta^2 \hbar^3 - 80 T \delta^2 \hbar^3 + 40 T^2 \delta^2 \hbar^3 - 20 \delta \hbar^4 + 20 T \delta \hbar^4 + 4 \hbar^5) +$$

$$\left. \frac{(\dots 1 \dots)}{\dots 1 \dots} + O[\epsilon]^3 \right\}, \dots 1 \dots, \text{True} \}$$

large output	show less	show more	show all	set size limit...
--------------	-----------	-----------	----------	-------------------

$\{\text{tt} = \text{ComposeSeries}[(1 + t \delta) \text{Last}[\Delta_{\text{Cu},2}[\{\xi, \eta, \delta\}, \{x, y\}]], (1 + t \delta)^4 \epsilon + O[\epsilon]^{18}];$
 $\text{Together}@\text{Log}[\text{tt}],$
 $\text{Exponent}[\text{Normal}@\text{Together}@\text{Log}[\text{tt}] /. \{\xi \rightarrow d \xi, \eta \rightarrow d \eta, x \rightarrow d x, y \rightarrow d y\}, d],$
 $\text{Exponent}[\text{Normal}@\text{Together}@\text{Log}[\text{tt}] /. \{x \rightarrow d x, y \rightarrow d y\}, d]$
 $\} // \text{Expand}$

$$\left\{ \left(2 a \delta + 6 a t \delta^2 + 2 a x y \delta^2 + t \gamma \delta^2 - 4 x y \gamma \delta^2 + 6 a t^2 \delta^3 + 4 a t x y \delta^3 + 2 t^2 \gamma \delta^3 - 6 t x y \gamma \delta^3 - \right. \right.$$

$$2 x^2 y^2 \gamma \delta^3 + 2 a t^3 \delta^4 + 2 a t^2 x y \delta^4 + t^3 \gamma \delta^4 - 2 t^2 x y \gamma \delta^4 - \frac{3}{2} t x^2 y^2 \gamma \delta^4 + 2 a y \delta \eta -$$

$$2 y \gamma \delta \eta + 4 a t y \delta^2 \eta - 2 t y \gamma \delta^2 \eta - 3 x y^2 \gamma \delta^2 \eta + 2 a t^2 y \delta^3 \eta - 2 t x y^2 \gamma \delta^3 \eta -$$

$$y^2 \gamma \delta \eta^2 - \frac{1}{2} t y^2 \gamma \delta^2 \eta^2 + 2 a x \delta \xi - 2 x \gamma \delta \xi + 4 a t x \delta^2 \xi - 2 t x \gamma \delta^2 \xi - 3 x^2 y \gamma \delta^2 \xi +$$

$$2 a t^2 x \delta^3 \xi - 2 t x^2 y \gamma \delta^3 \xi + 2 a \eta \xi + 4 a t \delta \eta \xi + 2 t \gamma \delta \eta \xi - 4 x y \gamma \delta \eta \xi + 2 a t^2 \delta^2 \eta \xi +$$

$$\left. \left. 2 t^2 \gamma \delta^2 \eta \xi - 2 t x y \gamma \delta^2 \eta \xi - y \gamma \eta^2 \xi - x^2 \gamma \delta \xi^2 - \frac{1}{2} t x^2 \gamma \delta^2 \xi^2 - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right) \epsilon + \right.$$

$$\left(2 a^2 \delta^2 - 2 a \gamma \delta^2 + 12 a^2 t \delta^3 + 4 a^2 x y \delta^3 - 8 a t \gamma \delta^3 - 20 a x y \gamma \delta^3 - 2 t \gamma^2 \delta^3 + 18 x y \gamma^2 \delta^3 + \right.$$

$$30 a^2 t^2 \delta^4 + 20 a^2 t x y \delta^4 - 10 a t^2 \gamma \delta^4 - 88 a t x y \gamma \delta^4 - 13 a x^2 y^2 \gamma \delta^4 - \frac{15}{2} t^2 \gamma^2 \delta^4 +$$

$$64 t x y \gamma^2 \delta^4 + 34 x^2 y^2 \gamma^2 \delta^4 + 40 a^2 t^3 \delta^5 + 40 a^2 t^2 x y \delta^5 - 152 a t^2 x y \gamma \delta^5 - 48 a t x^2 y^2 \gamma \delta^5 -$$

$$10 t^3 \gamma^2 \delta^5 + 86 t^2 x y \gamma^2 \delta^5 + 107 t x^2 y^2 \gamma^2 \delta^5 + 11 x^3 y^3 \gamma^2 \delta^5 + 30 a^2 t^4 \delta^6 + 40 a^2 t^3 x y \delta^6 +$$

$$10 a t^4 \gamma \delta^6 - 128 a t^3 x y \gamma \delta^6 - 66 a t^2 x^2 y^2 \gamma \delta^6 - 5 t^4 \gamma^2 \delta^6 + 54 t^3 x y \gamma^2 \delta^6 + \frac{247}{2} t^2 x^2 y^2 \gamma^2 \delta^6 +$$

$$\frac{80}{3} t x^3 y^3 \gamma^2 \delta^6 + 12 a^2 t^5 \delta^7 + 20 a^2 t^4 x y \delta^7 + 8 a t^5 \gamma \delta^7 - 52 a t^4 x y \gamma \delta^7 - 40 a t^3 x^2 y^2 \gamma \delta^7 +$$

$$16 t^4 x y \gamma^2 \delta^7 + 62 t^3 x^2 y^2 \gamma^2 \delta^7 + \frac{64}{3} t^2 x^3 y^3 \gamma^2 \delta^7 + 2 a^2 t^6 \delta^8 + 4 a^2 t^5 x y \delta^8 + 2 a t^6 \gamma \delta^8 -$$

$$8 a t^5 x y \gamma \delta^8 - 9 a t^4 x^2 y^2 \gamma \delta^8 + \frac{1}{2} t^6 \gamma^2 \delta^8 + 2 t^5 x y \gamma^2 \delta^8 + \frac{23}{2} t^4 x^2 y^2 \gamma^2 \delta^8 + \frac{17}{3} t^3 x^3 y^3 \gamma^2 \delta^8 +$$

$$4 a^2 y \delta^2 \eta - 12 a y \gamma \delta^2 \eta + 6 y \gamma^2 \delta^2 \eta + 20 a^2 t y \delta^3 \eta - 48 a t y \gamma \delta^3 \eta - 20 a x y^2 \gamma \delta^3 \eta +$$

$$14 t y \gamma^2 \delta^3 \eta + 40 x y^2 \gamma^2 \delta^3 \eta + 40 a^2 t^2 y \delta^4 \eta - 72 a t^2 y \gamma \delta^4 \eta - 72 a t x y^2 \gamma \delta^4 \eta + 6 t^2 y \gamma^2 \delta^4 \eta +$$

$$115 t x y^2 \gamma^2 \delta^4 \eta + 23 x^2 y^3 \gamma^2 \delta^4 \eta + 40 a^2 t^3 y \delta^5 \eta - 48 a t^3 y \gamma \delta^5 \eta - 96 a t^2 x y^2 \gamma \delta^5 \eta -$$

$$6 t^3 y \gamma^2 \delta^5 \eta + 118 t^2 x y^2 \gamma^2 \delta^5 \eta + 53 t x^2 y^3 \gamma^2 \delta^5 \eta + 20 a^2 t^4 y \delta^6 \eta - 12 a t^4 y \gamma \delta^6 \eta -$$

$$56 a t^3 x y^2 \gamma \delta^6 \eta - 4 t^4 y \gamma^2 \delta^6 \eta + 51 t^3 x y^2 \gamma^2 \delta^6 \eta + 40 t^2 x^2 y^3 \gamma^2 \delta^6 \eta + 4 a^2 t^5 y \delta^7 \eta -$$

$$\begin{aligned}
 & 12 a t^4 x y^2 \gamma \delta^7 \eta + 8 t^4 x y^2 \gamma^2 \delta^7 \eta + 10 t^3 x^2 y^3 \gamma^2 \delta^7 \eta - 7 a y^2 \gamma \delta^2 \eta^2 + 10 y^2 \gamma^2 \delta^2 \eta^2 - \\
 & 24 a t y^2 \gamma \delta^3 \eta^2 + 24 t y^2 \gamma^2 \delta^3 \eta^2 + 15 x y^3 \gamma^2 \delta^3 \eta^2 - 30 a t^2 y^2 \gamma \delta^4 \eta^2 + \frac{37}{2} t^2 y^2 \gamma^2 \delta^4 \eta^2 + \\
 & 32 t x y^3 \gamma^2 \delta^4 \eta^2 - 16 a t^3 y^2 \gamma \delta^5 \eta^2 + 5 t^3 y^2 \gamma^2 \delta^5 \eta^2 + 22 t^2 x y^3 \gamma^2 \delta^5 \eta^2 - 3 a t^4 y^2 \gamma \delta^6 \eta^2 + \\
 & \frac{1}{2} t^4 y^2 \gamma^2 \delta^6 \eta^2 + 5 t^3 x y^3 \gamma^2 \delta^6 \eta^2 + 3 y^3 \gamma^2 \delta^2 \eta^3 + \frac{17}{3} t y^3 \gamma^2 \delta^3 \eta^3 + \frac{10}{3} t^2 y^3 \gamma^2 \delta^4 \eta^3 + \\
 & \frac{2}{3} t^3 y^3 \gamma^2 \delta^5 \eta^3 + 4 a^2 x \delta^2 \xi - 12 a x \gamma \delta^2 \xi + 6 x \gamma^2 \delta^2 \xi + 20 a^2 t x \delta^3 \xi - 48 a t x \gamma \delta^3 \xi - \\
 & 20 a x^2 y \gamma \delta^3 \xi + 14 t x \gamma^2 \delta^3 \xi + 40 x^2 y \gamma^2 \delta^3 \xi + 40 a^2 t^2 x \delta^4 \xi - 72 a t^2 x \gamma \delta^4 \xi - 72 a t x^2 y \gamma \delta^4 \xi + \\
 & 6 t^2 x \gamma^2 \delta^4 \xi + 115 t x^2 y \gamma^2 \delta^4 \xi + 23 x^3 y^2 \gamma^2 \delta^4 \xi + 40 a^2 t^3 x \delta^5 \xi - 48 a t^3 x \gamma \delta^5 \xi - \\
 & 96 a t^2 x^2 y \gamma \delta^5 \xi - 6 t^3 x \gamma^2 \delta^5 \xi + 118 t^2 x^2 y \gamma^2 \delta^5 \xi + 53 t x^3 y^2 \gamma^2 \delta^5 \xi + 20 a^2 t^4 x \delta^6 \xi - \\
 & 12 a t^4 x \gamma \delta^6 \xi - 56 a t^3 x^2 y \gamma \delta^6 \xi - 4 t^4 x \gamma^2 \delta^6 \xi + 51 t^3 x^2 y \gamma^2 \delta^6 \xi + 40 t^2 x^3 y^2 \gamma^2 \delta^6 \xi + \\
 & 4 a^2 t^5 x \delta^7 \xi - 12 a t^4 x^2 y \gamma \delta^7 \xi + 8 t^4 x^2 y \gamma^2 \delta^7 \xi + 10 t^3 x^3 y^2 \gamma^2 \delta^7 \xi + 4 a^2 \delta \eta \xi - 4 a \gamma \delta \eta \xi + \\
 & 20 a^2 t \delta^2 \eta \xi - 8 a t \gamma \delta^2 \eta \xi - 28 a x y \gamma \delta^2 \eta \xi - 6 t \gamma^2 \delta^2 \eta \xi + 38 x y \gamma^2 \delta^2 \eta \xi + 40 a^2 t^2 \delta^3 \eta \xi + \\
 & 8 a t^2 \gamma \delta^3 \eta \xi - 96 a t x y \gamma \delta^3 \eta \xi - 14 t^2 \gamma^2 \delta^3 \eta \xi + 88 t x y \gamma^2 \delta^3 \eta \xi + 44 x^2 y^2 \gamma^2 \delta^3 \eta \xi + \\
 & 40 a^2 t^3 \delta^4 \eta \xi + 32 a t^3 \gamma \delta^4 \eta \xi - 120 a t^2 x y \gamma \delta^4 \eta \xi - 6 t^3 \gamma^2 \delta^4 \eta \xi + 62 t^2 x y \gamma^2 \delta^4 \eta \xi + \\
 & 93 t x^2 y^2 \gamma^2 \delta^4 \eta \xi + 20 a^2 t^4 \delta^5 \eta \xi + 28 a t^4 \gamma \delta^5 \eta \xi - 64 a t^3 x y \gamma \delta^5 \eta \xi + 6 t^4 \gamma^2 \delta^5 \eta \xi + \\
 & 12 t^3 x y \gamma^2 \delta^5 \eta \xi + 63 t^2 x^2 y^2 \gamma^2 \delta^5 \eta \xi + 4 a^2 t^5 \delta^6 \eta \xi + 8 a t^5 \gamma \delta^6 \eta \xi - 12 a t^4 x y \gamma \delta^6 \eta \xi + \\
 & 4 t^5 \gamma^2 \delta^6 \eta \xi + 14 t^3 x^2 y^2 \gamma^2 \delta^6 \eta \xi - 8 a y \gamma \delta \eta^2 \xi + 6 y \gamma^2 \delta \eta^2 \xi - 24 a t y \gamma \delta^2 \eta^2 \xi + \\
 & 5 t y \gamma^2 \delta^2 \eta^2 \xi + 25 x y^2 \gamma^2 \delta^2 \eta^2 \xi - 24 a t^2 y \gamma \delta^3 \eta^2 \xi - 8 t^2 y \gamma^2 \delta^3 \eta^2 \xi + 45 t x y^2 \gamma^2 \delta^3 \eta^2 \xi - \\
 & 8 a t^3 y \gamma \delta^4 \eta^2 \xi - 7 t^3 y \gamma^2 \delta^4 \eta^2 \xi + 24 t^2 x y^2 \gamma^2 \delta^4 \eta^2 \xi + 4 t^3 x y^2 \gamma^2 \delta^5 \eta^2 \xi + 4 y^2 \gamma^2 \delta \eta^3 \xi + \\
 & 5 t y^2 \gamma^2 \delta^2 \eta^3 \xi + t^2 y^2 \gamma^2 \delta^3 \eta^3 \xi - 7 a x^2 \gamma \delta^2 \xi^2 + 10 x^2 \gamma^2 \delta^2 \xi^2 - 24 a t x^2 \gamma \delta^3 \xi^2 + \\
 & 24 t x^2 \gamma^2 \delta^3 \xi^2 + 15 x^3 y \gamma^2 \delta^3 \xi^2 - 30 a t^2 x^2 \gamma \delta^4 \xi^2 + \frac{37}{2} t^2 x^2 \gamma^2 \delta^4 \xi^2 + 32 t x^3 y \gamma^2 \delta^4 \xi^2 - \\
 & 16 a t^3 x^2 \gamma \delta^5 \xi^2 + 5 t^3 x^2 \gamma^2 \delta^5 \xi^2 + 22 t^2 x^3 y \gamma^2 \delta^5 \xi^2 - 3 a t^4 x^2 \gamma \delta^6 \xi^2 + \frac{1}{2} t^4 x^2 \gamma^2 \delta^6 \xi^2 + \\
 & 5 t^3 x^3 y \gamma^2 \delta^6 \xi^2 - 8 a x \gamma \delta \eta \xi^2 + 6 x \gamma^2 \delta \eta \xi^2 - 24 a t x \gamma \delta^2 \eta \xi^2 + 5 t x \gamma^2 \delta^2 \eta \xi^2 + \\
 & 25 x^2 y \gamma^2 \delta^2 \eta \xi^2 - 24 a t^2 x \gamma \delta^3 \eta \xi^2 - 8 t^2 x \gamma^2 \delta^3 \eta \xi^2 + 45 t x^2 y \gamma^2 \delta^3 \eta \xi^2 - 8 a t^3 x \gamma \delta^4 \eta \xi^2 - \\
 & 7 t^3 x \gamma^2 \delta^4 \eta \xi^2 + 24 t^2 x^2 y \gamma^2 \delta^4 \eta \xi^2 + 4 t^3 x^2 y \gamma^2 \delta^5 \eta \xi^2 - a \gamma \eta^2 \xi^2 - 3 t \gamma^2 \delta \eta^2 \xi^2 + \\
 & 11 x y \gamma^2 \delta \eta^2 \xi^2 + 6 a t^2 \gamma \delta^2 \eta^2 \xi^2 - \frac{5}{2} t^2 \gamma^2 \delta^2 \eta^2 \xi^2 + 12 t x y \gamma^2 \delta^2 \eta^2 \xi^2 + 8 a t^3 \gamma \delta^3 \eta^2 \xi^2 + \\
 & 4 t^3 \gamma^2 \delta^3 \eta^2 \xi^2 + 3 a t^4 \gamma \delta^4 \eta^2 \xi^2 + \frac{7}{2} t^4 \gamma^2 \delta^4 \eta^2 \xi^2 - t^3 x y \gamma^2 \delta^4 \eta^2 \xi^2 + y \gamma^2 \eta^3 \xi^2 - \\
 & t y \gamma^2 \delta \eta^3 \xi^2 - 2 t^2 y \gamma^2 \delta^2 \eta^3 \xi^2 + 3 x^3 \gamma^2 \delta^2 \xi^3 + \frac{17}{3} t x^3 \gamma^2 \delta^3 \xi^3 + \frac{10}{3} t^2 x^3 \gamma^2 \delta^4 \xi^3 + \\
 & \frac{2}{3} t^3 x^3 \gamma^2 \delta^5 \xi^3 + 4 x^2 \gamma^2 \delta \eta \xi^3 + 5 t x^2 \gamma^2 \delta^2 \eta \xi^3 + t^2 x^2 \gamma^2 \delta^3 \eta \xi^3 + x \gamma^2 \eta^2 \xi^3 - t x \gamma^2 \delta \eta^2 \xi^3 - \\
 & 2 t^2 x \gamma^2 \delta^2 \eta^2 \xi^3 - \frac{1}{3} t \gamma^2 \eta^3 \xi^3 + \frac{1}{3} t^2 \gamma^2 \delta \eta^3 \xi^3 + \frac{2}{3} t^3 \gamma^2 \delta^2 \eta^3 \xi^3 \Big) \epsilon^2 + 0[\epsilon]^3, 6, 6 \Big)
 \end{aligned}$$

```

{tt = Last[DeltaQu,2[{{xi, eta, delta}, {x, y}]]];
Log[tt],
Exponent[Normal@Together@Log[tt] /. {xi -> d xi, eta -> d eta, x -> d x, y -> d y}, d] // Expand

```

$$\left\{ \text{Log} \left[\frac{\hbar}{-\delta + T \delta + \hbar} \right] + \left(\frac{2 a T \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} - \frac{8 a T^2 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} + \frac{12 a T^3 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} - \frac{8 a T^4 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} + \dots 267 \dots + \frac{x^2 y^2 \gamma \delta^2 \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x y^2 \gamma \delta \eta \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x^2 y \gamma \delta \xi \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x y \gamma \eta \xi \hbar^6}{(-\delta + T \delta + \hbar)^5} \right) \epsilon + \left(- \frac{32 a^2 T^2 \delta^{10} \hbar^2}{(\dots 1 \dots)^2} + \dots 8307 \dots + \dots 1 \dots + \frac{144 x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^{11}}{\dots 1 \dots} \right) \epsilon^2 + O[\epsilon]^3, 6 \right\}$$

large output | show less | show more | show all | set size limit...

Reorderings with Rord

Rord

In[*]:=

```
Rordui, wj → ki [CU[L---, {L---, ui, wj, r---}s, R---, Q-, P-]] :=
Simp@Module[{u, w, δ, Δ1, yax, q, p, kk = P[[5]], δ1 = ∂ui, wj Q},
{yax, q, p} = Echo[List@@ If[δ1 === 0, ΔU, kk [{u, w}, {u, w}],
ΔU, kk [{u, w, δ}, {u, w}]] /. {y → yk, a → ak, x → xk, t → ts, T → Ts};
CU[L, {L, Sequence@@ yax, r}s, R, q + (Q / . ui | wj → 0), e-q DPui → Du, wj → Dw [P] [p eq]] /.
{u → ∂ui Q / . wj → 0, w → ∂wj Q / . ui → 0, δ → δ1}];
```

Rord

In[*]:=

```
Rordui, wj → ki [CU[L---, {L---, ui, wj, r---}s, R---, Q-, P-]] :=
Simp@Module[{u, w, δ, Δ1, yax, q, p, n, kk = P[[5]], δ1 = ∂ui, wj Q},
{yax, q, p} = List@@ If[δ1 === 0, ΔU, kk [{u, w}, {u, w}], ΔU, kk [{u, w, δ}, {u, w}]] /.
{y → yn, a → an, x → xn, t → ts, T → Ts};
(*Echo@{{ui, v}, {wj, ω}}, P, p eq}; *)
CU[L, {L, Sequence@@ yax, r}s, R, q + (Q / . ui | wj → 0), e-q SP{ui → v, wj → ω} [P p eq]] /.
{n → k, v → ∂ui Q / . wj → 0, w → ∂wj Q / . ui → 0, δ → δ1}];
```

```
With[{co = CCU[{y1, x1}1, {x2, a2, y2}2, ħ t1 a2 + ħ t1-1 (et1 - 1) y1 x2, 12 + ε x1 y2]}],
{Short[rhs = co // Rordx2, a2 → 3, 3], HL[CU[co] == CU[rhs]]}]
{CCU[{y1, x1}1, {a3, x3, y2}2,  $\frac{e^{-\gamma \hbar t_1} (e^{\gamma \hbar t_1} \hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1} \hbar x_3 y_1)}{t_1}$ , 1 + x1 y2 ∈ + O[ε]3], True}
```

```
With[{co = CCU[{y1, a1, a2}1, {x2, x1, y2}2,
ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
12 + ε (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2) ]},
{Short[rhs = co // Rorda1, a2 → 3 // Rordx2, x1 → 4, 3], HL[CU[co] == CU[rhs]]}]
{CCU[{y1, a3}1, {x4, y2}2,
ħ a3 l11 t1 + ħ a3 l12 t1 + ħ a3 l21 t2 + ħ a3 l22 t2 + ħ x4 y1 γ11 + ħ x4 y2 γ12 + ħ x4 y1 γ21 + ħ x4 y2 γ22,
1 + (a3 l1 + a3 l2 + p11 x4 y1 + p21 x4 y1 + p12 x4 y2 + p22 x4 y2) ∈ + O[ε]3], True}
ħ a3 l11 t1 + ħ a3 l12 t1 + ħ a3 l21 t2 + ħ a3 l22 t2 +
ħ x4 y1 γ11 + ħ x4 y2 γ12 + ħ x4 y1 γ21 + ħ x4 y2 γ22 // Simplify
ħ (a3 (l11 t1 + l12 t1 + (l21 + l22) t2) + x4 (y1 (γ11 + γ21) + y2 (γ12 + γ22)))
```

With [{ $\mathbf{c0} = \mathbb{C}_{\text{CU}}[\{y_1, a_1, x_1\}_1, \{x_2, a_2, y_2\}_2]$,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$,
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)]]$,
{Short[rhs = $\mathbf{c0}$ // Rord $_{x_2, a_2 \rightarrow 3}$, 3], HL[CU[$\mathbf{c0}$] = CU[rhs]]}]]
{ $\mathbb{C}_{\text{CU}}[\{y_1, a_1, x_1\}_1, \ll 1 \gg_2, \ll 1 \gg \ll 1 \gg]$,
 $1 + e^{-\gamma \hbar l_{12} t_1 - \gamma \hbar l_{22} t_2} (e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_1 l_1 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_3 l_2 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +$
 $e^{\ll 1 \gg + \ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \in + O[\epsilon]^3$], True }

With [{ $\mathbf{q0} = \mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \{x_2, a_2, y_2\}_2]$,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$,
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)]]$,
{Short[rhs = $\mathbf{q0}$ // Rord $_{x_2, a_2 \rightarrow 3}$, 3], HL[QU[$\mathbf{q0}$] = QU[rhs]]}]]
{ $\mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \ll 1 \gg_2, \ll 1 \gg \ll 1 \gg]$,
 $1 + e^{-\gamma \hbar l_{12} t_1 - \gamma \hbar l_{22} t_2} (e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_1 l_1 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_3 l_2 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +$
 $e^{\ll 1 \gg + \ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \in + O[\epsilon]^3$], True }

With [{ $\mathbf{q0} = \mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \{x_2, a_2, y_2\}_2]$,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$,
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)]]$,
{Short[rhs = $\mathbf{q0}$ // Rord $_{a_2, y_2 \rightarrow 3}$, 3], HL[QU[$\mathbf{q0}$] = QU[rhs]]}]]
{ $\ll 1 \gg$, True }

Timing@With [{ $\mathbf{q0} = \mathbb{C}_{\text{QU}}[\{x_1, y_1\}_1, \{x_2, a_2, y_2\}_2]$,
 $\hbar (l_{12} t_1 a_2 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$,
 $\theta_2 + \epsilon (l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)]]$,
{Short[rhs = $\mathbf{q0}$ // Rord $_{x_1, y_1 \rightarrow 3}$, 5]}]]

{116.156, { $\mathbb{C}_{\text{QU}}[\{y_3, a_3, x_3\}_1, \ll 1 \gg_2, \frac{\ll 1 \gg}{1 - \ll 1 \gg + \ll 1 \gg}]$,
 $((\hbar a_2 l_2 + p_{11} - p_{11} T_1 + \hbar p_{22} x_2 y_2 + \hbar p_{12} x_3 y_2 + \ll 46 \gg + 2 \hbar p_{12} T_1 x_2 y_2 \gamma_{11} \gamma_{21} -$
 $\hbar p_{12} T_1^2 x_2 y_2 \gamma_{11} \gamma_{21} + \hbar p_{11} x_2 y_2 \gamma_{12} \gamma_{21} - 2 \hbar p_{11} T_1 x_2 y_2 \gamma_{12} \gamma_{21} + \hbar p_{11} T_1^2 x_2 y_2 \gamma_{12} \gamma_{21}) \epsilon) /$
 $(\hbar - 3 \hbar \gamma_{11} + 3 \hbar T_1 \gamma_{11} + 3 \hbar \gamma_{11}^2 - 6 \hbar T_1 \gamma_{11}^2 + 3 \hbar T_1^2 \gamma_{11}^2 - \hbar \gamma_{11}^3 + 3 \hbar T_1 \gamma_{11}^3 - 3 \hbar T_1^2 \gamma_{11}^3 + \hbar T_1^3 \gamma_{11}^3) +$
 $(8 a_3 p_{11} T_1 + \ll 1 \gg + \ll 2726 \gg + 3 \gamma \ll 6 \gg \gamma_{21}^3) \ll 1 \gg) /$
 $(4 - 28 \gamma_{11} + \ll 48 \gg + 4 T_1^7 \gamma_{11}^7) + O[\epsilon]^3]] }$

$$\begin{aligned}
 & \text{Timing@With}[\{\text{qo} = \mathbb{C}_{\text{QU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \{\mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_2, \\
 & \quad \hbar (\mathbf{l}_{12} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2), \\
 & \quad \mathbf{1}_2 + \epsilon (\mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2) \}], \\
 & \{\text{Short}[\text{rhs} = \text{qo} // \text{Rord}_{\mathbf{x}_1, \mathbf{y}_1 \rightarrow 3}, 5], \text{HL@SimpT}[\text{QU}[\text{qo}] = \text{QU}[\text{rhs}]]\}] \\
 & \{388.922, \\
 & \left\{ \mathbb{C}_{\text{QU}}[\{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_1, \{\ll 1 \gg\}_2, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + \left((4 \hbar \mathbf{a}_2 \mathbf{l}_2 + 4 \mathbf{p}_{11} - 4 \mathbf{p}_{11} T_1 + 4 \hbar \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2 + \right. \right. \\
 & \quad \left. \left. \ll 339 \gg + \gamma \hbar^4 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4 \gamma \hbar^4 T_1 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2) \epsilon \right) / \right. \\
 & \quad \left. (4 \hbar - 20 \hbar \gamma_{11} + 20 \hbar T_1 \gamma_{11} + 40 \hbar \gamma_{11}^2 - 80 \hbar T_1 \gamma_{11}^2 + 40 \hbar T_1^2 \gamma_{11}^2 - 40 \hbar \gamma_{11}^3 + \ll 13 \gg + \right. \\
 & \quad \left. 20 \hbar T_1 \gamma_{11}^5 - 40 \hbar T_1^2 \gamma_{11}^5 + 40 \hbar T_1^3 \gamma_{11}^5 - 20 \hbar T_1^4 \gamma_{11}^5 + 4 \hbar T_1^5 \gamma_{11}^5) + \right. \\
 & \quad \left. (576 \mathbf{a}_3 \mathbf{p}_{11} T_1 + \ll 8073 \gg + \ll 1 \gg) \frac{\ll 1 \gg}{\ll 79 \gg + 288 T_1^9 \gamma_{11}^9} + 0[\epsilon]^3 \right\}, \text{True} \} \}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Timing@With}[\{\text{qo} = \mathbb{C}_{\text{QU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \{\mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_2, \\
 & \quad \hbar (\mathbf{l}_{12} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2), \\
 & \quad \mathbf{1}_2 + \epsilon (\mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2) \}], \\
 & \{\text{Short}[\text{rhs} = \text{qo} // \text{Rord}_{\mathbf{x}_1, \mathbf{y}_1 \rightarrow 1}, 5], \text{HL@SimpT}[\text{QU}[\text{qo}] = \text{QU}[\text{rhs}]]\}] \\
 & \{336.781, \\
 & \left\{ \mathbb{C}_{\text{QU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\ll 1 \gg\}_2, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + \left((4 \hbar \mathbf{a}_2 \mathbf{l}_2 + 4 \mathbf{p}_{11} - 4 \mathbf{p}_{11} T_1 + 4 \hbar \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \right. \right. \\
 & \quad \left. \left. 4 \hbar \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \ll 338 \gg + \gamma \hbar^4 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4 \gamma \hbar^4 T_1 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2) \epsilon \right) / \right. \\
 & \quad \left. (4 \hbar - 20 \hbar \gamma_{11} + 20 \hbar T_1 \gamma_{11} + 40 \hbar \gamma_{11}^2 - 80 \hbar T_1 \gamma_{11}^2 + 40 \hbar T_1^2 \gamma_{11}^2 - 40 \hbar \gamma_{11}^3 + \ll 10 \gg + \right. \\
 & \quad \left. 20 \hbar T_1^4 \gamma_{11}^4 - 4 \hbar \gamma_{11}^5 + 20 \hbar T_1 \gamma_{11}^5 - 40 \hbar T_1^2 \gamma_{11}^5 + 40 \hbar T_1^3 \gamma_{11}^5 - 20 \hbar T_1^4 \gamma_{11}^5 + 4 \hbar T_1^5 \gamma_{11}^5) + \right. \\
 & \quad \left. (576 \mathbf{a}_1 \mathbf{p}_{11} T_1 + \ll 8073 \gg + \ll 1 \gg) \frac{\ll 1 \gg}{\ll 79 \gg + 288 T_1^9 \gamma_{11}^9} + 0[\epsilon]^3 \right\}, \text{True} \} \}
 \end{aligned}$$

Canonical ordering with Cord

Cord

In[*]:=

```

Cord[ $\mathbb{C}_U[L\_], \{L\_ , u\_i, w\_j, r\_s\}_s, R\_ , Q\_ , P\_ ] /;$ 
```

$$\text{OrderedQ}[\{w, u\} /. \{\mathbf{y} \rightarrow 1, \mathbf{a} \rightarrow 2, \mathbf{x} \rightarrow 3\}] :=
 \left((*\text{Echo}@\{u_i, w_j\}; *) \text{Cord}[\text{Rord}_{u_i, w_j \rightarrow \text{Unique}[]}[\mathbb{C}_U[L, \{L, u_i, w_j, r\}_s, R, Q, P]]] \right);
 \text{Cord}[\mathbb{C}_U[\text{specs}_ , Q_ , P_] := \mathbb{C}_U[\text{Sequence}@\text{Sort}@\{\text{specs}\}, Q, P] /.
 \text{Flatten}[\{\text{specs}\} /. \{\mathbf{yax}_s \Rightarrow (\{\mathbf{yax}\} /. u_i \Rightarrow (u_i \rightarrow u_s)))]$$

$\text{Cord}@\mathbb{C}_{\text{CU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \mathbf{0}, \mathbf{0}_1 + \mathbf{x}_1 \mathbf{y}_1]$

$\mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \mathbf{0}, (-\mathbf{t}_1 + \mathbf{x}_1 \mathbf{y}_1) + 2 \mathbf{a}_1 \epsilon + 0[\epsilon]^2]$

Block [{ \$p = 4, \$k = 0, c0 = CU [{y1, a1, x1, x2, a2, y2}]1,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_0 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] } ,$
Timing @ { Short [Cord [c0], 8], HL @ Simp [CU [c0] - CU [Cord [c0]]] } }]

{ 4.53125,
 $\left\{ \text{CU} [\{y_1, a_1, x_1\}]_1, \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \right. \right.$
 $\left. \hbar a_1 l_{12} t_1 + \ll 12 \gg + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar^2 a_1 l_{22} t_1 t_2 \gamma_{22} + \hbar x_1 y_1 \gamma_{22} \right) /$
 $\left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{12} + \right.$
 $\left. e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{22} \right), \frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + O[\epsilon]^1, \mathbf{0} \} \}$

Block [{ \$p = 4, \$k = 1, c0 = CU [{y1, a1, x1, x2, a2, y2}]1,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_1 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] } ,$
Timing @ { Short [Cord [c0], 8], HL @ Simp [CU [c0] - CU [Cord [c0]]] } }]

{ 81.2656, { CU [{y1, a1, x1}]1,
 $\left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + \ll 14 \gg + \hbar x_1 y_1 \gamma_{22} \right) /$
 $\left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} + e^{\gamma \hbar l_{11} t_1 + \ll 2 \gg + \gamma \hbar \ll 1 \gg} t_2 \hbar t_1 \gamma_{12} + \right.$
 $\left. e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{22} \right),$
 $\frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + \left(\left(2 e^{2 \gamma \hbar l_{11} t_1 + 6 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 6 \gamma \hbar l_{22} t_2} a_1 l_1 + \ll 419 \gg \right) \epsilon \right) /$
 $\left(2 e^{2 \gamma \hbar l_{11} t_1 + 6 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 6 \gamma \hbar l_{22} t_2} + 1_0 e^{2 \gamma \hbar l_{11} t_1 + 5 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 5 \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{12} + \right.$
 $\left. \ll 18 \gg + 2 e^{2 \gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar^5 t_1^5 \gamma_{22}^5 \right) + O[\epsilon]^2, \mathbf{0} \} \}$

With [{ q0 = CU [{y1, a1, x1, x2, a2, y2}]1,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_0 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] } ,$
Cord [
 q0]

CU [{y1, a1, x1}]1,
 $\left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 + \right.$
 $e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 +$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{11} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \gamma_{12} -$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \gamma_{12} -$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 T_1 \gamma_{12} +$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 T_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 T_1 \gamma_{12} +$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 T_1 \gamma_{12} + \hbar x_1 y_1 \gamma_{12} +$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{21} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \gamma_{22} -$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \gamma_{22} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \gamma_{22} -$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 T_1 \gamma_{22} +$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 T_1 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 T_1 \gamma_{22} +$
 $e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 T_1 \gamma_{22} + \hbar x_1 y_1 \gamma_{22} \left. \right) /$
 $\left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} t_2 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \right.$
 $T_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} t_2 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} T_1 \gamma_{22} \left. \right),$
 $\frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} - \gamma_{12} + T_1 \gamma_{12} - \gamma_{22} + T_1 \gamma_{22}} + O[\epsilon]^1$

Stitching \mathbb{C} 's.

StitchingOEs

```
In[ ]:= mj→k[ $\mathbb{C}_U$ [specs__, Q_, P_]] := Cord[ $\mathbb{C}_U$ [Sequence@@Append[DeleteCases[{specs}, {__}_j|k], Flatten[{Cases[{specs}, {us__}_j => {us}], Cases[{specs}, {us__}_k => {us}]}}]_k], Q, P] /. {tj → tk, Tj → Tk}
```

```
co =  $\mathbb{C}_{CU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y3, a3, x3}3,  $\hbar$  Sum[l10i+j ti aj +  $\gamma$ 10i+j yi xj, {i, 3}, {j, 3}], 12];
co // m3→4, HL@Simp[CU[m3→4[co]] - m3→4[CU[co]]]
{ $\mathbb{C}_{CU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y4, a4, x4}4,  $\hbar$  ( a1 l11 t1 + a2 l12 t1 + a4 l13 t1 + a1 l21 t2 + a2 l22 t2 + a4 l23 t2 + a1 l31 t4 + a2 l32 t4 + a4 l33 t4 + x1 y1  $\gamma$ 11 + x2 y1  $\gamma$ 12 + x4 y1  $\gamma$ 13 + x1 y2  $\gamma$ 21 + x2 y2  $\gamma$ 22 + x4 y2  $\gamma$ 23 + x1 y4  $\gamma$ 31 + x2 y4  $\gamma$ 32 + x4 y4  $\gamma$ 33 ), 1 + O[ $\epsilon$ ]3], 0 }
```

Verifying that *m* commutes with evaluation, in CU:

```
co =  $\mathbb{C}_{CU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y3, a3, x3}3,  $\hbar$  Sum[l10i+j ti aj +  $\gamma$ 10i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{co // m2→3, HL@Simp[CU[m2→3[co]] - m2→3[CU[co]]]}
```

$$\left\{ 513.453, \left\{ \mathbb{C}_{CU} \left[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\dots 1 \dots}{e^{\dots 1 \dots} \dots 1 \dots \dots 1 \dots}, \frac{1}{1 + \hbar t_3 \gamma_{32}} + \left(\left(4 e^{2\gamma \hbar l_{12} t_1 + \dots 4 \dots} + 2\gamma \hbar l_{33} t_3 \hbar^2 a_3 x_1 y_1 \gamma_{12} \gamma_{31} - 2 \dots 7 \dots \gamma_{31} + \dots 154 \dots \right) \epsilon \right) / \left(2 e^{2\gamma \hbar l_{12} t_1 + 2\gamma \hbar l_{13} t_1 + \dots 3 \dots} + 2\gamma \hbar l_{33} t_3 + 10 e^{\dots 1 \dots} \hbar t_3 \gamma_{32} + \dots 2 \dots + \dots 1 \dots + 2 e^{2\gamma \hbar l_{12} t_1 + \dots 4 \dots} + \dots 1 \dots \hbar^5 t_3^5 \gamma_{32}^5 \right) + \frac{(\dots 1 \dots)^2}{\dots 1 \dots} + O[\epsilon]^3, \mathbf{0} \right\} \right\}$$

large output show less show more show all set size limit...

Verifying that *m* commutes with evaluation, in QU:

```
qo =  $\mathbb{C}_{QU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y3, a3, x3}3,  $\hbar$  Sum[l10i+j ti aj +  $\gamma$ 10i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{qo // m2→3, HL@SimpT[QU[m2→3[qo]] - m2→3[QU[qo]]]}
```

$$\left\{ 7831.47, \left\{ \mathbb{C}_{QU} \left[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\dots 1 \dots}{\dots 1 \dots}, \frac{1}{1 - \gamma_{32} + T_3 \gamma_{32}} + \left(\left(8 e^{2\gamma \hbar l_{12} t_1 + \dots 4 \dots} + 2\gamma \hbar l_{33} t_3 \hbar^2 a_3 T_3 x_1 y_1 \gamma_{12} \gamma_{31} + 4 \dots 8 \dots \gamma_{31} + \dots 371 \dots \right) \epsilon \right) / \left(4 e^{2\gamma \hbar l_{12} t_1 + 2\gamma \hbar l_{13} t_1 + \dots 3 \dots} + 2\gamma \hbar l_{33} t_3 - 20 e^{\dots 1 \dots} \gamma_{32} + \dots 26 \dots + 4 e^{2\gamma \hbar l_{12} t_1 + \dots 4 \dots} + \dots 1 \dots T_3^5 \gamma_{32}^5 \right) + \frac{(\dots 1 \dots)^2}{\dots 79 \dots + \dots 1 \dots} + O[\epsilon]^3, \mathbf{0} \right\} \right\}$$

large output show less show more show all set size limit...

In[*]:=

```
CU_[sp1_, Q1_, P1_] ≡ CU_[sp2_, Q2_, P2_] :=  
Sort[{sp1}] == Sort[{sp2}] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
co = CU[{y1, a1, x1}1, {y2, a2, x2}2,  
  {y3, a3, x3}3, ħ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 10];  
Timing@HL[(lhs = co // m1,2→1 // m1,3→1) ≡ (rhs = co // m2,3→2 // m1,2→1)]  
{41.9219, True}
```

```
co = CU[{y1, a1, x1}1, {y2, a2, x2}2,  
  {y3, a3, x3}3, ħ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 11];  
Timing@HL[(lhs = co // m1,2→1 // m1,3→1) ≡ (rhs = co // m2,3→2 // m1,2→1)]  
{30119.8, True}
```

mexamples

```
co = CU[{y1, a1, x1}1, {y2, a2, x2}2, ħ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 2}, {j, 2}], 11];  
Short[Simplify /@ (cexample = co // m1→2), 12]  
Short[Simplify /@ (qexample = (qo = co /. CU → QU) // m1→2), 12]
```

mexamples

$$\begin{aligned} & \mathbb{C}_{\text{CU}}[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11} + l_{12} + l_{21} + l_{22}) t_2 + \frac{1}{1 + \hbar t_2 \gamma_{21}} \\ & e^{-\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 (\gamma_{21} + e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \gamma_{12} (1 + \hbar t_2 \gamma_{21})) + \\ & e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{11} (e^{\gamma \hbar (l_{11} + l_{21}) t_2} - e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar t_2 \gamma_{22})) , \\ & \frac{1}{1 + \hbar t_2 \gamma_{21}} + \frac{1}{2 (1 + \hbar t_2 \gamma_{21})^5} e^{-2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar (4 a_2 (1 + \hbar t_2 \gamma_{21})^2 \\ & (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} t_2 + x_2 y_2) \gamma_{21}^2 + e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{11} \gamma_{22} + \\ & \gamma_{21} (e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \hbar x_2 y_2 (e^{\gamma \hbar (2 l_{11} + l_{12} + 2 l_{21} + l_{22}) t_2} \gamma_{11} + e^{\gamma \hbar (l_{11} + 2 l_{12} + l_{21} + 2 l_{22}) t_2} \gamma_{22}))) - \\ & \gamma \hbar (-2 e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} t_2 \gamma_{21}^2 (1 + \hbar t_2 \gamma_{21})^2 + 4 \ll 5 \gg (\ll 1 \gg) + \\ & \hbar \ll 4 \gg (3 \hbar t_2 \gamma_{21}^2 + 2 e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{21} (4 + e^{\gamma \ll 3 \gg} \hbar t_2 \gamma_{22}) + \\ & e^{\gamma \hbar (l_{11} + l_{21}) t_2} \gamma_{11} (2 + \hbar t_2 (\gamma_{21} - e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22})))) \in + 0[\epsilon]^2 \end{aligned}$$

mexamples

$$\begin{aligned} & \mathbb{C}_{\text{QU}}[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11} + l_{12} + l_{21} + l_{22}) t_2 + \\ & \frac{1}{1 + (-1 + T_2) \gamma_{21}} e^{-\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 (\gamma_{21} + e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \gamma_{12} (1 + (-1 + T_2) \gamma_{21})) + \\ & e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{11} (e^{\gamma \hbar (l_{11} + l_{21}) t_2} - e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (-1 + T_2) \gamma_{22})) , \\ & \frac{1}{1 + (-1 + T_2) \gamma_{21}} + \frac{1}{4 (1 + (-1 + T_2) \gamma_{21})^5} e^{-2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar \\ & (8 a_2 T_2 (1 + (-1 + T_2) \gamma_{21})^2 (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (-e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \\ & e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} T_2 + \hbar x_2 y_2) \gamma_{21}^2 + e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{11} \gamma_{22} + \\ & \gamma_{21} (e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \hbar x_2 y_2 (e^{\gamma \hbar (2 l_{11} + l_{12} + 2 l_{21} + l_{22}) t_2} \gamma_{11} + e^{\gamma \hbar (l_{11} + 2 l_{12} + l_{21} + 2 l_{22}) t_2} \gamma_{22}))) + \\ & \gamma (2 e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (1 - 4 T_2 + 3 T_2^2) \gamma_{21}^2 (1 + (-1 + T_2) \gamma_{21})^2 + \\ & 4 e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{21} (1 + (-1 + T_2) \gamma_{21}) (\ll 1 \gg) - \ll 1 \gg)) \in + 0[\epsilon]^2 \end{aligned}$$

R in QU.

The Faddeev-Quesne formula:

Faddeev

$$\text{In[*]:= } e_{q_-,k_-}[X_-] := e^{\sum_{j=1}^{k+1} \frac{(1-q)^j x^j}{j(1-q^j)}}; e_{q,-}[X_-] := e_{q,\$k}[X]$$

Table[Series[e_{q_n,k}[X], {ε, 0, 4}], {k, 0, 5}] // Column

$$e^x$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384} (e^x x^2 (-8 + x^4) \gamma^3 \hbar^3) \epsilon^3 + \frac{e^x x^4 (-32 + x^4) \gamma^4 \hbar^4 \epsilon^4}{6144} + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24 + 32x^3 + 3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{e^x x^3 (-4608 - 864x + 1024x^3 + 576x^4 + 27x^5) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24 + 72x^2 + 32x^3 + 3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{e^x x^3 (-4608 - 864x + 3616x^3 + 576x^4 + 27x^5) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24 + 72x^2 + 32x^3 + 3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{1}{4147200} e^x x^3 (-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24 + 72x^2 + 32x^3 + 3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{1}{4147200} e^x x^3 (-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

Table[Together@SeriesCoefficient[e_{q₅}[X], {x, 0, n}], {n, 0, 5}]

$$\left\{ 1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2+q^3+q^4)} \right\}$$

Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q₅}[X], {x, 0, n}]], {n, 0, 5}]

{1, 1, 1, 1, 1, 1}

R

$$\text{In[*]:= } \text{QU}[R_{i,j}] := \text{OQU}[\{y_1, a_1\}_i, \{a_2, x_2\}_j, \text{SS}[e^{\hbar b_1 a_2} e_{q_h}[\hbar y_1 x_2] /. b_1 \rightarrow \gamma^{-1} (\epsilon a_1 - t_i)]]; \text{QU}[R_{i,j}^{-1}] := S_j @ \text{QU}[R_{i,j}];$$

QU[R_{3,4}] // Short

$$\text{QU}[] + \frac{\epsilon \hbar \text{QU}[a_3, a_4]}{\gamma} + \hbar \text{QU}[y_3, x_4] + \frac{\epsilon \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{\gamma} + \frac{1}{2} \frac{\langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{\gamma} - \frac{\langle\langle 1 \rangle\rangle}{\gamma} - \frac{\epsilon \hbar^2 \langle\langle 1 \rangle\rangle t_3}{\gamma^2} - \frac{\hbar^2 \text{QU}[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 \text{QU}[a_4, a_4] t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ \$p=4, \$k=2):

QU[R_{1,2} ** R_{1,2}⁻¹] // Simp // HL // Timing

{0.078125, QU[]}

Verifying R3 (~156 secs @ \$p=4, \$k=2):

{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing

$$\left\{0.203125, \left\{QU\left[\right] + \frac{\epsilon \hbar QU[a_1, a_2]}{\gamma} + \frac{\epsilon \hbar QU[a_1, a_3]}{\gamma} + \ll 73 \gg + 2 \epsilon \hbar^2 QU[y_1, a_2, x_3] T_2 + QU[y_1, x_3] (\hbar - \hbar T_2), \mathbf{0}\right\}\right\}$$

R in \mathbb{C}_{QU} .

RinOE

$$\text{In[*]:= } \mathbb{C}_{QU,k}[R_{i,j}] := \mathbb{C}_{QU}\left[\{y_i, a_i, x_i\}_i, \{y_j, a_j, x_j\}_j, -\hbar \gamma^{-1} t_i a_j + \hbar y_i x_j, \text{Series}\left[e^{\hbar \gamma^{-1} t_i a_j - \hbar y_i x_j} \left(e^{\hbar b_i a_j} e_{q_n,k}[\hbar y_i x_j] / . b_i \rightarrow \gamma^{-1} (\epsilon a_i - t_i)\right), \{\epsilon, \mathbf{0}, k\}\right]\right]$$

{ $\mathbb{C}_{QU,1}[R_{1,2}]$, $\mathbb{C}_{QU,2}[R_{1,2}]$ }

$$\left\{\mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + O[\epsilon]^2\right], \mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + \frac{1}{288 \gamma^2} (144 \hbar^2 a_1^2 a_2^2 - 72 \gamma^2 \hbar^4 a_1 a_2 x_2^2 y_1^2 + 32 \gamma^4 \hbar^5 x_2^3 y_1^3 + 9 \gamma^4 \hbar^6 x_2^4 y_1^4) \epsilon^2 + O[\epsilon]^3\right]\right\}$$

The morphism $\mathbb{C}_{U,k}$.

MorphismOE

$$\text{In[*]:= } \mathbb{C}_{U,k}[a_* b_*] := \mathbb{C}_{U,k}[a] \mathbb{C}_{U,k}[b]; \mathbb{C}_{U,k}[m_{iS}[a_*]] := m_{iS}[\mathbb{C}_{U,k}[a_*]];$$

$\mathbb{C}_{QU,1}[R_{1,2} R_{3,4} R_{5,6}]$

$$\mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar a_2 t_1}{\gamma} - \frac{\hbar a_4 t_3}{\gamma} - \frac{\hbar a_6 t_5}{\gamma} + \hbar x_2 y_1 + \hbar x_4 y_3 + \hbar x_6 y_5, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} + \frac{\hbar a_3 a_4}{\gamma} + \frac{\hbar a_5 a_6}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2 - \frac{1}{4} \gamma \hbar^3 x_4^2 y_3^2 - \frac{1}{4} \gamma \hbar^3 x_6^2 y_5^2\right) \epsilon + O[\epsilon]^2\right]$$

$\mathbb{C}_{QU,1}[R_{1,2} R_{3,4} R_{5,6} // m_{1,3 \rightarrow 1} // m_{2,5 \rightarrow 2} // m_{4,6 \rightarrow 4}]$

$$\mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma} (-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_2} \gamma \hbar x_4 y_1 - \gamma \hbar T_2 x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2), 1 + \frac{1}{4 \gamma} (4 \hbar a_1 a_2 + 4 \hbar a_1 a_4 + 4 \hbar a_2 a_4 - 4 \gamma \hbar^2 a_4 x_2 y_1 - 8 e^{\hbar t_2} \gamma \hbar^2 a_2 x_4 y_1 + 8 \gamma \hbar^2 a_2 T_2 x_4 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 + 4 e^{\hbar t_2} \gamma^2 \hbar^3 x_2 x_4 y_1^2 - 4 \gamma^2 \hbar^3 T_2 x_2 x_4 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - e^{2 \hbar t_2} \gamma^2 \hbar^3 x_4^2 y_1^2 + \gamma^2 \hbar^3 T_2^2 x_4^2 y_1^2 - 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 + 4 e^{\hbar t_1 + \hbar t_2} \gamma^2 \hbar^3 x_4^2 y_1 y_2 - 4 e^{\hbar t_1} \gamma^2 \hbar^3 T_2 x_4^2 y_1 y_2 - e^{2 \hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2) \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{3,1 \rightarrow 1} // \mathbf{m}_{5,2 \rightarrow 2} // \mathbf{m}_{6,4 \rightarrow 4}]$$

$$\mathbb{C}_{\text{QU}} \left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma} \right. \\ \left. \left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2 \right), \right. \\ \left. 1 + \frac{1}{4\gamma} \left(4 \hbar a_1 a_2 + 4 \hbar a_1 a_4 + 4 \hbar a_2 a_4 - 4 \gamma \hbar^2 a_4 x_2 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - \right. \right. \\ \left. \left. 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 - e^{2\hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2 \right) \in + \mathbf{O}[\epsilon]^2 \right]$$

$$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{1,3 \rightarrow 1} // \mathbf{m}_{2,5 \rightarrow 2} // \mathbf{m}_{4,6 \rightarrow 4}] \equiv \mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{3,1 \rightarrow 1} // \mathbf{m}_{5,2 \rightarrow 2} // \mathbf{m}_{6,4 \rightarrow 4}]$$

$$\hbar \left(e^{\hbar t_2} - T_2 \right) x_4 y_1 = \mathbf{0} \&\& \in \hbar \left(e^{\hbar t_2} - T_2 \right) x_4 y_1 \left(8 a_2 + \gamma \hbar \left(-4 x_2 y_1 + x_4 \left(\left(e^{\hbar t_2} + T_2 \right) y_1 - 4 e^{\hbar t_1} y_2 \right) \right) \right) = \mathbf{0}$$

Exponentials as needed.

```
In[ ]:= Block[{$p = 2, $k = 2}, TableForm[StringSplit[
  "y | a | x | C@y_CU | C@a_CU | C@x_CU | Q@y_QU | Q@a_QU | Q@x_QU | AD@y_QU | AD@a_QU | AD@x_QU | SD@y_QU | SD@a_QU | SD
  @x_QU | S@y_CU | S@a_CU | S@x_CU | S@y_QU | S@a_QU | S@x_QU | Δ@y_CU | Δ@a_CU | Δ@x_CU | Δ@y_QU | Δ@a_QU | Δ@x_QU",
  "" ] /. s_String =>
  {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ε, 0, $k}]}]]
```

Out[]//TableForm=

y	y
a	a
x	x
C@y_CU	-x
C@a_CU	-a
C@x_CU	-y
Q@y_QU	$-\frac{x}{\sqrt{T}} - \frac{ax\epsilon\hbar}{\sqrt{T}} - \frac{a^2x\epsilon^2\hbar^2}{2\sqrt{T}}$
Q@a_QU	-a
Q@x_QU	$-\frac{y}{\sqrt{T}} + \frac{y(-a+\gamma)\epsilon\hbar}{\sqrt{T}} - \frac{y(a-\gamma)^2\epsilon^2\hbar^2}{2\sqrt{T}}$
AD@y_QU	$\frac{2}{3}a^2y\epsilon^2\hbar^2 + \frac{1}{6}y(6+3t\hbar+t^2\hbar^2) + \frac{1}{12}y\epsilon\hbar(xy\gamma\hbar - 4a(3+2t\hbar))$
AD@a_QU	a
AD@x_QU	x
SD@y_QU	$y + \frac{1}{48}t^2y\hbar^2 + \frac{1}{24}y(-2at+xy\gamma)\epsilon\hbar^2 + \frac{1}{12}a^2y\epsilon^2\hbar^2$
SD@a_QU	a
SD@x_QU	$\frac{7}{12}a^2x\epsilon^2\hbar^2 + x\left(1 + \frac{t\hbar}{2} + \frac{7t^2\hbar^2}{48}\right) + \frac{1}{24}x\epsilon\hbar(xy\gamma\hbar - 2a(12+7t\hbar))$
S@y_CU	-y
S@a_CU	-a
S@x_CU	-x
S@y_QU	$-\frac{y}{T} + \frac{y(-a+\gamma)\epsilon\hbar}{T} - \frac{y(a-\gamma)^2\epsilon^2\hbar^2}{2T}$
S@a_QU	-a
S@x_QU	$-x - a x \epsilon \hbar - \frac{1}{2} a^2 x \epsilon^2 \hbar^2$
Δ@y_CU	y ₁ + y ₂
Δ@a_CU	a ₁ + a ₂
Δ@x_CU	x ₁ + x ₂
Δ@y_QU	y ₁ + T ₁ y ₂ - εħ a ₁ T ₁ y ₂ + $\frac{1}{2}\epsilon^2\hbar^2 a_1^2 T_1 y_2$
Δ@a_QU	a ₁ + a ₂
Δ@x_QU	x ₁ + x ₂ - εħ a ₁ x ₂ + $\frac{1}{2}\epsilon^2\hbar^2 a_1^2 x_2$

Exp

Task. Define $\text{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi Q(P)}$ to ϵ^k in the algebra U_i , where ξ is a scalar, X is x_i or y_i , and P is an ϵ -dependent near-docile element, giving the answer in $\mathbb{C}\epsilon$ -form. Should satisfy

$$U @ \text{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U[e^{\xi X}, x \rightarrow Q(P)].$$

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi Q(P)} = Q(e^{\xi P_0} F(\xi))$, then $F(\xi = 0) = 1$ and we have:

$$Q(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F)) = Q(\partial_\xi e^{\xi P_0} F(\xi)) = \partial_\xi Q(e^{\xi P_0} F(\xi)) = \partial_\xi e^{\xi Q(P_0)} = e^{\xi Q(P_0)} Q(P) = Q(e^{\xi P_0} F(\xi)) Q(P).$$

This is an ODE for F . Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ .

Exp

```
In[*]:= (* Bug: The first line is valid only if 0 (e^P0) == e^0 (P0). *)
(* Bug: ξ must be a symbol. *)
ExpU_{i,0}[ξ_, P_] := CU[{y_i, a_i, x_i}_i, Normal@P /. ε → 0, 1 + 0_0];
ExpU_{i,k}[ξ_, P_] := Module[{yax = {y_i, a_i, x_i}, P0, φ, φS, F, j, rhs, at0, atξ},
  P0 = Normal@P /. ε → 0;
  φS =
  Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
  F = Normal@Last@ExpU_{i,k-1}[ξ, P] + e^k φS. (φS /. φ_{js}_[ξ] := Times@@yax^{js});
  rhs = Normal@Last@m_{i,j→i}[CU[yax_i, ξ P0, F + 0_k] m_{i→j}@CU[{y_i, a_i, x_i}_i, 0, P + 0_k]];
  at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, yax];
  atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, yax];
  CU[yax_i, ξ P0, F + 0_k] /. DSolve[And@@(at0 ∪ atξ), φS, ξ] [[1]] ]
```

```
In[*]:= Timing@Block[{$p = 4, $k = 2}, {
  s = S1[QU[y1]] /. QU → Times,
  exps = ExpQU_{i,$k}[η, s], (* Warning: wrong unless $p>=$k+1! *)
  HL@Simp[S1@OQU[{y1}_1, SS[e^h η y1]] - QU@(exps /. η → h η)]
}]
```

$$Out[*]= \left\{ 35.8281, \left\{ a_1 \left(-\frac{\epsilon \hbar}{T_1} + \frac{\gamma \epsilon^2 \hbar^2}{T_1} \right) y_1 + \left(-\frac{1}{T_1} + \frac{\gamma \epsilon \hbar}{T_1} - \frac{\gamma^2 \epsilon^2 \hbar^2}{2 T_1} \right) y_1 - \frac{\epsilon^2 \hbar^2 a_1^2 y_1}{2 T_1}, \right. \right.$$

$$\left. \mathbb{C}_{QU} \left[\{y_1, a_1, x_1\}_1, -\frac{\eta y_1}{T_1}, 1 + \frac{(2 \gamma \eta \hbar T_1 y_1 - 2 \eta \hbar a_1 T_1 y_1 - \gamma \eta^2 \hbar y_1^2) \epsilon}{2 T_1^2} + \right. \right.$$

$$\left. \left(-\frac{\gamma^2 \eta \hbar^2 y_1}{2 T_1} + \frac{\gamma \eta \hbar^2 a_1 y_1}{T_1} - \frac{\eta \hbar^2 a_1^2 y_1}{2 T_1} + \frac{7 \gamma^2 \eta^2 \hbar^2 y_1^2}{4 T_1^2} - \frac{2 \gamma \eta^2 \hbar^2 a_1 y_1^2}{T_1^2} + \right. \right.$$

$$\left. \left. \frac{\eta^2 \hbar^2 a_1^2 y_1^2}{2 T_1^2} - \frac{\gamma^2 \eta^3 \hbar^2 y_1^3}{T_1^3} + \frac{\gamma \eta^3 \hbar^2 a_1 y_1^3}{2 T_1^3} + \frac{\gamma^2 \eta^4 \hbar^2 y_1^4}{8 T_1^4} \right) \epsilon^2 + O[\epsilon]^3, \mathbf{0} \right\}$$

```
In[*]:= Timing@Block[{$p = 4, $k = 2}, {
  s = S1[QU[a1]] /. QU → Times,
  exps = ExpQU_{i,$k}[α, s], (* Warning: wrong unless $p>=$k+1! *)
  HL@Simp[S1@OQU[{a1}_1, SS[e^h α a1]] - QU@(exps /. α → h α)]
}]
```

$$Out[*]= \left\{ 33.5938, \left\{ -a_1, \mathbb{C}_{QU} \left[\{y_1, a_1, x_1\}_1, -\alpha a_1, 1 + O[\epsilon]^3, \mathbf{0} \right] \right\} \right\}$$

```
In[*]:= Timing@Block[{$p = 4, $k = 2}, {
  s = S1[QU[x1]] /. QU → Times,
  exps = ExpQu1,$k[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
  HL@Simp[S1@OQu[{x1}1, SS[eħ ξ x1]] - QU@(exps /. ξ → ħ ξ)]
}]
```

```
Out[*]:= {34.0625, {-x1 - ε ħ a1 x1 - 1/2 ε2 ħ2 a12 x1,
  EQu[{y1, a1, x1}1, -ξ x1, 1 + (-ξ ħ a1 x1 - 1/2 γ ξ2 ħ x12) ε + (-1/2 ξ ħ2 a12 x1 + 1/4 γ2 ξ2 ħ2 x12 -
  γ ξ2 ħ2 a1 x12 + 1/2 ξ2 ħ2 a12 x12 - 1/2 γ2 ξ3 ħ2 x13 + 1/2 γ ξ3 ħ2 a1 x13 + 1/8 γ2 ξ4 ħ2 x14) ε2 + O[ε]3], 0}}
```

$S(e^{\eta y} e^{\alpha a} e^{\xi x})$

LinearLambda

```
In[*]:= Timing@Block[{$p = 3, $k = 1}, {
  (sexp = m3,2,1→1[ExpQu1,$k[η, S1[QU[y1]] /. QU → Times] ExpQu2,$k[α, S2[QU[a2]] /. QU → Times]
  ExpQu3,$k[ξ, S3[QU[x3]] /. QU → Times]]) /. u-1 ⇒ u,
  HL@SimpT[QU@(sexp /. {η → ħ η, α → ħ α, ξ → ħ ξ}) -
  S1@OQu[{y1, a1, x1}1, SS[eħ (η y1 + α a1 + ξ x1)]]]
}]
```

LinearLambda

```
Out[*]:= {15.2969, {EQu[{y1, a1, x1}, 1/T ħ (eα γ η ξ - eα γ T η ξ - a T α ħ - eα γ y η ħ - eα γ T x ξ ħ),
  1 + 1/(4 T2 ħ) (-3 e2 α γ γ η2 ξ2 + 4 e2 α γ T γ η2 ξ2 - e2 α γ T2 γ η2 ξ2 + 8 a eα γ T η ξ ħ - 4 eα γ T γ η ξ ħ +
  4 eα γ T2 γ η ξ ħ + 6 e2 α γ y γ η2 ξ ħ - 2 e2 α γ T y γ η2 ξ ħ + 6 e2 α γ T x γ η ξ2 ħ -
  2 e2 α γ T2 x γ η ξ2 ħ - 4 a eα γ T y η ħ2 + 4 eα γ T y γ η ħ2 - 2 e2 α γ y2 γ η2 ħ2 -
  4 a eα γ T2 x ξ ħ2 - 4 e2 α γ T x y γ η ξ ħ2 - 2 e2 α γ T2 x2 γ ξ2 ħ2) ε + O[ε]2], 0}}
```

$$\Delta_{1 \rightarrow 1,2} (e^{\eta y_1} e^{\alpha a_1} e^{\xi x_1})$$

LinearLambda

```
Timing@Block[{$p = 4, $k = 2}, {
  sexp = m1,3,5→1@m2,4,6→2@Times[ (* Warning: wrong unless $p>=$k+1! *)
    Prepend[{y2}2]@ExpQU, $k[η, Δ1→1,2[QU[y1]] /. QU → Times],
    Prepend[{a4}4]@ExpQU, $k[α, Δ3→3,4[QU[a3]] /. QU → Times],
    Prepend[{x6}6]@ExpQU, $k[ξ, Δ5→5,6[QU[x5]] /. QU → Times]
  ] /. {η → ħ η, α → ħ α, ξ → ħ ξ},
  HL@SimpT[QU@sexp - Δ1→1,2@OQU[{y1, a1, x1}1, SS[eħ(η y1+α a1+ξ x1)]]]
}]
```

LinearLambda

```
Out[*]= {162., {CQU[{y2, a2, x2}2, {y1, a1, x1}1, α ħ a1 + α ħ a2 + ξ ħ x1 + ξ ħ x2 + η ħ y1 + η ħ T1 y2,
  1 +  $\frac{1}{2} (-2 \xi \hbar^2 a_1 x_2 + \gamma \xi^2 \hbar^3 x_1 x_2 - 2 \eta \hbar^2 a_1 T_1 y_2 + \gamma \eta^2 \hbar^3 T_1 y_1 y_2) \epsilon +$ 
   $\frac{1}{24} (12 \xi \hbar^3 a_1^2 x_2 + 6 \gamma^2 \xi^2 \hbar^4 x_1 x_2 - 12 \gamma \xi^2 \hbar^4 a_1 x_1 x_2 + 4 \gamma^2 \xi^3 \hbar^5 x_1^2 x_2 + 12 \xi^2 \hbar^4 a_1^2 x_2^2 +$ 
   $4 \gamma^2 \xi^3 \hbar^5 x_1 x_2^2 - 12 \gamma \xi^3 \hbar^5 a_1 x_1 x_2^2 + 3 \gamma^2 \xi^4 \hbar^6 x_1^2 x_2^2 + 12 \eta \hbar^3 a_1^2 T_1 y_2 +$ 
   $24 \eta \xi \hbar^4 a_1^2 T_1 x_2 y_2 - 12 \gamma \eta \xi^2 \hbar^5 a_1 T_1 x_1 x_2 y_2 + 6 \gamma^2 \eta^2 \hbar^4 T_1 y_1 y_2 - 12 \gamma \eta^2 \hbar^4 a_1 T_1 y_1 y_2 -$ 
   $12 \gamma \eta^2 \xi \hbar^5 a_1 T_1 x_2 y_1 y_2 + 6 \gamma^2 \eta^2 \xi^2 \hbar^6 T_1 x_1 x_2 y_1 y_2 + 4 \gamma^2 \eta^3 \hbar^5 T_1 y_1^2 y_2 + 12 \eta^2 \hbar^4 a_1^2 T_1^2 y_2^2 +$ 
   $4 \gamma^2 \eta^3 \hbar^5 T_1^2 y_1 y_2^2 - 12 \gamma \eta^3 \hbar^5 a_1 T_1^2 y_1 y_2^2 + 3 \gamma^2 \eta^4 \hbar^6 T_1^2 y_1^2 y_2^2) \epsilon^2 + O[\epsilon]^3, \mathbf{0}}$ 
```

Zip and Bind

E

```
In[*]= E /: E[L1_, Q1_, P1_] ≡ E[L2_, Q2_, P2_] :=
  Simplify[L1 == L2] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0];
E /: E[L1_, Q1_, P1_] E[L2_, Q2_, P2_] := E[L1 + L2, Q1 + Q2, P1 * P2];
```

Zip

```
In[*]= {t*, y*, a*, x*, z*} = {τ, η, α, ξ, ζ};
{t*, η*, α*, ξ*, ζ*} = {t, y, a, x, z}; (u-i)* := (u*)i;
```

Zip

```
In[*]= Zip{}[P_] := P; Zip{ξ, ζ, ...}[P_] := (Expand[P // Zip{ξ, ζ, ...}] /. f-. Sd- → ∂{ξ*, d}f) /. ζ* → 0
```

QZip implements the “Q-level zips” on $E(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Zip

```
In[*]= E /: QZipξs_List@E[L_, Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, qt, zruler, Q1, Q2},
  zs = Table[ξ*, {ξ, ζs}];
  c = Q /. Alternatives@@(ξs ∪ zs) → 0;
  ys = Table[∂ξ(Q /. Alternatives@@zs → 0), {ξ, ζs}];
  ηs = Table[∂z(Q /. Alternatives@@ξs → 0), {z, zs}];
  qt = Inverse@Table[Kδz, ξ* - ∂z, ξQ, {ξ, ζs}, {z, zs}];
  zruler = Thread[zs → qt.(zs + ys)];
  Q2 = (Q1 = c + ηs.zs /. zruler) /. Alternatives@@zs → 0;
  Simplify /@ E[L, Q2, Det[qt] e-Q2 Zipξs[eQ1(P /. zruler)]]];
```



```
In[ ]:= Timing@{E0 = E[0, Sum[a10i+j xi ξj, {i, 3}, {j, 3}],
  1 + e Sum[fi[x1, x2, x3] ξi, {i, 3}] + e Sum[f10i+j[x1, x2, x3] ξi ξj, {i, 3}, {j, 3}]],
  lhs = QZip[ξ1, ξ2]@E0,
  HL[lhs == QZip[ξ1]@QZip[ξ2]@E0]}
```

```
Out[ ]:= {38.6875, {E[0, ... 1 ..., 1 + e (ξ1 ... 1 ... + ... 1 ... + ... 1 ...) +
  e (ξ12 f11[x1, x2, x3] + ... 7 ... + ξ32 f33[x1, x2, x3)]}, ... 1 ..., True}}
```

large output show less show more show all set size limit...

```
In[ ]:= Timing@{
  Eh = E[0, h Sum[a10i+j xi ξj, {i, 3}, {j, 3}],
  1 + e Sum[fi[x1, x2, x3] ξi, {i, 3}] + e Sum[f10i+j[x1, x2, x3] ξi ξj, {i, 3}, {j, 3}]],
  lhs = Normal[Eh /. E[L_, Q_, P_] => Series[P eL+Q, {h, 0, 2}]] // Zip[ξ1],
  HL@Simplify[lhs == Normal[QZip[ξ1][Eh] /. E[L_, Q_, P_] => Series[P eL+Q, {h, 0, 2}]]]}
```

```
Out[ ]:= {18.4375, {E[0, h ... 1 ..., 1 + e (ξ1 ... 1 ... + ... 1 ... + ... 1 ...) +
  e (ξ12 f11[x1, x2, x3] + ... 7 ... + ξ32 f33[x1, x2, x3)]}, ... 1 ..., True}}
```

large output show less show more show all set size limit...

LZip implements the “L-level zips” on $E(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single “P”. Here the z’s are t and α and the ζ ’s are τ and a .

Zip

```
In[ ]:= E /. LZip[ξs_List@E[L_, Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
  zs = Table[ξ*, {ξ, ξs}];
  c = L /. Alternatives@@(ξs ∪ zs) → 0;
  ys = Table[∂ξ(L /. Alternatives@@zs → 0), {ξ, ξs}];
  ηs = Table[∂z(L /. Alternatives@@ξs → 0), {z, zs}];
  lt = Inverse@Table[Kδz, ξ* - ∂z, ξL, {ξ, ξs}, {z, zs}];
  zrule = Thread[zs → lt.(zs + ys)];
  L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives@@zs → 0;
  Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives@@zs → 0;
  Simplify /@ E[L2, Q2, Det[lt] e-L2-Q2 Zip[ξs][eL1+Q1(P /. T2t /. zrule)]] // . t2T];
```

Bind

```
In[ ]:= Bind[is___][L_E, R_E] := Module[{n},
  Times[
    L /. Table[(v : T | t | a | x | y)i → vnei, {i, {is}}],
    R /. Table[(v : τ | α | ξ | η)i → vnei, {i, {is}}]
  ] // LZipFlatten@Table[{τnei, anei}, {i, {is}}] // QZipFlatten@Table[{ξnei, ynei}, {i, {is}}] ];
  BL_List := BindL; Bis___ := Bind{is};
  Bind[ξ_E] := ξ;
  Bind[LS___, ξs_List, R_] := Bindξs[Bind[LS], R];
```

In[*]:= **Bind**_{2}[**E**[**0**, ξ ($x_1 + x_2$), **1**], **E**[**0**, ξ_2 ($x_2 + x_3$), **1**]]

Out[*]:= **E**[**0**, ξ ($x_1 + x_2 + x_3$), **1**]

In[*]:= **Bind**_{2}[**E**[**0**, $(\xi_2 + \xi_3) x_2$, **1**], **E**[**0**, $(\xi_1 + \xi_2) x$, **1**]]

Out[*]:= **E**[**0**, x ($\xi_1 + \xi_2 + \xi_3$), **1**]

In[*]:= **Bind**_{1,2}[**E**[**0**, $(\xi_2 + \xi_3) x_2 + \xi_1 x_1$, **1**], **E**[**0**, $(\xi_1 + \xi_2) x$, **1**]]

Out[*]:= **E**[**0**, x ($\xi_1 + \xi_2 + \xi_3$), **1**]

An $xy \rightarrow axy \rightarrow ayx \rightarrow yax \equiv xay \rightarrow xya \rightarrow yxa \rightarrow yax$ test:

In[*]:= **Bind**[**E**[$\alpha_1 a_1 + \tau_1 t_1$, $e^{\gamma \alpha_1} \xi_1 x_1 + \eta_1 y_1$, **1**], {**1**}, **E**[$\tau_1 t_1 + \alpha_1 a_1$, $\xi_1 x_1 + \eta_1 y_1 + \xi_1 \eta_1 t_1$, **1**]]

Out[*]:= **E**[$\alpha_1 \alpha_1 + \tau_1 \tau_1$, $y_1 \eta_1 + e^{\gamma \alpha_1} (x_1 + \tau_1 \eta_1) \xi_1$, **1**]

In[*]:= **Column**@{**Cord**[**CCU**[{ x_1 , a_1 }]₁, $\xi_1 x_1 + \alpha_1 a_1$, **1** + θ_0]],
Cord[**CCU**[{ x_1 , y_1 }]₁, $\xi_1 x_1 + \eta_1 y_1$, **1** + θ_0]],
Cord[**CCU**[{ a_1 , y_1 }]₁, $\alpha_1 a_1 + \eta_1 y_1$, **1** + θ_0]]}

CCU[{ a_1 , x_1 }]₁, $e^{-\gamma \alpha_1} (e^{\gamma \alpha_1} a_1 \alpha_1 + x_1 \xi_1)$, **1** + $0[\epsilon]^1$]
Out[*]:= **CCU**[{ y_1 , a_1 , x_1 }]₁, $y_1 \eta_1 + x_1 \xi_1 - \tau_1 \eta_1 \xi_1$, **1** + $0[\epsilon]^1$]
CCU[{ y_1 , a_1 }]₁, $e^{-\gamma \alpha_1} (e^{\gamma \alpha_1} a_1 \alpha_1 + y_1 \eta_1)$, **1** + $0[\epsilon]^1$]

In[*]:= {**rx****a** = **E**[$\tau_1 t_1 + \alpha_1 a_1$, $e^{-\gamma \alpha_1} \xi_1 x_1 + \eta_1 y_1$, **1**];
rx**y** = **E**[$\tau_1 t_1 + \alpha_1 a_1$, $\xi_1 x_1 + \eta_1 y_1 - \xi_1 \eta_1 t_1$, **1**];
ra**y** = **E**[$\tau_1 t_1 + \alpha_1 a_1$, $e^{-\gamma \alpha_1} \eta_1 y_1 + \xi_1 x_1$, **1**];
lhs = **Expand** /@ **Bind**[**rx****a**, {**1**}, **rx****y**, {**1**}, **ra****y**],
HL[**lhs** == **Expand** /@ **Bind**[**ra****y**, {**1**}, **rx****y**, {**1**}, **rx****a**]]}

Out[*]:= {**E**[$a_1 \alpha_1 + \tau_1 \tau_1$, $e^{-\gamma \alpha_1} y_1 \eta_1 + e^{-\gamma \alpha_1} x_1 \xi_1 - e^{-\gamma \alpha_1} \tau_1 \eta_1 \xi_1$, **1**], **True**}

In[*]:= **Simplify** /@ **m** _{$i, j \rightarrow k$} @**CCU**[{ y_i , a_i , x_i }] _{i} , { y_j , a_j , x_j }] _{j} , $\eta_i y_i + \alpha_i a_i + \xi_i x_i + \eta_j y_j + \alpha_j a_j + \xi_j x_j$, **1** + θ_{rk}]

Out[*]:= **CCU**[{ y_k , a_k , x_k }] _{k} , $a_k (\alpha_i + \alpha_j) + y_k (\eta_i + e^{-\gamma \alpha_i} \eta_j) + e^{-\gamma \alpha_j} x_k \xi_i - \tau_k \eta_j \xi_i + x_k \xi_j$,
 $1 + \frac{1}{2} \eta_j \xi_i (4 a_k - 2 e^{-\gamma \alpha_i} \gamma y_k \eta_j + \gamma (-2 e^{-\gamma \alpha_j} x_k + \tau_k \eta_j) \xi_i) \in + 0[\epsilon]^2$]

Tensorial Representations

t1

In[*]:= **t1** := **E**[**0**, **0**, **1** + $\theta_{\$k}$]; **t1** _{i} := **t1**;

tm

In[*]:= **m**[**U**₋, **kk**₋] _{$i, j \rightarrow k$} := **m**[**U**, **kk**] _{$i, j \rightarrow k$} = **Module**[{**OE**},
OE = **Simplify** /@
m _{$i, j \rightarrow k$} @**CCU**[{ y_i , a_i , x_i }] _{i} , { y_j , a_j , x_j }] _{j} , $\eta_i y_i + \alpha_i a_i + \xi_i x_i + \eta_j y_j + \alpha_j a_j + \xi_j x_j$, **1** + θ_{rk}];
E[$\tau_k (\tau_i + \tau_j) + (\text{OE}[[2]] /. (\xi | \eta)_{i|j} \rightarrow \theta)$, **OE**[[2]] /. $a_k \rightarrow \theta$, **OE**[[3]]];
tm _{$i, j \rightarrow k$} := **m**[**\$U**, **\$k**] _{$i, j \rightarrow k$} ;

tm_{1,2→3}

$$\text{Out[*]} = \mathbb{E} \left[\mathbf{a}_3 (\alpha_1 + \alpha_2) + \mathbf{t}_3 (\tau_1 + \tau_2), \mathbf{y}_3 (\eta_1 + e^{-\gamma \alpha_1} \eta_2) - \frac{(-1 + T_3) \eta_2 \xi_1}{\hbar} + \mathbf{x}_3 (e^{-\gamma \alpha_2} \xi_1 + \xi_2), \right. \\ \left. 1 + \frac{1}{4 \hbar} e^{-\gamma (\alpha_1 + \alpha_2)} \eta_2 \xi_1 (8 e^{\gamma (\alpha_1 + \alpha_2)} \hbar \mathbf{a}_3 T_3 + e^{\gamma \alpha_2} \gamma (-1 + 3 T_3) \eta_2 (-2 \hbar \mathbf{y}_3 + e^{\gamma \alpha_1} (-1 + T_3) \xi_1) + \right. \\ \left. 2 \gamma \hbar \mathbf{x}_3 (2 \hbar \mathbf{y}_3 - e^{\gamma \alpha_1} (-1 + 3 T_3) \xi_1)) \right] \in +O[\epsilon]^2$$

Associativity of tm.

```
In[*]:= Table[Block[{$U = U, $k = kk},
  {lhs = Bind[tm1,2→2, {2}, tm2,3→1];
  {$U, $k} -> HL[lhs ≡ Bind[tm2,3→2, {2}, tm1,2→1]]}
], {U, {CU, QU}}, {kk, 0, 1}]

Out[*]:= {{{{CU, 0} → True}, {{CU, 1} → True}}, {{{QU, 0} → True}, {{QU, 1} → True}}}]

In[*]:= Block[{$U = CU, $k = 2}, Timing@{lhs = Bind[tm1,2→2, {2}, tm2,3→1];
  HL[lhs ≡ Bind[tm2,3→2, {2}, tm1,2→1]]}]

Out[*]:= {157.141, {True}}
```

tS

```
In[*]:= S[U_, kk_]i_ := S[U, kk]i = Module[{OE},
  OE = m3,2,1→i[ExpU1, $k[η, S1[U[y1]]] /. U → Times]
  ExpU2, $k[α, S2[U[a2]]] /. U → Times] ExpU3, $k[ξ, S3[U[x3]]] /. U → Times];
  E[-ti τi + (OE[[2]] /. ξ | η → 0), OE[[2]] /. ai → 0, OE[[3]]] /. {η → ηi, α → αi, ξ → ξi};
  tSi := S[$U, $k]i;
```

In[*]:= **tS₁**

$$\text{Out[*]} = \mathbb{E} \left[-\mathbf{a}_1 \alpha_1 - \mathbf{t}_1 \tau_1, \frac{1}{\hbar T_1} (-e^{\gamma \alpha_1} \hbar \mathbf{y}_1 \eta_1 - e^{\gamma \alpha_1} \hbar T_1 \mathbf{x}_1 \xi_1 + e^{\gamma \alpha_1} \eta_1 \xi_1 - e^{\gamma \alpha_1} T_1 \eta_1 \xi_1), \right. \\ \left. 1 + \frac{1}{4 \hbar T_1^2} (4 e^{\gamma \alpha_1} \gamma \hbar^2 T_1 \mathbf{y}_1 \eta_1 - 4 e^{\gamma \alpha_1} \hbar^2 \mathbf{a}_1 T_1 \mathbf{y}_1 \eta_1 - 2 e^{2\gamma \alpha_1} \gamma \hbar^2 \mathbf{y}_1^2 \eta_1^2 - 4 e^{\gamma \alpha_1} \hbar^2 \mathbf{a}_1 T_1^2 \mathbf{x}_1 \xi_1 - \right. \\ \left. 4 e^{\gamma \alpha_1} \gamma \hbar T_1 \eta_1 \xi_1 + 8 e^{\gamma \alpha_1} \hbar \mathbf{a}_1 T_1 \eta_1 \xi_1 + 4 e^{\gamma \alpha_1} \gamma \hbar T_1^2 \eta_1 \xi_1 - 4 e^{2\gamma \alpha_1} \gamma \hbar^2 T_1 \mathbf{x}_1 \mathbf{y}_1 \eta_1 \xi_1 + \right. \\ \left. 6 e^{2\gamma \alpha_1} \gamma \hbar \mathbf{y}_1 \eta_1^2 \xi_1 - 2 e^{2\gamma \alpha_1} \gamma \hbar T_1 \mathbf{y}_1 \eta_1^2 \xi_1 - 2 e^{2\gamma \alpha_1} \gamma \hbar^2 T_1^2 \mathbf{x}_1^2 \xi_1^2 + 6 e^{2\gamma \alpha_1} \gamma \hbar T_1 \mathbf{x}_1 \eta_1 \xi_1^2 - \right. \\ \left. 2 e^{2\gamma \alpha_1} \gamma \hbar T_1^2 \mathbf{x}_1 \eta_1 \xi_1^2 - 3 e^{2\gamma \alpha_1} \gamma \eta_1^2 \xi_1^2 + 4 e^{2\gamma \alpha_1} \gamma T_1 \eta_1^2 \xi_1^2 - e^{2\gamma \alpha_1} \gamma T_1^2 \eta_1^2 \xi_1^2) \right] \in +O[\epsilon]^2$$

tS is an anti-homomorphism for tm.

```
In[*]:= HL[(tS1 tS2) ~ B1,2 ~ tm1,2→1 ≡ tm2,1→1 ~ B1 ~ tS1]

Out[*]:= True
```

tR

```
R[QU, kk_]i_,j_ := R[QU, kk]i,j→k = Module[{OE},
  OE = Simplify /@ @QU, kk@Ri,j;
  E[- $\frac{\hbar \mathbf{a}_j \mathbf{t}_i}{\gamma}$ ,  $\hbar \mathbf{x}_j \mathbf{y}_i$ , Last@OE]];
  tRi,j := R[$U, $k]i,j;
  tRi,j := tRi,j = tRi,j ~ Bj ~ tSj;
```

In[*]:= $\{\mathbf{tR}_{1,2}, \overline{\mathbf{tR}}_{1,2}\}$

Out[*]:= $\left\{ \mathbb{E} \left[-\frac{\hbar a_2 t_1}{\gamma}, \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2 \right) \epsilon + \mathcal{O}[\epsilon^2] \right], \mathbb{E} \left[\frac{\hbar a_2 t_1}{\gamma}, -\frac{\hbar x_2 y_1}{T_1}, \right. \right.$
 $\left. 1 - \frac{1}{4 (\gamma T_1^2)} \left(\hbar (4 a_1 T_1 (a_2 T_1 + \gamma \hbar x_2 y_1) + \gamma \hbar x_2 y_1 (4 a_2 T_1 + 3 \gamma \hbar x_2 y_1)) \right) \epsilon + \mathcal{O}[\epsilon^2] \right\}$

Testing R2

In[*]:= $\mathbf{HL} \left[\left(\overline{\mathbf{tR}}_{1,2} \mathbf{tR}_{3,4} \right) \sim \mathbf{B}_{1,2,3,4} \sim \left(\mathbf{tm}_{1,3 \rightarrow 1} \mathbf{tm}_{2,4 \rightarrow 2} \right) = \mathbf{t1}_1 \mathbf{t1}_2 \right]$

Out[*]:= **True**

Testing R3

In[*]:= $\mathbf{HL} \left[\left(\mathbf{tR}_{2,3} \mathbf{tR}_{1,4} \mathbf{tR}_{5,6} \right) \sim \mathbf{B}_{\text{Range@6}} \sim \left(\mathbf{tm}_{1,5 \rightarrow 1} \mathbf{tm}_{2,6 \rightarrow 2} \mathbf{tm}_{3,4 \rightarrow 3} \right) \equiv \right.$
 $\left. \left(\mathbf{tR}_{1,2} \mathbf{tR}_{5,3} \mathbf{tR}_{6,4} \right) \sim \mathbf{B}_{\text{Range@6}} \sim \left(\mathbf{tm}_{1,5 \rightarrow 1} \mathbf{tm}_{2,6 \rightarrow 2} \mathbf{tm}_{3,4 \rightarrow 3} \right) \right]$

Out[*]:= **True**

In[*]:= $\mathbf{m}_{1,3,5 \rightarrow j} @ \mathbf{m}_{2,4,6 \rightarrow k} @ \mathbf{Times} \left[\left(* \text{Warning: wrong unless } \$p \geq \$k + 1! * \right) \right.$
 $\mathbf{Prepend} \left[\{y_2\}_2 \right] @ \mathbf{Exp}_{\text{QU}, \$k} \left[\eta, \Delta_{1 \rightarrow 1, 2} [\text{QU}[y_1]] \right] /. \text{QU} \rightarrow \mathbf{Times},$
 $\mathbf{Prepend} \left[\{a_4\}_4 \right] @ \mathbf{Exp}_{\text{QU}, \$k} \left[\alpha, \Delta_{3 \rightarrow 3, 4} [\text{QU}[a_3]] \right] /. \text{QU} \rightarrow \mathbf{Times},$
 $\mathbf{Prepend} \left[\{x_6\}_6 \right] @ \mathbf{Exp}_{\text{QU}, \$k} \left[\xi, \Delta_{5 \rightarrow 5, 6} [\text{QU}[x_5]] \right] /. \text{QU} \rightarrow \mathbf{Times}$
 $\left. \right] /. \{\eta \rightarrow \eta_i, \alpha \rightarrow \alpha_i, \xi \rightarrow \xi_i\}$

Out[*]:= $\mathbb{E}_{\text{QU}} \left[\{y_j, a_j, x_j\}_j, \{y_k, a_k, x_k\}_k, a_j \alpha_i + a_k \alpha_i + y_j \eta_i + T_j y_k \eta_i + x_j \xi_i + x_k \xi_i, \right.$
 $\left. 1 + \frac{1}{2} \left(-2 \hbar a_j T_j y_k \eta_i + \gamma \hbar T_j y_j y_k \eta_i^2 - 2 \hbar a_j x_k \xi_i + \gamma \hbar x_j x_k \xi_i^2 \right) \epsilon + \mathcal{O}[\epsilon^2] \right]$

tDelta

```
In[*]:=  $\Delta[U_, kk_] := \Delta[U, kk] = \text{Module} \left[ \{ \text{OE} \}, \right.$   

 $\text{OE} = \text{Block} \left[ \{ \$k = kk, \$p = kk + 1 \}, \mathbf{m}_{1,3,5 \rightarrow 1} @ \mathbf{m}_{2,4,6 \rightarrow 2} @ \mathbf{Times} \left[ \right.$   

 $\mathbf{Prepend} \left[ \{y_2\}_2 \right] @ \mathbf{Exp}_{U, \$k} \left[ \eta, \Delta_{1 \rightarrow 1, 2} [U[y_1]] \right] /. U \rightarrow \mathbf{Times},$   

 $\mathbf{Prepend} \left[ \{a_4\}_4 \right] @ \mathbf{Exp}_{U, \$k} \left[ \alpha, \Delta_{3 \rightarrow 3, 4} [U[a_3]] \right] /. U \rightarrow \mathbf{Times},$   

 $\mathbf{Prepend} \left[ \{x_6\}_6 \right] @ \mathbf{Exp}_{U, \$k} \left[ \xi, \Delta_{5 \rightarrow 5, 6} [U[x_5]] \right] /. U \rightarrow \mathbf{Times}$   

 $\left. \right] /. \{\eta \rightarrow \eta_1, \alpha \rightarrow \alpha_1, \xi \rightarrow \xi_1\};$   

 $\mathbb{E} \left[ \tau_1 (t_1 + t_2) + \alpha_1 (a_1 + a_2), \text{OE}[[3]] \right] /. \alpha_1 \rightarrow \mathbf{0}, \text{OE}[[4]]];$   

 $\mathbf{t}\Delta_{i \rightarrow j, k} :=$   

 $\Delta[\$U, \$k] /. \left\{ (v : \tau | \eta | \alpha | \xi)_1 \rightarrow v_i, (v : t | T | y | a | x)_1 \rightarrow v_j, (v : t | T | y | a | x)_2 \rightarrow v_k \right\};$ 
```

In[*]:= $\mathbf{t}\Delta_{1 \rightarrow 1, 2}$

Out[*]:= $\mathbb{E} \left[(a_1 + a_2) \alpha_1 + (t_1 + t_2) \tau_1, y_1 \eta_1 + T_1 y_2 \eta_1 + x_1 \xi_1 + x_2 \xi_1, \right.$
 $\left. 1 + \frac{1}{2} \left(-2 \hbar a_1 T_1 y_2 \eta_1 + \gamma \hbar T_1 y_1 y_2 \eta_1^2 - 2 \hbar a_1 x_2 \xi_1 + \gamma \hbar x_1 x_2 \xi_1^2 \right) \epsilon + \mathcal{O}[\epsilon^2] \right]$

Testing co-associativity.

In[*]:= $\mathbf{HL} \left[\mathbf{t}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_2 \sim \mathbf{t}\Delta_{2 \rightarrow 2, 3} \equiv \mathbf{t}\Delta_{1 \rightarrow 1, 3} \sim \mathbf{B}_1 \sim \mathbf{t}\Delta_{1 \rightarrow 1, 2} \right]$

Out[*]:= **True**

Alternative Algorithms

AltLogos

In[]:=

```

λalt,k[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
  eq = ρ@eξxcu.ρ@eηycu == ρ@edycu.ρ@ec(t1cu - 2εacu).ρ@ebxcu;
  {so} = Solve[Thread[Flatten/@eq], {d, b, c}] /. C@1 → 0;
  Series[e-ηy-ξx+ηξt+c t+d y-2εca+bx /. so, {ε, 0, k}]]];

```

```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[Δcu,2[{ξ, η}, {x, y}]]]}
```

$$\left\{ 1 + \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right) \epsilon + \frac{1}{2} \left(\left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right)^2 + 2 \left(-a \gamma \eta^2 \xi^2 + y \gamma^2 \eta^3 \xi^2 + x \gamma^2 \eta^2 \xi^3 - \frac{1}{3} t \gamma^2 \eta^3 \xi^3 \right) \right) \epsilon^2 + O[\epsilon]^3, \text{True} \right\}$$
