

Pensieve header: A unified verification notebook for the \$sl_2\$-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

Go;

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
In[ ]:= HL[ε_] := Style[ε, Background → Yellow];
```

DocileQ

DocileQ

```
In[ ]:= DQ[ε_] := (Exponent[Normal@ε /.
  {a → a / ε, a_i_ → a_i / ε, (u : x | y) ⇒ ε-1/2 u, (u : x | y)_i_ ⇒ ε-1/2 u_i}, ε, Min] ≥ 0);
```

```
In[ ]:= DQ /@ {ε2 x y a2, ε2 x2 y3}
```

```
Out[ ]:= {True, False}
```

Initialization / Utilities

It is verification-risky to work with low \$E\$!

TD

```
In[ ]:= $p = 2; $k = 1; $E := {$k, $p};
$trim := {ħp / ; p > $p → 0, εk / ; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → eħ t_i, T → eħ t}; q_ħ = eγ ε ħ;
SS[ε_, op_] := Collect[
  Normal@Series[If[$p > 0, ε, ε /. TRule], {ħ, 0, $p}],
  ħ, op];
SS[ε_] := SS[ε, Together];
SST[ε_, op___] := SS[ε /. TRule, op];
Simp[ε_, op_] := Collect[ε, _CU | _QU, op];
Simp[ε_] := Simp[ε, SS[#, Expand] &];
SimpT[ε_] := Collect[ε, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

```
In[ ]:= DPα→Dx, β→Dy[P_][λ_] :=
  Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) ⇒ c ∂{x,m}, {y,n} λ]
```

$$\text{HL}[\text{DP}_{x \rightarrow D_\epsilon, y \rightarrow D_\eta}[x^2 y^3][e^{\delta \xi \eta}] == 6 e^{\delta \eta \xi} \delta^3 \xi + 6 e^{\delta \eta \xi} \delta^4 \eta \xi^2 + e^{\delta \eta \xi} \delta^5 \eta^2 \xi^3]$$

True

CF

```
In[ ]:= CF[ $\mathcal{E}$ _] := ExpandDenominator@
ExpandNumerator@Together[Expand[ $\mathcal{E}$ ] /. ex- ey- => ex+y /. ex- => eCF[x]];
```

SeriesData

```
In[ ]:= Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

SP

```
In[ ]:= SP[{}][P_] := P; SP[ $\{\xi \rightarrow x, ps\_ \}$ ][P_] := Expand[P // SP[ps]] /. f_ .  $\xi^{d_}$  =>  $\partial_{\{x,d\}}$  f
```

$$\text{SP}_{\{\xi \rightarrow x\}}[(\xi^2 + \xi + 3)(x^5 e^x + 7x) + 99a]$$

$$7 + 99a + 21x + 20e^x x^3 + 15e^x x^4 + 5e^x x^5$$

$$\text{SP}_{\{\xi \rightarrow x, \eta \rightarrow y\}}[(\xi^2 + \xi + 3 + 2\xi\eta)(x^5 e^x + 7x) + 99a + e^{\delta xy} \xi \eta]$$

$$7 + 99a + 21x + 20e^x x^3 + 15e^x x^4 + 5e^x x^5 + e^{xy\delta} \delta + e^{xy\delta} xy \delta^2$$

DeclareAlgebra

QLImplementation

```
In[ ]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

In[*]:=

```

DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
  gs = Generators /. {opts},
  cs = Centrals /. {opts} /. Centrals -> {}},
  (#u = U@#) & /@gs;
  gp = Alternatives @@ gs; gp = gp | gp; (* gens *)
  sr = Flatten@Table[{g -> ++k, gi -> {i, k}}, {g, gs}]; (* sorting -> *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[_] := Collect[_U, Expand] /. $trim;
  Ui[_] := _ /. {t : cp -> ti, u_U -> (#i &) /@u};
  Ui[NCM[]] = pow[_] = U@{ } = 1_U = U[];
  B[U@(x_)i_, U@(y_)i_] := Ui@B[U@x, U@y];
  B[U@(x_)i_, U@(y_)j_] /; i != j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy];
  U@{c_. * (L : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[L, {n}] ** U@{r}];
  U@{c_. * L : gp, r___} := CE[c U[L] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
  U@{L_, r___} := U@{Expand[L], r};
  U[_NonCommutativeMultiply] := U /@ _;
  OU[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, L_List -> L_null, {1}];
    vs = Join@@(First /@ sp);
    us = Join@@(sp /. L_s_ -> (L /. x_i_ -> x_s));
    CE[Total[
      CoefficientRules[poly, vs] /. (p_ -> c_) -> c U@(us^p)
    ] /; x_null -> x];
  pow[_] = pow[_] := pow[_ - 1] ** _;
  SU[_] := CE@Total[
    CoefficientRules[_] /.
      (p_ -> c_) -> c NCM@@MapThread[pow, {Last /@ {ss}, p}]];
  m_j -> k [c_. * u_U] := CE[(c /. (t : cp)_j -> tk) DeleteCases[u, _j|k]] **
    U@@Cases[u, w_j -> wk] ** U@@Cases[u, _k];
  U /: c_. * u_U * v_U := CE[c u ** v];
  S_i [c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i]] **
    U_i[NCM@@Reverse@Cases[u, x_i -> S@U@x]]] ]

```

DeclareMorphism

QLImplementation

```
In[*]:= DeclareMorphism[m_, U_ -> V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
    {(g_ -> img_) -> (m[U[g]] = img), (g_ -> img_) -> (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM@@(m/@U/@{vs});
  m[E_] := Simp[E /. oncs /. u_U -> m[u]] /. $trim;
```

Meta-Operations

QLImplementation

```
In[*]:= m_{j->k} = Identity;
m_{j->k}[E_Plus] := Simp[m_{j->k}/@E];
m_{i_s, i, j->k}[E_] := m_{j->k}@m_{i_s, i->j}@E;
S_i[E_Plus] := Simp[S_i/@E];
```

Implementing $CU = \mathcal{U}(\mathfrak{sl}_2^{\mathbb{C}})$

CU

```
In[*]:= DeclareAlgebra[CU, Generators -> {y, a, x}, CentralS -> {t}];
B[a_CU, y_CU] = -y y_CU; B[x_CU, a_CU] = -x x_CU;
B[x_CU, y_CU] = 2 e a_CU - t 1_CU;
(S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU);
S_i[CU, CentralS] = {t_i -> -t_i};
Delta@y_CU = CU@y_1 + T_1 CU@y_2; Delta@a_CU = CU@a_1 + CU@a_2; Delta@x_CU = CU@x_1 + CU@x_2;
```

Verifying associativity on triples of generators:

```
With[{bas = CU/@{y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp // HL
}] // Timing
{1.32813,
 {(28 t^2 \gamma^4 + 116 t \gamma^5 e) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
In[ ]:= With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
    {z1, bas}, {z2, bas} ] ]
```

```
Out[ ]:= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
In[ ]:= With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
Out[ ]:= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

$$t_1 t_3 CU[y_1, a_1, x_2] + t_1 t_1 CU[y_1, a_2, x_2] // m_{1 \rightarrow 3}$$

$$CU[a_2, x_2, y_3] t_3^2 + CU[x_2, y_3, a_3] t_3^2$$

Verifying meta-associativity:

```
Module[{z, u},
  Table[u = CU[z[[1]]1, z[[2]]2, z[[3]]3]; z → HL[m1,3→3@m2,3→3@u == m2,3→3@m1,2→2@u],
    {z, Tuples[{y, a, x}, 3]}]]
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

Implementing QU = $\mathcal{U}_q(\mathfrak{sl}_2^{\hbar\epsilon})$

Aside

```
Series[(1 - T e^{-2\epsilon a \hbar}) / \hbar, {a, 0, 3}]
```

Aside

$$\frac{1 - T}{\hbar} + 2 T \epsilon a - 2 (T \epsilon^2 \hbar) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

```
In[ ]:= HL /@ DQ /@ Series[{{(1 - T e^{-2\epsilon a \hbar}) / \hbar, e^{\hbar \epsilon a}}, {\epsilon, 0, 5}}
```

```
Out[ ]:= {True, True}
```

QU

```
In[ ]:= DeclareAlgebra[QU, Generators → {y, a, x}, CentralS → {t, T}];
B[aQU, yQU] = -\gamma yQU; B[xQU, aQU] = -\gamma QU@x;
B[xQU, yQU] := SS[qh - 1] QU@{y, x} + OQU[{a}, SS[(1 - T e^{-2\epsilon a \hbar}) / \hbar]];
(S@yQU := OQU[{a, y}, SS[-T^{-1} e^{\hbar \epsilon a} y]]; S@aQU = -aQU; S@xQU := OQU[{a, x}, SS[-e^{\hbar \epsilon a} x]];
S_i_[QU, CentralS] = {t_i → -t_i, T_i → T_i^{-1}};
\Delta@yQU := OQU[{y1, a1}_1, {y2}_2, SS[y1 + T1 e^{-\hbar \epsilon a1} y2]];
\Delta@aQU = QU@a1 + QU@a2; \Delta@xQU := OQU[{a1, x1}_1, {x2}_2, SS[x1 + e^{-\hbar \epsilon a1} x2]];
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}]]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
 {QU[y], QU[x]} →  $\frac{(-1 + T) QU[]}{\hbar} - 2 T \in QU[a] - \gamma \in \hbar QU[y, x]$ },
 {{QU[a], QU[y]} → -γ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} →  $\frac{(1 - T) QU[]}{\hbar} + 2 T \in QU[a] + \gamma \in \hbar QU[y, x]$ ,
 {QU[x], QU[a]} → -γ QU[x], {QU[x], QU[x]} → 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
 {z1, bas}, {z2, bas}, {z3, bas}]]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ \$p=5, \$k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
 (rhs = (z1 ** z2) ** z3 // Simp) // Short,
 HL[z1 ** (z2 ** z3) - rhs // Simp]
}] // Timing
{3.78125, { $\left(\frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \in - 280 T \ll 1 \gg \in + 198 T^2 \gamma^5 \in}{\hbar}\right) QU[y, y, y, x, x] +$ 
 <<18>> + (1 + 8 γ ∈ ħ) QU[y, <<11>>, x], 0}}
```

Verifying that S is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
 {z1, bas}, {z2, bas}]]
{{{QU[y1], QU[y1]} → 0, {QU[y1], QU[a1]} → 0, {QU[y1], QU[x1]} → 0},
 {{QU[a1], QU[y1]} → 0, {QU[a1], QU[a1]} → 0, {QU[a1], QU[x1]} → 0},
 {{QU[x1], QU[y1]} → 0, {QU[x1], QU[a1]} → 0, {QU[x1], QU[x1]} → 0}}
```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs @ \$p=5, \$k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
 Short[lhs = z1 ** (z2 ** z3)],
 Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
 Expand[Limit[rhs /. TRule[QU → CU], ħ → 0] - lhs] // HL
}] // Timing
{10.125, {28 t^2 γ^4 CU[y, y, y, x, x] +
 116 t γ^5 ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
 2  $\left(\frac{\gamma^4}{\hbar^2} - \frac{2 T \gamma^4}{\hbar^2} + \frac{T^2 \gamma^4}{\hbar^2} + \frac{\gamma^5 \in}{\hbar} - \frac{2 T \gamma^5 \in}{\hbar} + \frac{T^2 \gamma^5 \in}{\hbar}\right) QU[y, y, y, x, x] +$ 
 <<209>> + (1 + 8 γ ∈ ħ) QU[y, y, y, <<7>>, x, x, x], 0}}
```

Implementing θ

theta

In[]:=

```
DeclareMorphism[Cθ, CU → CU, {y → -xCU, a → -aCU, x → -yCU}, {t → -t, T → T-1}]];
DeclareMorphism[Qθ, QU → QU, {y ↦ QQU[{a, x}, SS[-T-1/2 eħε a x]],
a → -aQU, x ↦ QQU[{a, y}, SS[-T-1/2 eħε a y]]}], {t → -t, T → T-1}]];
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}]]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}]]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}]]
{QU[y] → - $\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon \hbar QU[a, x]}{\sqrt{T}}$  → QU[y], QU[a] → -QU[a] → QU[a],
QU[x] →  $\left(-\frac{1}{\sqrt{T}} + \frac{\gamma \epsilon \hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon \hbar QU[y, a]}{\sqrt{T}}$  → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
{{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
{{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

In[]:=

$$AD\$\mathbf{f} = \gamma \left(\left(\text{Cosh} \left[\hbar \left(a \epsilon + \frac{\gamma \epsilon}{2} - \frac{t}{2} \right) \right] - \text{Cosh} \left[\hbar \sqrt{\left(\frac{t - \gamma \epsilon}{2} \right)^2 + \epsilon \omega} \right] \right) / \right. \\ \left. \left(\hbar e^{\hbar((a+\gamma)\epsilon - t/2)} \text{Sinh} \left[\frac{\gamma \epsilon \hbar}{2} \right] (a^2 \epsilon + a \gamma \epsilon - a t - \omega) \right) \right);$$

Docility of AD\$f:

```
In[*]:= HL@DQ@Block[{$p = 4}, Collect[SS@AD$f /.  $\omega \rightarrow a_1, \epsilon$ ]]
```

```
Out[*]:= True
```

Scaling behaviour of AD\$f:

```
HL@Simplify[AD$f == ((AD$f /.  $\gamma \rightarrow 1$ ) /. { $\epsilon \rightarrow \gamma \epsilon, a \rightarrow \gamma^{-1} a, \omega \rightarrow \gamma^{-1} \omega$ })]
```

```
True
```

```
HL@FullSimplify[
  AD$f == ((AD$f /.  $\gamma \rightarrow 1$ ) /. { $\hbar \rightarrow \gamma^2 \hbar, \epsilon \rightarrow \epsilon / \gamma, a \rightarrow a / \gamma, t \rightarrow \gamma^{-2} t, \omega \rightarrow \gamma^{-3} \omega$ })]
```

```
True
```

ADeq

```
In[*]:= AD$ $\omega$  =  $\gamma$  CU[y, x] +  $\epsilon$  CU[a, a] - (t -  $\gamma \epsilon$ ) CU[a];
```

ADeq

```
In[*]:= DeclareMorphism[AD, QU  $\rightarrow$  CU,
  {a  $\rightarrow$  aCU, x  $\rightarrow$  CU@x, y  $\rightarrow$  SCU[SS[AD$f], a  $\rightarrow$  aCU,  $\omega \rightarrow$  AD$ $\omega$ ] ** yCU}]
```

Verifying that the asymmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2}  $\rightarrow$  HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{QU[y], QU[y]}  $\rightarrow$  0, {QU[y], QU[a]}  $\rightarrow$  0, {QU[y], QU[x]}  $\rightarrow$  0},
{{QU[a], QU[y]}  $\rightarrow$  0, {QU[a], QU[a]}  $\rightarrow$  0, {QU[a], QU[x]}  $\rightarrow$  0},
{{QU[x], QU[y]}  $\rightarrow$  0, {QU[x], QU[a]}  $\rightarrow$  0, {QU[x], QU[x]}  $\rightarrow$  0}
```

The Symmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

```
In[*]:= SD$g =  $\sqrt{\left(2 \gamma \left(\cosh\left[\frac{\hbar}{2} \sqrt{t^2 + \gamma^2 \epsilon^2 + 4 \epsilon \varpi}\right] - \cosh\left[\frac{t - \epsilon \gamma - 2 \epsilon a}{2 / \hbar}\right]\right)\right) /$ 
 $\left(\sinh\left[\frac{\gamma \epsilon \hbar}{2}\right] (t (2 a + \gamma) - 2 a (a + \gamma) \epsilon + 2 \varpi) \hbar\right)}$ ;
```

SDeq

```
In[*]:= SD$f = Simplify[e $\hbar (t/2 - \epsilon a)$  (SD$g /. {a  $\rightarrow$  -a, t  $\rightarrow$  -t})];
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:


```

In[ ]:= {SD$P = 
$$\frac{\text{Cosh}[\hbar \left( \frac{\epsilon-t}{2} + \epsilon a \right)] - \text{Cosh}[\hbar \sqrt{\frac{t^2+\epsilon^2}{4} + \epsilon w}]}{\hbar \text{Sinh}[\frac{-\epsilon \hbar}{2}] (w - \epsilon a^2 + (t-\epsilon) a + t/2)},$$

Simplify[SD$P == (SD$P /. {a -> -a-1, t -> -t})] // HL,
PowerExpand@Simplify[(SD$P /. {h -> \gamma^2 h, \epsilon -> \epsilon/\gamma, a -> a/\gamma, t -> \gamma^{-2} t, w -> \gamma^{-3} w}) ==
SD$g (SD$g /. {a -> -a-\gamma, t -> -t})] // HL,
SD$Q = Simplify[SD$P /. {a -> c-1/2}],
Simplify[SD$Q == (SD$Q /. {c -> -c, t -> -t})] // HL,
FullSimplify[SD$g == FullSimplify[

$$\sqrt{\text{SD$Q}} /. c -> a+1/2 /. \{h -> \gamma^2 h, \epsilon -> \epsilon/\gamma, a -> a/\gamma, t -> \gamma^{-2} t, w -> \gamma^{-3} w\}] // HL,$$

HL@DQ@Block[{$p = 4}, Collect[SS@SD$g /. w -> a1, \epsilon]],
HL@DQ@Block[{$p = 4}, Collect[SS@SD$f /. w -> a1, \epsilon]]
}

```

$$\text{Out[]:= } \left\{ - \left(\left(\left(\text{Cosh} \left[\left(a \epsilon + \frac{1}{2} (-t + \epsilon) \right) \hbar \right] - \text{Cosh} \left[\sqrt{\frac{1}{4} (t^2 + \epsilon^2) + \epsilon w} \hbar \right] \right) \text{Csch} \left[\frac{\epsilon \hbar}{2} \right] \right) / \right. \\
\left. \left(\left(\frac{t}{2} + a (t - \epsilon) - a^2 \epsilon + w \right) \hbar \right) \right), \text{True, True}, \\
- \left(\left(4 \left(\text{Cosh} \left[\frac{1}{2} (t - 2 c \epsilon) \hbar \right] - \text{Cosh} \left[\frac{1}{2} \sqrt{t^2 + \epsilon^2 + 4 \epsilon w} \hbar \right] \right) \text{Csch} \left[\frac{\epsilon \hbar}{2} \right] \right) / \right. \\
\left. \left((4 c t + \epsilon - 4 c^2 \epsilon + 4 w) \hbar \right) \right), \text{True, True, True, True} \}$$

SDeq

```

In[ ]:= SD$w = \gamma CU[y, x] + \epsilon CU[a, a] - (t - \gamma \epsilon) CU[a] - t \gamma 1_{CU} / 2;

```

SDeq

```

In[ ]:= DeclareMorphism[SD, QU -> CU, {a -> a_{CU},
x -> S_{CU}[SS[SD$f], a -> a_{CU}, w -> SD$w] ** X_{CU},
y -> S_{CU}[SS[SD$g], a -> a_{CU}, w -> SD$w] ** Y_{CU}}]

```

Verifying the θ -symmetry:

```

Table[HL@SimpT[C@SD[z]] == SD[Q@z]], {z, QU /@ {y, a, x}}]
{True, True, True}

```

Verifying that the symmetric dequantizer is a homomorphism:

```

With[{bas = QU /@ {y, a, x}},
Table[{z1, z2} -> HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}]]
{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> 0, {QU[y], QU[x]} -> 0,
{QU[a], QU[y]} -> 0, {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> 0,
{QU[x], QU[y]} -> 0, {QU[x], QU[a]} -> 0, {QU[x], QU[x]} -> 0}

```

The representation ρ

rho

```
In[ ]:=
  rho@yCU = rho@yQU =  $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$ ; rho@aCU = rho@aQU =  $\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ ;
  rho@xCU =  $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ ; rho@xQU =  $\begin{pmatrix} 0 & (1 - e^{-\gamma \epsilon \hbar}) / (\epsilon \hbar) \\ 0 & 0 \end{pmatrix}$ ;
  rho[e^epsilon] := MatrixExp[rho[epsilon]];
  rho[epsilon] := (epsilon /. TRule /. t -> gamma /. (U : CU | QU) [u___] => Fold[Dot,  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , rho /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[rho[z1 ** z2] == rho[z1].rho[z2]] /. e^k_ /; k > $k -> 0],
  {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[rho[e^xi U ex].rho[e^alpha U ea] == rho[e^alpha U ea].rho[e^e^{-gamma} xi U ex]], {U, {CU, QU}}]
{True, True}
```

\mathbb{E} and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

Multiplying OEs

```
In[ ]:=
  E_U[s1_, Q1_, P1_] E_U[s2_, Q2_, P2_] ^:= E_U[s1, s2, Q1 + Q2, P1 P2];
```

CdsO

```
In[ ]:=
  CU@E_CU[specs___, Q_, P_] := O_CU[specs, SS[e^Q P]];
  QU@E_QU[specs___, Q_, P_] := O_QU[specs, SS[e^Q P]];
```

Logos

```
In[ ]:=
  c_Integer^k_Integer := c + O[epsilon]^{k+1};
  Lambda_U,k[{alpha_, beta_}, {x_, x_}] := E_U[{x}, (alpha + beta) x, 1_k];
  Lambda_U,k[{xi_, alpha_}, {x, a}] := E_U[{a, x}, alpha a + e^{-gamma alpha} xi x, 1_k];
  Lambda_U,k[{alpha_, eta_}, {a, y}] := E_U[{y, a}, alpha a + e^{-gamma alpha} eta y, 1_k];
```

Table[

$$\{\Lambda_{U,1}[\{\alpha, \beta\}, \{u, u\}],$$

$$\text{lhs} = U @ \mathbb{E}_U[\{u_1, u_2\}, \hbar(\alpha u_1 + \beta u_2), 1], \text{HL}[\text{lhs} = U @ \Lambda_{U,1}[\hbar\{\alpha, \beta\}, \{u, u\}]],$$

$$\{U, \{\text{CU}, \text{QU}\}\}, \{u, \{y, a, x\}\}$$

$$\{\{\{\mathbb{E}_{\text{CU}}[\{y\}, y(\alpha + \beta), 1 + 0[\epsilon]^2],$$

$$\text{CU}[] + (\alpha \hbar + \beta \hbar) \text{CU}[y] + \left(\frac{\alpha^2 \hbar^2}{2} + \alpha \beta \hbar^2 + \frac{\beta^2 \hbar^2}{2}\right) \text{CU}[y, y], \text{True}\},$$

$$\{\mathbb{E}_{\text{CU}}[\{a\}, a(\alpha + \beta), 1 + 0[\epsilon]^2], \text{CU}[] + (\alpha \hbar + \beta \hbar) \text{CU}[a] + \left(\frac{\alpha^2 \hbar^2}{2} + \alpha \beta \hbar^2 + \frac{\beta^2 \hbar^2}{2}\right) \text{CU}[a, a],$$

$$\text{True}\}, \{\mathbb{E}_{\text{CU}}[\{x\}, x(\alpha + \beta), 1 + 0[\epsilon]^2],$$

$$\text{CU}[] + (\alpha \hbar + \beta \hbar) \text{CU}[x] + \left(\frac{\alpha^2 \hbar^2}{2} + \alpha \beta \hbar^2 + \frac{\beta^2 \hbar^2}{2}\right) \text{CU}[x, x], \text{True}\}\},$$

$$\{\{\{\mathbb{E}_{\text{QU}}[\{y\}, y(\alpha + \beta), 1 + 0[\epsilon]^2], \text{QU}[] + (\alpha \hbar + \beta \hbar) \text{QU}[y] + \left(\frac{\alpha^2 \hbar^2}{2} + \alpha \beta \hbar^2 + \frac{\beta^2 \hbar^2}{2}\right) \text{QU}[y, y],$$

$$\text{True}\}, \{\mathbb{E}_{\text{QU}}[\{a\}, a(\alpha + \beta), 1 + 0[\epsilon]^2], \text{QU}[] + (\alpha \hbar + \beta \hbar) \text{QU}[a] +$$

$$\left(\frac{\alpha^2 \hbar^2}{2} + \alpha \beta \hbar^2 + \frac{\beta^2 \hbar^2}{2}\right) \text{QU}[a, a], \text{True}\}, \{\mathbb{E}_{\text{QU}}[\{x\}, x(\alpha + \beta), 1 + 0[\epsilon]^2],$$

$$\text{QU}[] + (\alpha \hbar + \beta \hbar) \text{QU}[x] + \left(\frac{\alpha^2 \hbar^2}{2} + \alpha \beta \hbar^2 + \frac{\beta^2 \hbar^2}{2}\right) \text{QU}[x, x], \text{True}\}\}\}$$

$$\{\Lambda_{\#,1}[\{\xi, \alpha\}, \{x, a\}], \text{lhs} = \# @ \mathbb{E}_{\#}[\{x, a\}, \hbar(\xi x + \alpha a), 1],$$

$$\text{HL}[\text{lhs} = \# @ \Lambda_{\#,1}[\hbar\{\xi, \alpha\}, \{x, a\}]] \& /@ \{\text{CU}, \text{QU}\}$$

$$\{\{\{\mathbb{E}_{\text{CU}}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x \xi, 1 + 0[\epsilon]^2],$$

$$\text{CU}[] + \alpha \hbar \text{CU}[a] + (\xi \hbar - \alpha \gamma \xi \hbar^2) \text{CU}[x] + \frac{1}{2} \alpha^2 \hbar^2 \text{CU}[a, a] + \alpha \xi \hbar^2 \text{CU}[a, x] + \frac{1}{2} \xi^2 \hbar^2 \text{CU}[x, x],$$

$$\text{True}\}, \{\mathbb{E}_{\text{QU}}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x \xi, 1 + 0[\epsilon]^2], \text{QU}[] + \alpha \hbar \text{QU}[a] +$$

$$(\xi \hbar - \alpha \gamma \xi \hbar^2) \text{QU}[x] + \frac{1}{2} \alpha^2 \hbar^2 \text{QU}[a, a] + \alpha \xi \hbar^2 \text{QU}[a, x] + \frac{1}{2} \xi^2 \hbar^2 \text{QU}[x, x], \text{True}\}\}$$

$$\{\Lambda_{\#,2}[\{\alpha, \eta\}, \{a, y\}], \text{lhs} = \# @ \mathbb{E}_{\#}[\{a, y\}, \hbar(\eta y + \alpha a), 1],$$

$$\text{HL}[\text{lhs} = \# @ \Lambda_{\#,2}[\hbar\{\alpha, \eta\}, \{a, y\}]] \& /@ \{\text{CU}, \text{QU}\}$$

$$\{\{\{\mathbb{E}_{\text{CU}}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y \eta, 1 + 0[\epsilon]^3],$$

$$\text{CU}[] + \alpha \hbar \text{CU}[a] + (\eta \hbar - \alpha \gamma \eta \hbar^2) \text{CU}[y] + \frac{1}{2} \alpha^2 \hbar^2 \text{CU}[a, a] + \alpha \eta \hbar^2 \text{CU}[y, a] + \frac{1}{2} \eta^2 \hbar^2 \text{CU}[y, y],$$

$$\text{True}\}, \{\mathbb{E}_{\text{QU}}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y \eta, 1 + 0[\epsilon]^3], \text{QU}[] + \alpha \hbar \text{QU}[a] +$$

$$(\eta \hbar - \alpha \gamma \eta \hbar^2) \text{QU}[y] + \frac{1}{2} \alpha^2 \hbar^2 \text{QU}[a, a] + \alpha \eta \hbar^2 \text{QU}[y, a] + \frac{1}{2} \eta^2 \hbar^2 \text{QU}[y, y], \text{True}\}\}$$

Goal. In either U , compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_{\eta} F = \partial_{\eta}(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```

If[$k > 0, With[{U = CU},
Module[{G, F, fs, bs, e, b, es, sol},
G = Echo@Simp[Table[$xi^k/k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
fs = Echo@Flatten@Table[f_{1,i,j,k}[\eta], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
F = Echo[fs.(bs = fs /. f_{L_,i_,j_,k_}[\eta] => e^L U@{y^i, a^j, x^k})];
es = Flatten[
Table[Coefficient[e, b] == 0, {e, {F - 1_U /. \eta -> 0, F ** G - y_U ** F - \partial_\eta F}}, {b, bs}]]];
sol = Echo@First[F /. DSolve[es, fs, \eta]];
Echo[sol /. {e -> 1, U -> Times}];
Collect[sol /. {e -> 1, U -> Times}, e, Simplify]
]]]
" -t \xi CU[] + 2 e \xi CU[a] - \gamma e \xi^2 CU[x] + CU[y]
" {f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta],
f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]}
" CU[] f_{0,0,0,0}[\eta] + e CU[] f_{1,0,0,0}[\eta] + e CU[x] f_{1,0,0,1}[\eta] + e CU[a] f_{1,0,1,0}[\eta] + e CU[a, x] f_{1,0,1,1}[\eta] +
e CU[y] f_{1,1,0,0}[\eta] + e CU[y, x] f_{1,1,0,1}[\eta] + e CU[y, a] f_{1,1,1,0}[\eta] + e CU[y, a, x] f_{1,1,1,1}[\eta]
» e^{-t\eta\xi} CU[] + \frac{1}{2} e^{-t\eta\xi} t \gamma e \eta^2 \xi^2 CU[] + 2 e^{-t\eta\xi} e \eta \xi CU[a] - e^{-t\eta\xi} \gamma e \eta \xi^2 CU[x] - e^{-t\eta\xi} \gamma e \eta^2 \xi CU[y]
» 1 + 2 a e \eta \xi - y \gamma e \eta^2 \xi - x \gamma e \eta \xi^2 + \frac{1}{2} t \gamma e \eta^2 \xi^2
1 + \frac{1}{2} e \eta \xi (4 a + \gamma (-2 y \eta - 2 x \xi + t \eta \xi))

```

Logos

```

In[ ]:= \Lambda_{U,kk}[\{\xi 1, \eta 1\}, {x, y}] := \Lambda_{U,kk}[\{\xi 1, \eta 1\}, {x, y}] =
Block[{$k = kk, $p = kp}, Module[{\xi, \eta, G, F, fs, f, bs, e, b, es},
G = Simp[Table[$xi^k/k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
fs = Flatten@Table[f_{1,i,j,k}[\eta], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
F = fs.(bs = fs /. f_{L_,i_,j_,k_}[\eta] => e^L U@{y^i, a^j, x^k});
es = Flatten[
Table[Coefficient[e, b] == 0, {e, {F - 1_U /. \eta -> 0, F ** G - y_U ** F - \partial_\eta F}}, {b, bs}]]];
F = F /. DSolve[es, fs, \eta][[1]];
\mathcal{U}_U[{y, a, x},
\xi x + \eta y + (U /. {CU -> -t \eta \xi, QU -> \eta \xi (1 - T) / \hbar}),
F + 0_{$k} /. {e -> 1, U -> Times}
] /. {\xi -> \xi 1, \eta -> \eta 1}];

```

```

In[ ]:= Timing@\Lambda_{QU,3}[\{\xi, \eta\}, {x, y}]

```

$$\{0., \mathcal{U}_U[\{y, a, x\}, y \eta + x \xi + \frac{(1-T) \eta \xi}{\hbar}, 1 + \frac{1}{4 \hbar} \eta \xi (\gamma \eta \xi - 4 T \gamma \eta \xi + 3 T^2 \gamma \eta \xi + 8 a T \hbar + 2 y \gamma \eta \hbar - 6 T y \gamma \eta \hbar + 2 x \gamma \xi \hbar - 6 T x \gamma \xi \hbar + 4 x y \gamma \hbar^2) e + (-a T y \gamma \eta^2 \xi (-\eta \xi + 3 T \eta \xi - 3 \hbar) - a T x \gamma \eta \xi^2 (-\eta \xi + 3 T \eta \xi - 3 \hbar) + 2 a^2 T \eta \xi (T \eta \xi - \hbar) + 2 a T x y \gamma \eta^2 \xi^2 \hbar - \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi (-\eta \xi + 3 T \eta \xi - \hbar) \hbar - \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 (-\eta \xi + 3 T \eta \xi - \hbar) \hbar +$$

$$\begin{aligned}
& \frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 + \frac{1}{24} y^2 \gamma^2 \eta^3 \xi (3 \eta \xi - 18 T \eta \xi + 27 T^2 \eta \xi + 4 \hbar - 28 T \hbar) + \frac{1}{24} x^2 \gamma^2 \eta \xi^3 \\
& (3 \eta \xi - 18 T \eta \xi + 27 T^2 \eta \xi + 4 \hbar - 28 T \hbar) + \frac{1}{2 \hbar} a T \gamma \eta^2 \xi^2 (\eta \xi - 4 T \eta \xi + 3 T^2 \eta \xi + 4 \hbar - 6 T \hbar) + \\
& \frac{1}{4} x y \gamma^2 \eta \xi (2 \eta^2 \xi^2 - 10 T \eta^2 \xi^2 + 12 T^2 \eta^2 \xi^2 + 5 \eta \xi \hbar - 21 T \eta \xi \hbar + 2 \hbar^2) - \frac{1}{24 \hbar} \\
& y \gamma^2 \eta^2 \xi (-3 \eta^2 \xi^2 + 21 T \eta^2 \xi^2 - 45 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 10 \eta \xi \hbar + 68 T \eta \xi \hbar - 82 T^2 \eta \xi \hbar - \\
& 6 \hbar^2 + 30 T \hbar^2) - \frac{1}{24 \hbar} x \gamma^2 \eta \xi^2 (-3 \eta^2 \xi^2 + 21 T \eta^2 \xi^2 - 45 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 10 \eta \xi \hbar + \\
& 68 T \eta \xi \hbar - 82 T^2 \eta \xi \hbar - 6 \hbar^2 + 30 T \hbar^2) + \frac{1}{288 \hbar^2} (-1 + T) \gamma^2 \eta^2 \xi^2 (-9 \eta^2 \xi^2 + 63 T \eta^2 \xi^2 - \\
& 135 T^2 \eta^2 \xi^2 + 81 T^3 \eta^2 \xi^2 - 40 \eta \xi \hbar + 272 T \eta \xi \hbar - 328 T^2 \eta \xi \hbar - 36 \hbar^2 + 180 T \hbar^2) \Big) \epsilon^2 + \\
& (-a T x y^2 \gamma^2 \eta^3 \xi^2 (-\eta \xi + 3 T \eta \xi - 4 \hbar) \hbar - a T x^2 y \gamma^2 \eta^2 \xi^3 (-\eta \xi + 3 T \eta \xi - 4 \hbar) \hbar + \\
& 2 a^2 T x y \gamma \eta^2 \xi^2 (T \eta \xi - \hbar) \hbar + a T x^2 y^2 \gamma^2 \eta^3 \xi^3 \hbar^2 - \frac{1}{4} x^2 y^3 \gamma^3 \eta^3 \xi^2 (-\eta \xi + 3 T \eta \xi - 2 \hbar) \hbar^2 - \\
& \frac{1}{4} x^3 y^2 \gamma^3 \eta^2 \xi^3 (-\eta \xi + 3 T \eta \xi - 2 \hbar) \hbar^2 + \frac{1}{6} x^3 y^3 \gamma^3 \eta^3 \xi^3 \hbar^3 + \\
& \frac{4}{3} a^3 T \eta \xi (T^2 \eta^2 \xi^2 - 3 T \eta \xi \hbar + \hbar^2) - a^2 T y \gamma \eta^2 \xi (-T \eta^2 \xi^2 + 3 T^2 \eta^2 \xi^2 + \eta \xi \hbar - 9 T \eta \xi \hbar + 3 \hbar^2) - \\
& a^2 T x \gamma \eta \xi^2 (-T \eta^2 \xi^2 + 3 T^2 \eta^2 \xi^2 + \eta \xi \hbar - 9 T \eta \xi \hbar + 3 \hbar^2) + \\
& \frac{1}{24} x y^3 \gamma^3 \eta^3 \xi \hbar (3 \eta^2 \xi^2 - 18 T \eta^2 \xi^2 + 27 T^2 \eta^2 \xi^2 + 10 \eta \xi \hbar - 46 T \eta \xi \hbar + 4 \hbar^2) + \\
& \frac{1}{24} x^3 y \gamma^3 \eta \xi^3 \hbar (3 \eta^2 \xi^2 - 18 T \eta^2 \xi^2 + 27 T^2 \eta^2 \xi^2 + 10 \eta \xi \hbar - 46 T \eta \xi \hbar + 4 \hbar^2) + \\
& \frac{1}{8} x^2 y^2 \gamma^3 \eta^2 \xi^2 \hbar (3 \eta^2 \xi^2 - 16 T \eta^2 \xi^2 + 21 T^2 \eta^2 \xi^2 + 14 \eta \xi \hbar - 54 T \eta \xi \hbar + 12 \hbar^2) + \\
& \frac{1}{2} a T x y \gamma^2 \eta^2 \xi^2 (2 \eta^2 \xi^2 - 10 T \eta^2 \xi^2 + 12 T^2 \eta^2 \xi^2 + 15 \eta \xi \hbar - 45 T \eta \xi \hbar + 23 \hbar^2) + \\
& \frac{1}{12} a T y^2 \gamma^2 \eta^3 \xi (3 \eta^2 \xi^2 - 18 T \eta^2 \xi^2 + 27 T^2 \eta^2 \xi^2 + 22 \eta \xi \hbar - 82 T \eta \xi \hbar + 28 \hbar^2) + \\
& \frac{1}{12} a T x^2 \gamma^2 \eta \xi^3 (3 \eta^2 \xi^2 - 18 T \eta^2 \xi^2 + 27 T^2 \eta^2 \xi^2 + 22 \eta \xi \hbar - 82 T \eta \xi \hbar + 28 \hbar^2) + \frac{1}{2 \hbar} \\
& a^2 T \gamma \eta^2 \xi^2 (T \eta^2 \xi^2 - 4 T^2 \eta^2 \xi^2 + 3 T^3 \eta^2 \xi^2 - \eta \xi \hbar + 12 T \eta \xi \hbar - 15 T^2 \eta \xi \hbar - 4 \hbar^2 + 12 T \hbar^2) - \\
& \frac{1}{48} y^3 \gamma^3 \eta^4 \xi (-\eta^2 \xi^2 + 9 T \eta^2 \xi^2 - 27 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - \\
& 4 \eta \xi \hbar + 40 T \eta \xi \hbar - 84 T^2 \eta \xi \hbar - 2 \hbar^2 + 30 T \hbar^2) - \frac{1}{48} x^3 \gamma^3 \eta \xi^4 \\
& (-\eta^2 \xi^2 + 9 T \eta^2 \xi^2 - 27 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 4 \eta \xi \hbar + 40 T \eta \xi \hbar - 84 T^2 \eta \xi \hbar - 2 \hbar^2 + 30 T \hbar^2) - \\
& \frac{1}{12 \hbar} a T y \gamma^2 \eta^2 \xi (-3 \eta^3 \xi^3 + 21 T \eta^3 \xi^3 - 45 T^2 \eta^3 \xi^3 + 27 T^3 \eta^3 \xi^3 - 31 \eta^2 \xi^2 \hbar + \\
& 158 T \eta^2 \xi^2 \hbar - 163 T^2 \eta^2 \xi^2 \hbar - 74 \eta \xi \hbar^2 + 194 T \eta \xi \hbar^2 - 30 \hbar^3) - \\
& \frac{1}{12 \hbar} a T x \gamma^2 \eta \xi^2 (-3 \eta^3 \xi^3 + 21 T \eta^3 \xi^3 - 45 T^2 \eta^3 \xi^3 + 27 T^3 \eta^3 \xi^3 - 31 \eta^2 \xi^2 \hbar + \\
& 158 T \eta^2 \xi^2 \hbar - 163 T^2 \eta^2 \xi^2 \hbar - 74 \eta \xi \hbar^2 + 194 T \eta \xi \hbar^2 - 30 \hbar^3) -
\end{aligned}$$

$$\begin{aligned}
 & \frac{1}{48} x y^2 \gamma^3 \eta^2 \xi \left(-9 \eta^3 \xi^3 + 69 T \eta^3 \xi^3 - 171 T^2 \eta^3 \xi^3 + 135 T^3 \eta^3 \xi^3 - 60 \eta^2 \xi^2 \hbar + \right. \\
 & \quad \left. 416 T \eta^2 \xi^2 \hbar - 644 T^2 \eta^2 \xi^2 \hbar - 92 \eta \xi \hbar^2 + 508 T \eta \xi \hbar^2 - 24 \hbar^3 \right) - \\
 & \frac{1}{48} x^2 y \gamma^3 \eta \xi^2 \left(-9 \eta^3 \xi^3 + 69 T \eta^3 \xi^3 - 171 T^2 \eta^3 \xi^3 + 135 T^3 \eta^3 \xi^3 - 60 \eta^2 \xi^2 \hbar + \right. \\
 & \quad \left. 416 T \eta^2 \xi^2 \hbar - 644 T^2 \eta^2 \xi^2 \hbar - 92 \eta \xi \hbar^2 + 508 T \eta \xi \hbar^2 - 24 \hbar^3 \right) + \\
 & \frac{1}{96 \hbar} y^2 \left(3 \gamma^3 \eta^6 \xi^4 - 30 T \gamma^3 \eta^6 \xi^4 + 108 T^2 \gamma^3 \eta^6 \xi^4 - 162 T^3 \gamma^3 \eta^6 \xi^4 + 81 T^4 \gamma^3 \eta^6 \xi^4 + \right. \\
 & \quad \left. 24 \gamma^3 \eta^5 \xi^3 \hbar - 240 T \gamma^3 \eta^5 \xi^3 \hbar + 696 T^2 \gamma^3 \eta^5 \xi^3 \hbar - 576 T^3 \gamma^3 \eta^5 \xi^3 \hbar + 46 \gamma^3 \eta^4 \xi^2 \hbar^2 - \right. \\
 & \quad \left. 472 T \gamma^3 \eta^4 \xi^2 \hbar^2 + 858 T^2 \gamma^3 \eta^4 \xi^2 \hbar^2 + 16 \gamma^3 \eta^3 \xi \hbar^3 - 192 T \gamma^3 \eta^3 \xi \hbar^3 \right) + \\
 & \frac{1}{144 \hbar^2} a \left(9 T \gamma^2 \eta^5 \xi^5 - 72 T^2 \gamma^2 \eta^5 \xi^5 + 198 T^3 \gamma^2 \eta^5 \xi^5 - 216 T^4 \gamma^2 \eta^5 \xi^5 + 81 T^5 \gamma^2 \eta^5 \xi^5 + \right. \\
 & \quad \left. 112 T \gamma^2 \eta^4 \xi^4 \hbar - 708 T^2 \gamma^2 \eta^4 \xi^4 \hbar + 1248 T^3 \gamma^2 \eta^4 \xi^4 \hbar - 652 T^4 \gamma^2 \eta^4 \xi^4 \hbar + 348 T \gamma^2 \eta^3 \xi^3 \hbar^2 - \right. \\
 & \quad \left. 1416 T^2 \gamma^2 \eta^3 \xi^3 \hbar^2 + 1164 T^3 \gamma^2 \eta^3 \xi^3 \hbar^2 + 216 T \gamma^2 \eta^2 \xi^2 \hbar^3 - 360 T^2 \gamma^2 \eta^2 \xi^2 \hbar^3 \right) + \\
 & \frac{1}{96 \hbar} x^2 \left(3 \gamma^3 \eta^4 \xi^6 - 30 T \gamma^3 \eta^4 \xi^6 + 108 T^2 \gamma^3 \eta^4 \xi^6 - 162 T^3 \gamma^3 \eta^4 \xi^6 + 81 T^4 \gamma^3 \eta^4 \xi^6 + \right. \\
 & \quad \left. 24 \gamma^3 \eta^3 \xi^5 \hbar - 240 T \gamma^3 \eta^3 \xi^5 \hbar + 696 T^2 \gamma^3 \eta^3 \xi^5 \hbar - 576 T^3 \gamma^3 \eta^3 \xi^5 \hbar + 46 \gamma^3 \eta^2 \xi^4 \hbar^2 - \right. \\
 & \quad \left. 472 T \gamma^3 \eta^2 \xi^4 \hbar^2 + 858 T^2 \gamma^3 \eta^2 \xi^4 \hbar^2 + 16 \gamma^3 \eta \xi^3 \hbar^3 - 192 T \gamma^3 \eta \xi^3 \hbar^3 \right) + \\
 & \frac{1}{288 \hbar} x y \left(27 \gamma^3 \eta^5 \xi^5 - 252 T \gamma^3 \eta^5 \xi^5 + 846 T^2 \gamma^3 \eta^5 \xi^5 - 1188 T^3 \gamma^3 \eta^5 \xi^5 + 567 T^4 \gamma^3 \eta^5 \xi^5 + \right. \\
 & \quad \left. 250 \gamma^3 \eta^4 \xi^4 \hbar - 2226 T \gamma^3 \eta^4 \xi^4 \hbar + 5814 T^2 \gamma^3 \eta^4 \xi^4 \hbar - 4414 T^3 \gamma^3 \eta^4 \xi^4 \hbar + 624 \gamma^3 \eta^3 \xi^3 \hbar^2 - \right. \\
 & \quad \left. 5112 T \gamma^3 \eta^3 \xi^3 \hbar^2 + 7944 T^2 \gamma^3 \eta^3 \xi^3 \hbar^2 + 432 \gamma^3 \eta^2 \xi^2 \hbar^3 - 2880 T \gamma^3 \eta^2 \xi^2 \hbar^3 + 48 \gamma^3 \eta \xi \hbar^4 \right) + \\
 & \frac{1}{576 \hbar^2} y \left(9 \gamma^3 \eta^6 \xi^5 - 99 T \gamma^3 \eta^6 \xi^5 + 414 T^2 \gamma^3 \eta^6 \xi^5 - 810 T^3 \gamma^3 \eta^6 \xi^5 + 729 T^4 \gamma^3 \eta^6 \xi^5 - 243 \right. \\
 & \quad \left. T^5 \gamma^3 \eta^6 \xi^5 + 100 \gamma^3 \eta^5 \xi^4 \hbar - 1080 T \gamma^3 \eta^5 \xi^4 \hbar + 3840 T^2 \gamma^3 \eta^5 \xi^4 \hbar - 5320 T^3 \gamma^3 \eta^5 \xi^4 \hbar + 2460 \right. \\
 & \quad \left. T^4 \gamma^3 \eta^5 \xi^4 \hbar + 312 \gamma^3 \eta^4 \xi^3 \hbar^2 - 3288 T \gamma^3 \eta^4 \xi^3 \hbar^2 + 8808 T^2 \gamma^3 \eta^4 \xi^3 \hbar^2 - 6408 T^3 \gamma^3 \eta^4 \xi^3 \hbar^2 + \right. \\
 & \quad \left. 288 \gamma^3 \eta^3 \xi^2 \hbar^3 - 2880 T \gamma^3 \eta^3 \xi^2 \hbar^3 + 4320 T^2 \gamma^3 \eta^3 \xi^2 \hbar^3 + 48 \gamma^3 \eta^2 \xi \hbar^4 - 432 T \gamma^3 \eta^2 \xi \hbar^4 \right) + \\
 & \frac{1}{576 \hbar^2} x \left(9 \gamma^3 \eta^5 \xi^6 - 99 T \gamma^3 \eta^5 \xi^6 + 414 T^2 \gamma^3 \eta^5 \xi^6 - 810 T^3 \gamma^3 \eta^5 \xi^6 + 729 T^4 \gamma^3 \eta^5 \xi^6 - 243 \right. \\
 & \quad \left. T^5 \gamma^3 \eta^5 \xi^6 + 100 \gamma^3 \eta^4 \xi^5 \hbar - 1080 T \gamma^3 \eta^4 \xi^5 \hbar + 3840 T^2 \gamma^3 \eta^4 \xi^5 \hbar - 5320 T^3 \gamma^3 \eta^4 \xi^5 \hbar + 2460 \right. \\
 & \quad \left. T^4 \gamma^3 \eta^4 \xi^5 \hbar + 312 \gamma^3 \eta^3 \xi^4 \hbar^2 - 3288 T \gamma^3 \eta^3 \xi^4 \hbar^2 + 8808 T^2 \gamma^3 \eta^3 \xi^4 \hbar^2 - 6408 T^3 \gamma^3 \eta^3 \xi^4 \hbar^2 + \right. \\
 & \quad \left. 288 \gamma^3 \eta^2 \xi^3 \hbar^3 - 2880 T \gamma^3 \eta^2 \xi^3 \hbar^3 + 4320 T^2 \gamma^3 \eta^2 \xi^3 \hbar^3 + 48 \gamma^3 \eta \xi^2 \hbar^4 - 432 T \gamma^3 \eta \xi^2 \hbar^4 \right) + \\
 & \frac{1}{1152 \hbar^3} \left(3 \gamma^3 \eta^6 \xi^6 - 36 T \gamma^3 \eta^6 \xi^6 + 171 T^2 \gamma^3 \eta^6 \xi^6 - 408 T^3 \gamma^3 \eta^6 \xi^6 + 513 T^4 \gamma^3 \eta^6 \xi^6 - \right. \\
 & \quad \left. 324 T^5 \gamma^3 \eta^6 \xi^6 + 81 T^6 \gamma^3 \eta^6 \xi^6 + 40 \gamma^3 \eta^5 \xi^5 \hbar - 472 T \gamma^3 \eta^5 \xi^5 \hbar + 1968 T^2 \gamma^3 \eta^5 \xi^5 \hbar - \right. \\
 & \quad \left. 3664 T^3 \gamma^3 \eta^5 \xi^5 \hbar + 3112 T^4 \gamma^3 \eta^5 \xi^5 \hbar - 984 T^5 \gamma^3 \eta^5 \xi^5 \hbar + 156 \gamma^3 \eta^4 \xi^4 \hbar^2 - \right. \\
 & \quad \left. 1800 T \gamma^3 \eta^4 \xi^4 \hbar^2 + 6048 T^2 \gamma^3 \eta^4 \xi^4 \hbar^2 - 7608 T^3 \gamma^3 \eta^4 \xi^4 \hbar^2 + 3204 T^4 \gamma^3 \eta^4 \xi^4 \hbar^2 + \right. \\
 & \quad \left. 192 \gamma^3 \eta^3 \xi^3 \hbar^3 - 2112 T \gamma^3 \eta^3 \xi^3 \hbar^3 + 4800 T^2 \gamma^3 \eta^3 \xi^3 \hbar^3 - 2880 T^3 \gamma^3 \eta^3 \xi^3 \hbar^3 + \right. \\
 & \quad \left. 48 \gamma^3 \eta^2 \xi^2 \hbar^4 - 480 T \gamma^3 \eta^2 \xi^2 \hbar^4 + 432 T^2 \gamma^3 \eta^2 \xi^2 \hbar^4 \right) \Big) \epsilon^3 + O[\epsilon^4] \Big\}
 \end{aligned}$$

$\{\Delta_{\text{CU},1}[\{\xi, \eta\}, \{x, y\}], \text{lhs} = \text{CU}@\text{CU}[\{x, y\}, \hbar(\xi x + \eta y), 1],$
 $\text{HL}[\text{lhs} = \text{CU}@\Delta_{\text{CU},1}[\hbar\{\xi, \eta\}, \{x, y\}]]\}$

$$\begin{aligned}
 & \{\text{CU}[\{y, a, x\}, y \eta + x \xi - t \eta \xi, 1 + \frac{1}{2} \eta \xi (4a - 2y \gamma \eta - 2x \gamma \xi + t \gamma \eta \xi)] \epsilon + O[\epsilon^2], \\
 & \quad (1 - t \eta \xi \hbar^2) \text{CU}[] + 2 \epsilon \eta \xi \hbar^2 \text{CU}[a] + \xi \hbar \text{CU}[x] + \eta \hbar \text{CU}[y] + \\
 & \quad \frac{1}{2} \xi^2 \hbar^2 \text{CU}[x, x] + \eta \xi \hbar^2 \text{CU}[y, x] + \frac{1}{2} \eta^2 \hbar^2 \text{CU}[y, y], \text{True}\}
 \end{aligned}$$

```
{ΔQU,1[{ξ, η}, {x, y}], lhs = QU@EQU[{x, y}, ħ (ξ x + η y), 1],
HL@SimpT[lhs == QU@ΔQU,1[ħ {ξ, η}, {x, y}]]}
```

$$\left\{ \mathbb{E}_{QU} \left[\{y, a, x\}, y \eta + x \xi + \frac{(1-T) \eta \xi}{\hbar}, 1 + \frac{1}{4 \hbar} \right. \right. \\ \left. \left. \eta \xi (\gamma \eta \xi - 4 T \gamma \eta \xi + 3 T^2 \gamma \eta \xi + 8 a T \hbar + 2 y \gamma \eta \hbar - 6 T y \gamma \eta \hbar + 2 x \gamma \xi \hbar - 6 T x \gamma \xi \hbar + 4 x y \gamma \hbar^2) \epsilon + \right. \right. \\ \left. \left. O[\epsilon]^2 \right], \left(1 + \eta \xi \hbar - T \eta \xi \hbar \right) QU[] + 2 T \epsilon \eta \xi \hbar^2 QU[a] + \xi \hbar QU[x] + \right. \\ \left. \eta \hbar QU[y] + \frac{1}{2} \xi^2 \hbar^2 QU[x, x] + \eta \xi \hbar^2 QU[y, x] + \frac{1}{2} \eta^2 \hbar^2 QU[y, y], \text{True} \right\}$$

```
{tt = Last[ΔCU,2[{ξ, η}, {x, y}], Log[tt],
Exponent[Normal@Log[tt] /. {ξ → ħ ξ, η → ħ η, x → ħ x, y → ħ y}, ħ]} // Expand
```

$$\left\{ 1 + \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right) \epsilon + \right. \\ \left(2 a^2 \eta^2 \xi^2 - a \gamma \eta^2 \xi^2 - 2 a y \gamma \eta^3 \xi^2 + y \gamma^2 \eta^3 \xi^2 + \frac{1}{2} y^2 \gamma^2 \eta^4 \xi^2 - 2 a x \gamma \eta^2 \xi^3 + x \gamma^2 \eta^2 \xi^3 + a t \gamma \eta^3 \xi^3 - \right. \\ \left. \frac{1}{3} t \gamma^2 \eta^3 \xi^3 + x y \gamma^2 \eta^3 \xi^3 - \frac{1}{2} t y \gamma^2 \eta^4 \xi^3 + \frac{1}{2} x^2 \gamma^2 \eta^2 \xi^4 - \frac{1}{2} t x \gamma^2 \eta^3 \xi^4 + \frac{1}{8} t^2 \gamma^2 \eta^4 \xi^4 \right) \epsilon^2 + O[\epsilon]^3, \\ \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right) \epsilon + \left(-a \gamma \eta^2 \xi^2 + y \gamma^2 \eta^3 \xi^2 + x \gamma^2 \eta^2 \xi^3 - \frac{1}{3} t \gamma^2 \eta^3 \xi^3 \right) \epsilon^2 + \\ \left. O[\epsilon]^3, 6 \right\}$$

```
{tt = Last[ΔQU,2[{ξ, η}, {x, y}], Log[tt],
Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\begin{aligned}
 & \left\{ 1 + \left(2 a T \eta \xi + \frac{1}{2} y \gamma \eta^2 \xi - \frac{3}{2} T y \gamma \eta^2 \xi + \right. \right. \\
 & \quad \left. \frac{1}{2} x \gamma \eta \xi^2 - \frac{3}{2} T x \gamma \eta \xi^2 + \frac{\gamma \eta^2 \xi^2}{4 \hbar} - \frac{T \gamma \eta^2 \xi^2}{\hbar} + \frac{3 T^2 \gamma \eta^2 \xi^2}{4 \hbar} + x y \gamma \eta \xi \hbar \right) \epsilon + \\
 & \left(2 a^2 T^2 \eta^2 \xi^2 + 2 a T \gamma \eta^2 \xi^2 - 3 a T^2 \gamma \eta^2 \xi^2 + \frac{1}{8} \gamma^2 \eta^2 \xi^2 - \frac{3}{4} T \gamma^2 \eta^2 \xi^2 + \frac{5}{8} T^2 \gamma^2 \eta^2 \xi^2 + \right. \\
 & \quad a T y \gamma \eta^3 \xi^2 - 3 a T^2 y \gamma \eta^3 \xi^2 + \frac{5}{12} y \gamma^2 \eta^3 \xi^2 - \frac{17}{6} T y \gamma^2 \eta^3 \xi^2 + \frac{41}{12} T^2 y \gamma^2 \eta^3 \xi^2 + \frac{1}{8} y^2 \gamma^2 \eta^4 \xi^2 - \\
 & \quad \frac{3}{4} T y^2 \gamma^2 \eta^4 \xi^2 + \frac{9}{8} T^2 y^2 \gamma^2 \eta^4 \xi^2 + a T x \gamma \eta^2 \xi^3 - 3 a T^2 x \gamma \eta^2 \xi^3 + \frac{5}{12} x \gamma^2 \eta^2 \xi^3 - \frac{17}{6} T x \gamma^2 \eta^2 \xi^3 + \\
 & \quad \frac{41}{12} T^2 x \gamma^2 \eta^2 \xi^3 + \frac{1}{2} x y \gamma^2 \eta^3 \xi^3 - \frac{5}{2} T x y \gamma^2 \eta^3 \xi^3 + 3 T^2 x y \gamma^2 \eta^3 \xi^3 + \frac{1}{8} x^2 \gamma^2 \eta^2 \xi^4 - \\
 & \quad \frac{3}{4} T x^2 \gamma^2 \eta^2 \xi^4 + \frac{9}{8} T^2 x^2 \gamma^2 \eta^2 \xi^4 + \frac{\gamma^2 \eta^4 \xi^4}{32 \hbar^2} - \frac{T \gamma^2 \eta^4 \xi^4}{4 \hbar^2} + \frac{11 T^2 \gamma^2 \eta^4 \xi^4}{16 \hbar^2} - \frac{3 T^3 \gamma^2 \eta^4 \xi^4}{4 \hbar^2} + \\
 & \quad \frac{9 T^4 \gamma^2 \eta^4 \xi^4}{32 \hbar^2} + \frac{a T \gamma \eta^3 \xi^3}{2 \hbar} - \frac{2 a T^2 \gamma \eta^3 \xi^3}{\hbar} + \frac{3 a T^3 \gamma \eta^3 \xi^3}{2 \hbar} + \frac{5 \gamma^2 \eta^3 \xi^3}{36 \hbar} - \frac{13 T \gamma^2 \eta^3 \xi^3}{12 \hbar} + \\
 & \quad \frac{25 T^2 \gamma^2 \eta^3 \xi^3}{12 \hbar} - \frac{41 T^3 \gamma^2 \eta^3 \xi^3}{36 \hbar} + \frac{y \gamma^2 \eta^4 \xi^3}{8 \hbar} - \frac{7 T y \gamma^2 \eta^4 \xi^3}{8 \hbar} + \frac{15 T^2 y \gamma^2 \eta^4 \xi^3}{8 \hbar} - \frac{9 T^3 y \gamma^2 \eta^4 \xi^3}{8 \hbar} + \\
 & \quad x \gamma^2 \eta^3 \xi^4 - \frac{7 T x \gamma^2 \eta^3 \xi^4}{8 \hbar} + \frac{15 T^2 x \gamma^2 \eta^3 \xi^4}{8 \hbar} - \frac{9 T^3 x \gamma^2 \eta^3 \xi^4}{8 \hbar} - 2 a^2 T \eta \xi \hbar + 3 a T y \gamma \eta^2 \xi \hbar + \\
 & \quad \frac{1}{4} y \gamma^2 \eta^2 \xi \hbar - \frac{5}{4} T y \gamma^2 \eta^2 \xi \hbar + \frac{1}{6} y^2 \gamma^2 \eta^3 \xi \hbar - \frac{7}{6} T y^2 \gamma^2 \eta^3 \xi \hbar + 3 a T x \gamma \eta \xi^2 \hbar + \frac{1}{4} x \gamma^2 \eta \xi^2 \hbar - \\
 & \quad \frac{5}{4} T x \gamma^2 \eta \xi^2 \hbar + 2 a T x y \gamma \eta^2 \xi^2 \hbar + \frac{5}{4} x y \gamma^2 \eta^2 \xi^2 \hbar - \frac{21}{4} T x y \gamma^2 \eta^2 \xi^2 \hbar + \frac{1}{2} x y^2 \gamma^2 \eta^3 \xi^2 \hbar - \\
 & \quad \frac{3}{2} T x y^2 \gamma^2 \eta^3 \xi^2 \hbar + \frac{1}{6} x^2 \gamma^2 \eta \xi^3 \hbar - \frac{7}{6} T x^2 \gamma^2 \eta \xi^3 \hbar + \frac{1}{2} x^2 y \gamma^2 \eta^2 \xi^3 \hbar - \frac{3}{2} T x^2 y \gamma^2 \eta^2 \xi^3 \hbar + \\
 & \quad \left. \frac{1}{2} x y \gamma^2 \eta \xi \hbar^2 + \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi \hbar^2 + \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 \hbar^2 + \frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 \right) \epsilon^2 + 0[\epsilon]^3, \\
 & \left(2 a T \eta \xi + \frac{1}{2} y \gamma \eta^2 \xi - \frac{3}{2} T y \gamma \eta^2 \xi + \frac{1}{2} x \gamma \eta \xi^2 - \frac{3}{2} T x \gamma \eta \xi^2 + \frac{\gamma \eta^2 \xi^2}{4 \hbar} - \right. \\
 & \quad \left. \frac{T \gamma \eta^2 \xi^2}{\hbar} + \frac{3 T^2 \gamma \eta^2 \xi^2}{4 \hbar} + x y \gamma \eta \xi \hbar \right) \epsilon + \\
 & \left(2 a T \gamma \eta^2 \xi^2 - 3 a T^2 \gamma \eta^2 \xi^2 + \frac{1}{8} \gamma^2 \eta^2 \xi^2 - \frac{3}{4} T \gamma^2 \eta^2 \xi^2 + \frac{5}{8} T^2 \gamma^2 \eta^2 \xi^2 + \frac{5}{12} y \gamma^2 \eta^3 \xi^2 - \right. \\
 & \quad \frac{17}{6} T y \gamma^2 \eta^3 \xi^2 + \frac{41}{12} T^2 y \gamma^2 \eta^3 \xi^2 + \frac{5}{12} x \gamma^2 \eta^2 \xi^3 - \frac{17}{6} T x \gamma^2 \eta^2 \xi^3 + \frac{41}{12} T^2 x \gamma^2 \eta^2 \xi^3 + \\
 & \quad \frac{5 \gamma^2 \eta^3 \xi^3}{36 \hbar} - \frac{13 T \gamma^2 \eta^3 \xi^3}{12 \hbar} + \frac{25 T^2 \gamma^2 \eta^3 \xi^3}{12 \hbar} - \frac{41 T^3 \gamma^2 \eta^3 \xi^3}{36 \hbar} - 2 a^2 T \eta \xi \hbar + 3 a T y \gamma \eta^2 \xi \hbar + \\
 & \quad \frac{1}{4} y \gamma^2 \eta^2 \xi \hbar - \frac{5}{4} T y \gamma^2 \eta^2 \xi \hbar + \frac{1}{6} y^2 \gamma^2 \eta^3 \xi \hbar - \frac{7}{6} T y^2 \gamma^2 \eta^3 \xi \hbar + 3 a T x \gamma \eta \xi^2 \hbar + \\
 & \quad \frac{1}{4} x \gamma^2 \eta \xi^2 \hbar - \frac{5}{4} T x \gamma^2 \eta \xi^2 \hbar + \frac{5}{4} x y \gamma^2 \eta^2 \xi^2 \hbar - \frac{21}{4} T x y \gamma^2 \eta^2 \xi^2 \hbar + \frac{1}{6} x^2 \gamma^2 \eta \xi^3 \hbar - \\
 & \quad \left. \frac{7}{6} T x^2 \gamma^2 \eta \xi^3 \hbar + \frac{1}{2} x y \gamma^2 \eta \xi \hbar^2 + \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi \hbar^2 + \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 \hbar^2 \right) \epsilon^2 + 0[\epsilon]^3, 6 \}
 \end{aligned}$$

Logos

```
In[*]:= Simp[CU_[specs___, Q_, P_] := CU[specs, CF[Q], CF[P]];
```

Logos

```
In[*]:= ΔU_,k_[{u1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{u, ω, yax, q, p, Q, d},
  {yax, q, p} = List@@ΔU_,k_[{u, ω}, {u, w}];
  CU[yax, Q = (u u + ω w + δ u w + d v ω) / (1 - d δ),
  Expand[(1 - d δ)^-1 e^-Q DP_{u→D_u, ω→D_w}[p][e^Q] + θ_R] /. {d → ∂_{u, ω} q} /. {u → u1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
  {ΔCU, $k}[ħ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@ΔCU[{x, y}, ħ (ξ x + η y + δ x y), 1_{$k}], 5],
  HL@Simp[lhs - CU@ΔCU, $k][ħ {ξ, η, δ}, {x, y}]]
]
```

$$\left\{ \text{CU} \left[\{y, a, x\}, \frac{xy \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar}, \right. \right.$$

$$\frac{1}{1 + t \delta \hbar} + \left((4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 - \right.$$

$$4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 +$$

$$4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 -$$

$$6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 +$$

$$8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 +$$

$$4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 +$$

$$4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 +$$

$$4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4) \epsilon) /$$

$$(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5) + 0[\epsilon]^2],$$

$$\left(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 + \right.$$

$$2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 -$$

$$\left. 9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4 \right) \text{CU}[\] +$$

$$(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 +$$

$$18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4) \text{CU}[a] +$$

$$\llcorner 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 \text{CU}[y, y, y, y, x, x, x] +$$

$$\frac{1}{24} \delta^4 \hbar^4$$

$$\text{CU}[y, y, y, y, x, x, x], \theta \}$$

$\{\Delta_{\text{Qu},2}[\{\xi, \eta, \delta\}, \{x, y\}], \text{lhs} = \text{QU}@\text{CQu}[\{x, y\}, \hbar(\xi x + \eta y + \delta xy), 1],$
 $\text{HL}@\text{SimpT}[\text{lhs} = \text{QU}@\Delta_{\text{Qu},1}[\hbar\{\xi, \eta, \delta\}, \{x, y\}]]\}$

$$\left\{ \text{CQu}[\{y, a, x\}, \frac{\dots 1 \dots}{\dots 1 \dots}, \right.$$

$$\frac{\hbar}{-\delta + T \delta + \hbar} + \left((-8 a T \delta^4 \hbar^2 + 24 a T^2 \delta^4 \hbar^2 - 24 a T^3 \delta^4 \hbar^2 + 8 a T^4 \delta^4 \hbar^2 + \dots 149 \dots + \right.$$

$$4 x^2 y^2 \gamma \delta^2 \hbar^6 + 4 x y^2 \gamma \delta \eta \hbar^6 + 4 x^2 y \gamma \delta \xi \hbar^6 + 4 x y \gamma \eta \xi \hbar^6) \epsilon /$$

$$(-4 \delta^5 + 20 T \delta^5 - 40 T^2 \delta^5 + 40 T^3 \delta^5 - 20 T^4 \delta^5 + 4 T^5 \delta^5 + \dots 12 \dots + 40 T^3 \delta^3 \hbar^2 +$$

$$40 \delta^2 \hbar^3 - 80 T \delta^2 \hbar^3 + 40 T^2 \delta^2 \hbar^3 - 20 \delta \hbar^4 + 20 T \delta \hbar^4 + 4 \hbar^5) +$$

$$\left. \frac{(\dots 1 \dots)}{\dots 1 \dots} + O[\epsilon]^3 \right\}, \dots 1 \dots, \text{True} \}$$

large output	show less	show more	show all	set size limit...
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$\{\text{tt} = \text{ComposeSeries}[(1 + t \delta) \text{Last}[\Delta_{\text{Cu},2}[\{\xi, \eta, \delta\}, \{x, y\}]], (1 + t \delta)^4 \epsilon + O[\epsilon]^{18}];$
 $\text{Together}@\text{Log}[\text{tt}],$
 $\text{Exponent}[\text{Normal}@\text{Together}@\text{Log}[\text{tt}] /. \{\xi \rightarrow d \xi, \eta \rightarrow d \eta, x \rightarrow d x, y \rightarrow d y\}, d],$
 $\text{Exponent}[\text{Normal}@\text{Together}@\text{Log}[\text{tt}] /. \{x \rightarrow d x, y \rightarrow d y\}, d]$
 $\} // \text{Expand}$

$$\left\{ \left(2 a \delta + 6 a t \delta^2 + 2 a x y \delta^2 + t \gamma \delta^2 - 4 x y \gamma \delta^2 + 6 a t^2 \delta^3 + 4 a t x y \delta^3 + 2 t^2 \gamma \delta^3 - 6 t x y \gamma \delta^3 - \right. \right.$$

$$2 x^2 y^2 \gamma \delta^3 + 2 a t^3 \delta^4 + 2 a t^2 x y \delta^4 + t^3 \gamma \delta^4 - 2 t^2 x y \gamma \delta^4 - \frac{3}{2} t x^2 y^2 \gamma \delta^4 + 2 a y \delta \eta -$$

$$2 y \gamma \delta \eta + 4 a t y \delta^2 \eta - 2 t y \gamma \delta^2 \eta - 3 x y^2 \gamma \delta^2 \eta + 2 a t^2 y \delta^3 \eta - 2 t x y^2 \gamma \delta^3 \eta -$$

$$y^2 \gamma \delta \eta^2 - \frac{1}{2} t y^2 \gamma \delta^2 \eta^2 + 2 a x \delta \xi - 2 x \gamma \delta \xi + 4 a t x \delta^2 \xi - 2 t x \gamma \delta^2 \xi - 3 x^2 y \gamma \delta^2 \xi +$$

$$2 a t^2 x \delta^3 \xi - 2 t x^2 y \gamma \delta^3 \xi + 2 a \eta \xi + 4 a t \delta \eta \xi + 2 t \gamma \delta \eta \xi - 4 x y \gamma \delta \eta \xi + 2 a t^2 \delta^2 \eta \xi +$$

$$2 t^2 \gamma \delta^2 \eta \xi - 2 t x y \gamma \delta^2 \eta \xi - y \gamma \eta^2 \xi - x^2 \gamma \delta \xi^2 - \frac{1}{2} t x^2 \gamma \delta^2 \xi^2 - x \gamma \eta \xi^2 + \left. \frac{1}{2} t \gamma \eta^2 \xi^2 \right) \epsilon +$$

$$\left(2 a^2 \delta^2 - 2 a \gamma \delta^2 + 12 a^2 t \delta^3 + 4 a^2 x y \delta^3 - 8 a t \gamma \delta^3 - 20 a x y \gamma \delta^3 - 2 t \gamma^2 \delta^3 + 18 x y \gamma^2 \delta^3 + \right.$$

$$30 a^2 t^2 \delta^4 + 20 a^2 t x y \delta^4 - 10 a t^2 \gamma \delta^4 - 88 a t x y \gamma \delta^4 - 13 a x^2 y^2 \gamma \delta^4 - \frac{15}{2} t^2 \gamma^2 \delta^4 +$$

$$64 t x y \gamma^2 \delta^4 + 34 x^2 y^2 \gamma^2 \delta^4 + 40 a^2 t^3 \delta^5 + 40 a^2 t^2 x y \delta^5 - 152 a t^2 x y \gamma \delta^5 - 48 a t x^2 y^2 \gamma \delta^5 -$$

$$10 t^3 \gamma^2 \delta^5 + 86 t^2 x y \gamma^2 \delta^5 + 107 t x^2 y^2 \gamma^2 \delta^5 + 11 x^3 y^3 \gamma^2 \delta^5 + 30 a^2 t^4 \delta^6 + 40 a^2 t^3 x y \delta^6 +$$

$$10 a t^4 \gamma \delta^6 - 128 a t^3 x y \gamma \delta^6 - 66 a t^2 x^2 y^2 \gamma \delta^6 - 5 t^4 \gamma^2 \delta^6 + 54 t^3 x y \gamma^2 \delta^6 + \frac{247}{2} t^2 x^2 y^2 \gamma^2 \delta^6 +$$

$$\frac{80}{3} t x^3 y^3 \gamma^2 \delta^6 + 12 a^2 t^5 \delta^7 + 20 a^2 t^4 x y \delta^7 + 8 a t^5 \gamma \delta^7 - 52 a t^4 x y \gamma \delta^7 - 40 a t^3 x^2 y^2 \gamma \delta^7 +$$

$$16 t^4 x y \gamma^2 \delta^7 + 62 t^3 x^2 y^2 \gamma^2 \delta^7 + \frac{64}{3} t^2 x^3 y^3 \gamma^2 \delta^7 + 2 a^2 t^6 \delta^8 + 4 a^2 t^5 x y \delta^8 + 2 a t^6 \gamma \delta^8 -$$

$$8 a t^5 x y \gamma \delta^8 - 9 a t^4 x^2 y^2 \gamma \delta^8 + \frac{1}{2} t^6 \gamma^2 \delta^8 + 2 t^5 x y \gamma^2 \delta^8 + \frac{23}{2} t^4 x^2 y^2 \gamma^2 \delta^8 + \frac{17}{3} t^3 x^3 y^3 \gamma^2 \delta^8 +$$

$$4 a^2 y \delta^2 \eta - 12 a y \gamma \delta^2 \eta + 6 y \gamma^2 \delta^2 \eta + 20 a^2 t y \delta^3 \eta - 48 a t y \gamma \delta^3 \eta - 20 a x y^2 \gamma \delta^3 \eta +$$

$$14 t y \gamma^2 \delta^3 \eta + 40 x y^2 \gamma^2 \delta^3 \eta + 40 a^2 t^2 y \delta^4 \eta - 72 a t^2 y \gamma \delta^4 \eta - 72 a t x y^2 \gamma \delta^4 \eta + 6 t^2 y \gamma^2 \delta^4 \eta +$$

$$115 t x y^2 \gamma^2 \delta^4 \eta + 23 x^2 y^3 \gamma^2 \delta^4 \eta + 40 a^2 t^3 y \delta^5 \eta - 48 a t^3 y \gamma \delta^5 \eta - 96 a t^2 x y^2 \gamma \delta^5 \eta -$$

$$6 t^3 y \gamma^2 \delta^5 \eta + 118 t^2 x y^2 \gamma^2 \delta^5 \eta + 53 t x^2 y^3 \gamma^2 \delta^5 \eta + 20 a^2 t^4 y \delta^6 \eta - 12 a t^4 y \gamma \delta^6 \eta -$$

$$56 a t^3 x y^2 \gamma \delta^6 \eta - 4 t^4 y \gamma^2 \delta^6 \eta + 51 t^3 x y^2 \gamma^2 \delta^6 \eta + 40 t^2 x^2 y^3 \gamma^2 \delta^6 \eta + 4 a^2 t^5 y \delta^7 \eta -$$

$$\begin{aligned}
 & 12 a t^4 x y^2 \gamma \delta^7 \eta + 8 t^4 x y^2 \gamma^2 \delta^7 \eta + 10 t^3 x^2 y^3 \gamma^2 \delta^7 \eta - 7 a y^2 \gamma \delta^2 \eta^2 + 10 y^2 \gamma^2 \delta^2 \eta^2 - \\
 & 24 a t y^2 \gamma \delta^3 \eta^2 + 24 t y^2 \gamma^2 \delta^3 \eta^2 + 15 x y^3 \gamma^2 \delta^3 \eta^2 - 30 a t^2 y^2 \gamma \delta^4 \eta^2 + \frac{37}{2} t^2 y^2 \gamma^2 \delta^4 \eta^2 + \\
 & 32 t x y^3 \gamma^2 \delta^4 \eta^2 - 16 a t^3 y^2 \gamma \delta^5 \eta^2 + 5 t^3 y^2 \gamma^2 \delta^5 \eta^2 + 22 t^2 x y^3 \gamma^2 \delta^5 \eta^2 - 3 a t^4 y^2 \gamma \delta^6 \eta^2 + \\
 & \frac{1}{2} t^4 y^2 \gamma^2 \delta^6 \eta^2 + 5 t^3 x y^3 \gamma^2 \delta^6 \eta^2 + 3 y^3 \gamma^2 \delta^2 \eta^3 + \frac{17}{3} t y^3 \gamma^2 \delta^3 \eta^3 + \frac{10}{3} t^2 y^3 \gamma^2 \delta^4 \eta^3 + \\
 & \frac{2}{3} t^3 y^3 \gamma^2 \delta^5 \eta^3 + 4 a^2 x \delta^2 \xi - 12 a x \gamma \delta^2 \xi + 6 x \gamma^2 \delta^2 \xi + 20 a^2 t x \delta^3 \xi - 48 a t x \gamma \delta^3 \xi - \\
 & 20 a x^2 y \gamma \delta^3 \xi + 14 t x \gamma^2 \delta^3 \xi + 40 x^2 y \gamma^2 \delta^3 \xi + 40 a^2 t^2 x \delta^4 \xi - 72 a t^2 x \gamma \delta^4 \xi - 72 a t x^2 y \gamma \delta^4 \xi + \\
 & 6 t^2 x \gamma^2 \delta^4 \xi + 115 t x^2 y \gamma^2 \delta^4 \xi + 23 x^3 y^2 \gamma^2 \delta^4 \xi + 40 a^2 t^3 x \delta^5 \xi - 48 a t^3 x \gamma \delta^5 \xi - \\
 & 96 a t^2 x^2 y \gamma \delta^5 \xi - 6 t^3 x \gamma^2 \delta^5 \xi + 118 t^2 x^2 y \gamma^2 \delta^5 \xi + 53 t x^3 y^2 \gamma^2 \delta^5 \xi + 20 a^2 t^4 x \delta^6 \xi - \\
 & 12 a t^4 x \gamma \delta^6 \xi - 56 a t^3 x^2 y \gamma \delta^6 \xi - 4 t^4 x \gamma^2 \delta^6 \xi + 51 t^3 x^2 y \gamma^2 \delta^6 \xi + 40 t^2 x^3 y^2 \gamma^2 \delta^6 \xi + \\
 & 4 a^2 t^5 x \delta^7 \xi - 12 a t^4 x^2 y \gamma \delta^7 \xi + 8 t^4 x^2 y \gamma^2 \delta^7 \xi + 10 t^3 x^3 y^2 \gamma^2 \delta^7 \xi + 4 a^2 \delta \eta \xi - 4 a \gamma \delta \eta \xi + \\
 & 20 a^2 t \delta^2 \eta \xi - 8 a t \gamma \delta^2 \eta \xi - 28 a x y \gamma \delta^2 \eta \xi - 6 t \gamma^2 \delta^2 \eta \xi + 38 x y \gamma^2 \delta^2 \eta \xi + 40 a^2 t^2 \delta^3 \eta \xi + \\
 & 8 a t^2 \gamma \delta^3 \eta \xi - 96 a t x y \gamma \delta^3 \eta \xi - 14 t^2 \gamma^2 \delta^3 \eta \xi + 88 t x y \gamma^2 \delta^3 \eta \xi + 44 x^2 y^2 \gamma^2 \delta^3 \eta \xi + \\
 & 40 a^2 t^3 \delta^4 \eta \xi + 32 a t^3 \gamma \delta^4 \eta \xi - 120 a t^2 x y \gamma \delta^4 \eta \xi - 6 t^3 \gamma^2 \delta^4 \eta \xi + 62 t^2 x y \gamma^2 \delta^4 \eta \xi + \\
 & 93 t x^2 y^2 \gamma^2 \delta^4 \eta \xi + 20 a^2 t^4 \delta^5 \eta \xi + 28 a t^4 \gamma \delta^5 \eta \xi - 64 a t^3 x y \gamma \delta^5 \eta \xi + 6 t^4 \gamma^2 \delta^5 \eta \xi + \\
 & 12 t^3 x y \gamma^2 \delta^5 \eta \xi + 63 t^2 x^2 y^2 \gamma^2 \delta^5 \eta \xi + 4 a^2 t^5 \delta^6 \eta \xi + 8 a t^5 \gamma \delta^6 \eta \xi - 12 a t^4 x y \gamma \delta^6 \eta \xi + \\
 & 4 t^5 \gamma^2 \delta^6 \eta \xi + 14 t^3 x^2 y^2 \gamma^2 \delta^6 \eta \xi - 8 a y \gamma \delta \eta^2 \xi + 6 y \gamma^2 \delta \eta^2 \xi - 24 a t y \gamma \delta^2 \eta^2 \xi + \\
 & 5 t y \gamma^2 \delta^2 \eta^2 \xi + 25 x y^2 \gamma^2 \delta^2 \eta^2 \xi - 24 a t^2 y \gamma \delta^3 \eta^2 \xi - 8 t^2 y \gamma^2 \delta^3 \eta^2 \xi + 45 t x y^2 \gamma^2 \delta^3 \eta^2 \xi - \\
 & 8 a t^3 y \gamma \delta^4 \eta^2 \xi - 7 t^3 y \gamma^2 \delta^4 \eta^2 \xi + 24 t^2 x y^2 \gamma^2 \delta^4 \eta^2 \xi + 4 t^3 x y^2 \gamma^2 \delta^5 \eta^2 \xi + 4 y^2 \gamma^2 \delta \eta^3 \xi + \\
 & 5 t y^2 \gamma^2 \delta^2 \eta^3 \xi + t^2 y^2 \gamma^2 \delta^3 \eta^3 \xi - 7 a x^2 \gamma \delta^2 \xi^2 + 10 x^2 \gamma^2 \delta^2 \xi^2 - 24 a t x^2 \gamma \delta^3 \xi^2 + \\
 & 24 t x^2 \gamma^2 \delta^3 \xi^2 + 15 x^3 y \gamma^2 \delta^3 \xi^2 - 30 a t^2 x^2 \gamma \delta^4 \xi^2 + \frac{37}{2} t^2 x^2 \gamma^2 \delta^4 \xi^2 + 32 t x^3 y \gamma^2 \delta^4 \xi^2 - \\
 & 16 a t^3 x^2 \gamma \delta^5 \xi^2 + 5 t^3 x^2 \gamma^2 \delta^5 \xi^2 + 22 t^2 x^3 y \gamma^2 \delta^5 \xi^2 - 3 a t^4 x^2 \gamma \delta^6 \xi^2 + \frac{1}{2} t^4 x^2 \gamma^2 \delta^6 \xi^2 + \\
 & 5 t^3 x^3 y \gamma^2 \delta^6 \xi^2 - 8 a x \gamma \delta \eta \xi^2 + 6 x \gamma^2 \delta \eta \xi^2 - 24 a t x \gamma \delta^2 \eta \xi^2 + 5 t x \gamma^2 \delta^2 \eta \xi^2 + \\
 & 25 x^2 y \gamma^2 \delta^2 \eta \xi^2 - 24 a t^2 x \gamma \delta^3 \eta \xi^2 - 8 t^2 x \gamma^2 \delta^3 \eta \xi^2 + 45 t x^2 y \gamma^2 \delta^3 \eta \xi^2 - 8 a t^3 x \gamma \delta^4 \eta \xi^2 - \\
 & 7 t^3 x \gamma^2 \delta^4 \eta \xi^2 + 24 t^2 x^2 y \gamma^2 \delta^4 \eta \xi^2 + 4 t^3 x^2 y \gamma^2 \delta^5 \eta \xi^2 - a \gamma \eta^2 \xi^2 - 3 t \gamma^2 \delta \eta^2 \xi^2 + \\
 & 11 x y \gamma^2 \delta \eta^2 \xi^2 + 6 a t^2 \gamma \delta^2 \eta^2 \xi^2 - \frac{5}{2} t^2 \gamma^2 \delta^2 \eta^2 \xi^2 + 12 t x y \gamma^2 \delta^2 \eta^2 \xi^2 + 8 a t^3 \gamma \delta^3 \eta^2 \xi^2 + \\
 & 4 t^3 \gamma^2 \delta^3 \eta^2 \xi^2 + 3 a t^4 \gamma \delta^4 \eta^2 \xi^2 + \frac{7}{2} t^4 \gamma^2 \delta^4 \eta^2 \xi^2 - t^3 x y \gamma^2 \delta^4 \eta^2 \xi^2 + y \gamma^2 \eta^3 \xi^2 - \\
 & t y \gamma^2 \delta \eta^3 \xi^2 - 2 t^2 y \gamma^2 \delta^2 \eta^3 \xi^2 + 3 x^3 \gamma^2 \delta^2 \xi^3 + \frac{17}{3} t x^3 \gamma^2 \delta^3 \xi^3 + \frac{10}{3} t^2 x^3 \gamma^2 \delta^4 \xi^3 + \\
 & \frac{2}{3} t^3 x^3 \gamma^2 \delta^5 \xi^3 + 4 x^2 \gamma^2 \delta \eta \xi^3 + 5 t x^2 \gamma^2 \delta^2 \eta \xi^3 + t^2 x^2 \gamma^2 \delta^3 \eta \xi^3 + x \gamma^2 \eta^2 \xi^3 - t x \gamma^2 \delta \eta^2 \xi^3 - \\
 & 2 t^2 x \gamma^2 \delta^2 \eta^2 \xi^3 - \frac{1}{3} t \gamma^2 \eta^3 \xi^3 + \frac{1}{3} t^2 \gamma^2 \delta \eta^3 \xi^3 + \frac{2}{3} t^3 \gamma^2 \delta^2 \eta^3 \xi^3 \Big) \epsilon^2 + 0[\epsilon]^3, 6, 6 \}
 \end{aligned}$$

```

{tt = Last[DeltaQu,2[{xi, eta, delta}, {x, y}]];
Log[tt],
Exponent[Normal@Together@Log[tt] /. {xi -> d xi, eta -> d eta, x -> d x, y -> d y}, d] // Expand

```

$$\left\{ \text{Log} \left[\frac{\hbar}{-\delta + T \delta + \hbar} \right] + \left(\frac{2 a T \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} - \frac{8 a T^2 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} + \frac{12 a T^3 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} - \frac{8 a T^4 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} + \dots 267 \dots + \frac{x^2 y^2 \gamma \delta^2 \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x y^2 \gamma \delta \eta \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x^2 y \gamma \delta \xi \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x y \gamma \eta \xi \hbar^6}{(-\delta + T \delta + \hbar)^5} \right) \epsilon + \left(- \frac{32 a^2 T^2 \delta^{10} \hbar^2}{(\dots 1 \dots)^2} + \dots 8307 \dots + \dots 1 \dots + \frac{144 x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^{11}}{\dots 1 \dots} \right) \epsilon^2 + O[\epsilon]^3, 6 \right\}$$

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Reorderings with Rord

Rord

In[*]:=

```
Rordui, wj → ki [CU[L---, {L---, ui, wj, r---}s, R---, Q-, P-]] :=
Simp@Module[{u, w, δ, Δ1, yax, q, p, kk = P[[5]], δ1 = ∂ui, wj Q},
{yax, q, p} = Echo[List@@ If[δ1 === 0, ΔU, kk [{u, w}, {u, w}],
ΔU, kk [{u, w, δ}, {u, w}]] /. {y → yk, a → ak, x → xk, t → ts, T → Ts}}];
CU[L, {L, Sequence@@ yax, rs, R, q + (Q /. ui | wj → 0), e-q DPui → Du, wj → Dw [P] [p eq]] /.
{u → ∂ui Q /. wj → 0, w → ∂wj Q /. ui → 0, δ → δ1}];
```

Rord

In[*]:=

```
Rordui, wj → ki [CU[L---, {L---, ui, wj, r---}s, R---, Q-, P-]] :=
Simp@Module[{u, w, δ, Δ1, yax, q, p, n, kk = P[[5]], δ1 = ∂ui, wj Q},
{yax, q, p} = List@@ If[δ1 === 0, ΔU, kk [{u, w}, {u, w}], ΔU, kk [{u, w, δ}, {u, w}]] /.
{y → yn, a → an, x → xn, t → ts, T → Ts};
(*Echo@{{ui, v}, {wj, ω}}, P, p eq}; *)
CU[L, {L, Sequence@@ yax, rs, R, q + (Q /. ui | wj → 0), e-q SPui → v, wj → ω [P p eq]] /.
{n → k, v → ∂ui Q /. wj → 0, w → ∂wj Q /. ui → 0, δ → δ1}];
```

```
With[{co = CCU[{y1, x1}1, {x2, a2, y2}2, ħ t1 a2 + ħ t1-1 (et1 - 1) y1 x2, 12 + ε x1 y2]}],
{Short[rhs = co // Rordx2, a2 → 3, 3], HL[CU[co] = CU[rhs]]}]
{CCU[{y1, x1}1, {a3, x3, y2}2,  $\frac{e^{-\gamma \hbar t_1} (e^{\gamma \hbar t_1} \hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1} \hbar x_3 y_1)}{t_1}$ , 1 + x1 y2 ∈ + O[ε]3], True}
```

```
With[{co = CCU[{y1, a1, a2}1, {x2, x1, y2}2,
ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
12 + ε (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
{Short[rhs = co // Rorda1, a2 → 3 // Rordx2, x1 → 4, 3], HL[CU[co] = CU[rhs]]}]
{CCU[{y1, a3}1, {x4, y2}2,
ħ a3 l11 t1 + ħ a3 l12 t1 + ħ a3 l21 t2 + ħ a3 l22 t2 + ħ x4 y1 γ11 + ħ x4 y2 γ12 + ħ x4 y1 γ21 + ħ x4 y2 γ22,
1 + (a3 l1 + a3 l2 + p11 x4 y1 + p21 x4 y1 + p12 x4 y2 + p22 x4 y2) ∈ + O[ε]3], True}
ħ a3 l11 t1 + ħ a3 l12 t1 + ħ a3 l21 t2 + ħ a3 l22 t2 +
ħ x4 y1 γ11 + ħ x4 y2 γ12 + ħ x4 y1 γ21 + ħ x4 y2 γ22 // Simplify
ħ (a3 (l11 t1 + l12 t1 + (l21 + l22) t2) + x4 (y1 (γ11 + γ21) + y2 (γ12 + γ22)))
```

With [{ $\mathbf{c0} = \mathbb{C}_{\text{CU}} [\{y_1, a_1, x_1\}_1, \{x_2, a_2, y_2\}_2,$
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] \},$
 $\{\text{Short}[\text{rhs} = \mathbf{c0} // \text{Rord}_{x_2, a_2 \rightarrow 3}, 3], \text{HL}[\text{CU}[\mathbf{c0}] = \text{CU}[\text{rhs}]] \}$
 $\{\mathbb{C}_{\text{CU}} [\{y_1, a_1, x_1\}_1, \langle\langle 1 \rangle\rangle_2, \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle,$
 $1 + e^{-\gamma \hbar l_{12} t_1 - \gamma \hbar l_{22} t_2} (e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_1 l_1 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_3 l_2 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +$
 $e^{\langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \in + O[\epsilon]^3 \}, \text{True} \}$

With [{ $\mathbf{q0} = \mathbb{C}_{\text{QU}} [\{y_1, a_1, x_1\}_1, \{x_2, a_2, y_2\}_2,$
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] \},$
 $\{\text{Short}[\text{rhs} = \mathbf{q0} // \text{Rord}_{x_2, a_2 \rightarrow 3}, 3], \text{HL}[\text{QU}[\mathbf{q0}] = \text{QU}[\text{rhs}]] \}$
 $\{\mathbb{C}_{\text{QU}} [\{y_1, a_1, x_1\}_1, \langle\langle 1 \rangle\rangle_2, \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle,$
 $1 + e^{-\gamma \hbar l_{12} t_1 - \gamma \hbar l_{22} t_2} (e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_1 l_1 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_3 l_2 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +$
 $e^{\langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \in + O[\epsilon]^3 \}, \text{True} \}$

With [{ $\mathbf{q0} = \mathbb{C}_{\text{QU}} [\{y_1, a_1, x_1\}_1, \{x_2, a_2, y_2\}_2,$
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] \},$
 $\{\text{Short}[\text{rhs} = \mathbf{q0} // \text{Rord}_{a_2, y_2 \rightarrow 3}, 3], \text{HL}[\text{QU}[\mathbf{q0}] = \text{QU}[\text{rhs}]] \}$
 $\{\langle\langle 1 \rangle\rangle, \text{True} \}$

Timing@With [{ $\mathbf{q0} = \mathbb{C}_{\text{QU}} [\{x_1, y_1\}_1, \{x_2, a_2, y_2\}_2,$
 $\hbar (l_{12} t_1 a_2 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $\theta_2 + \epsilon (l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] \},$
 $\{\text{Short}[\text{rhs} = \mathbf{q0} // \text{Rord}_{x_1, y_1 \rightarrow 3}, 5] \}$

{ 116.156, { $\mathbb{C}_{\text{QU}} [\{y_3, a_3, x_3\}_1, \langle\langle 1 \rangle\rangle_2, \frac{\langle\langle 1 \rangle\rangle}{1 - \langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle},$
 $((\hbar a_2 l_2 + p_{11} - p_{11} T_1 + \hbar p_{22} x_2 y_2 + \hbar p_{12} x_3 y_2 + \langle\langle 46 \rangle\rangle + 2 \hbar p_{12} T_1 x_2 y_2 \gamma_{11} \gamma_{21} -$
 $\hbar p_{12} T_1^2 x_2 y_2 \gamma_{11} \gamma_{21} + \hbar p_{11} x_2 y_2 \gamma_{12} \gamma_{21} - 2 \hbar p_{11} T_1 x_2 y_2 \gamma_{12} \gamma_{21} + \hbar p_{11} T_1^2 x_2 y_2 \gamma_{12} \gamma_{21}) \epsilon) /$
 $(\hbar - 3 \hbar \gamma_{11} + 3 \hbar T_1 \gamma_{11} + 3 \hbar \gamma_{11}^2 - 6 \hbar T_1 \gamma_{11}^2 + 3 \hbar T_1^2 \gamma_{11}^2 - \hbar \gamma_{11}^3 + 3 \hbar T_1 \gamma_{11}^3 - 3 \hbar T_1^2 \gamma_{11}^3 + \hbar T_1^3 \gamma_{11}^3) +$
 $((8 a_3 p_{11} T_1 + \langle\langle 1 \rangle\rangle + \langle\langle 2726 \rangle\rangle + 3 \gamma \langle\langle 6 \rangle\rangle \gamma_{21}^3) \langle\langle 1 \rangle\rangle) /$
 $(4 - 28 \gamma_{11} + \langle\langle 48 \rangle\rangle + 4 T_1^7 \gamma_{11}^7) + O[\epsilon]^3 \}$

$$\begin{aligned}
 & \text{Timing@With}[\{\mathbf{q}_0 = \mathbb{C}_{\text{QU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \{\mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_2, \\
 & \quad \hbar (\mathbf{l}_{12} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2), \\
 & \quad \mathbf{1}_2 + \epsilon (\mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2)\}], \\
 & \{\text{Short}[\text{rhs} = \mathbf{q}_0 // \text{Rord}_{\mathbf{x}_1, \mathbf{y}_1 \rightarrow 3}, 5], \text{HL@SimpT}[\text{QU}[\mathbf{q}_0] == \text{QU}[\text{rhs}]]\}] \\
 & \{388.922, \\
 & \left\{ \mathbb{C}_{\text{QU}}[\{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_1, \{\ll 1 \gg\}_2, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + \left((4 \hbar \mathbf{a}_2 \mathbf{l}_2 + 4 \mathbf{p}_{11} - 4 \mathbf{p}_{11} T_1 + 4 \hbar \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2 + \right. \right. \\
 & \quad \left. \left. \ll 339 \gg + \gamma \hbar^4 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4 \gamma \hbar^4 T_1 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2) \epsilon \right) / \right. \\
 & \quad \left. (4 \hbar - 20 \hbar \gamma_{11} + 20 \hbar T_1 \gamma_{11} + 40 \hbar \gamma_{11}^2 - 80 \hbar T_1 \gamma_{11}^2 + 40 \hbar T_1^2 \gamma_{11}^2 - 40 \hbar \gamma_{11}^3 + \ll 13 \gg + \right. \\
 & \quad \left. 20 \hbar T_1 \gamma_{11}^5 - 40 \hbar T_1^2 \gamma_{11}^5 + 40 \hbar T_1^3 \gamma_{11}^5 - 20 \hbar T_1^4 \gamma_{11}^5 + 4 \hbar T_1^5 \gamma_{11}^5) + \right. \\
 & \quad \left. (576 \mathbf{a}_3 \mathbf{p}_{11} T_1 + \ll 8073 \gg + \ll 1 \gg) \frac{\ll 1 \gg}{\ll 79 \gg + 288 T_1^9 \gamma_{11}^9} + 0[\epsilon]^3 \right\}, \text{True} \} \}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Timing@With}[\{\mathbf{q}_0 = \mathbb{C}_{\text{QU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \{\mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_2, \\
 & \quad \hbar (\mathbf{l}_{12} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2), \\
 & \quad \mathbf{1}_2 + \epsilon (\mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2)\}], \\
 & \{\text{Short}[\text{rhs} = \mathbf{q}_0 // \text{Rord}_{\mathbf{x}_1, \mathbf{y}_1 \rightarrow 1}, 5], \text{HL@SimpT}[\text{QU}[\mathbf{q}_0] == \text{QU}[\text{rhs}]]\}] \\
 & \{336.781, \\
 & \left\{ \mathbb{C}_{\text{QU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\ll 1 \gg\}_2, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + \left((4 \hbar \mathbf{a}_2 \mathbf{l}_2 + 4 \mathbf{p}_{11} - 4 \mathbf{p}_{11} T_1 + 4 \hbar \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \right. \right. \\
 & \quad \left. \left. 4 \hbar \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \ll 338 \gg + \gamma \hbar^4 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4 \gamma \hbar^4 T_1 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2) \epsilon \right) / \right. \\
 & \quad \left. (4 \hbar - 20 \hbar \gamma_{11} + 20 \hbar T_1 \gamma_{11} + 40 \hbar \gamma_{11}^2 - 80 \hbar T_1 \gamma_{11}^2 + 40 \hbar T_1^2 \gamma_{11}^2 - 40 \hbar \gamma_{11}^3 + \ll 10 \gg + \right. \\
 & \quad \left. 20 \hbar T_1^4 \gamma_{11}^4 - 4 \hbar \gamma_{11}^5 + 20 \hbar T_1 \gamma_{11}^5 - 40 \hbar T_1^2 \gamma_{11}^5 + 40 \hbar T_1^3 \gamma_{11}^5 - 20 \hbar T_1^4 \gamma_{11}^5 + 4 \hbar T_1^5 \gamma_{11}^5) + \right. \\
 & \quad \left. (576 \mathbf{a}_1 \mathbf{p}_{11} T_1 + \ll 8073 \gg + \ll 1 \gg) \frac{\ll 1 \gg}{\ll 79 \gg + 288 T_1^9 \gamma_{11}^9} + 0[\epsilon]^3 \right\}, \text{True} \} \}
 \end{aligned}$$

Canonical ordering with Cord

Cord

In[*]:=

```

Cord[ $\mathbb{C}_U[L\_], \{L\_], u\_i, w\_j, r\_]\_s, R\_], Q_, P_] /;
  OrderedQ[\{w, u\} /. {y -> 1, a -> 2, x -> 3}] :=
  ((*Echo@{u_i, w_j}; *) Cord[Rord_{u_i, w_j -> Unique[]} [ $\mathbb{C}_U[L, \{L, u_i, w_j, r\]\_s, R, Q, P] ] ]];
Cord[ $\mathbb{C}_U[\text{specs}\_], Q_, P_] := \mathbb{C}_U[\text{Sequence} @@ \text{Sort} @ \{\text{specs}\}, Q, P] /.
  Flatten[\{\text{specs}\} /. \{yax\_]\_s -> (\{yax\} /. u\_i -> (u_i -> u_s))]$$$ 
```

$\mathbb{C}_{\text{CU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \mathbf{0}, \mathbf{0}_1 + \mathbf{x}_1 \mathbf{y}_1]$

$\mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \mathbf{0}, (-\mathbf{t}_1 + \mathbf{x}_1 \mathbf{y}_1) + 2 \mathbf{a}_1 \epsilon + 0[\epsilon]^2]$

Block [{ \$p = 4, \$k = 0, c0 = CU [{ y1, a1, x1, x2, a2, y2 } 1, $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$, $1_0 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)$] }, **Timing** @ { Short [Cord [c0], 8], HL @ Simp [CU [c0] - CU [Cord [c0]]] }]

{ 4.53125,
$$\left\{ \text{CU} \left[\{ y_1, a_1, x_1 \} 1, \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 + \ll 12 \gg + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar^2 a_1 l_{22} t_1 t_2 \gamma_{22} + \hbar x_1 y_1 \gamma_{22} \right) / \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{22} \right), \frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + O[\epsilon]^1, \mathbf{0} \right\}$$

Block [{ \$p = 4, \$k = 1, c0 = CU [{ y1, a1, x1, x2, a2, y2 } 1, $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$, $1_1 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)$] }, **Timing** @ { Short [Cord [c0], 8], HL @ Simp [CU [c0] - CU [Cord [c0]]] }]

{ 81.2656,
$$\left\{ \text{CU} \left[\{ y_1, a_1, x_1 \} 1, \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + \ll 14 \gg + \hbar x_1 y_1 \gamma_{22} \right) / \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} + e^{\gamma \hbar l_{11} t_1 + \ll 2 \gg + \gamma \hbar \ll 1 \gg} t_2 \hbar t_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{22} \right), \frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + \left(\left(2 e^{2 \gamma \hbar l_{11} t_1 + 6 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 6 \gamma \hbar l_{22} t_2} a_1 l_1 + \ll 419 \gg \right) \epsilon \right) / \left(2 e^{2 \gamma \hbar l_{11} t_1 + 6 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 6 \gamma \hbar l_{22} t_2} + 1_0 e^{2 \gamma \hbar l_{11} t_1 + 5 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 5 \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{12} + \ll 18 \gg + 2 e^{2 \gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar^5 t_1^5 \gamma_{22}^5 \right) + O[\epsilon]^2, \mathbf{0} \right\}$$

With [{ q0 = QU [{ y1, a1, x1, x2, a2, y2 } 1, $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2)$, $1_0 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)$] }, **Cord** [q0]]

$$\left\{ \text{QU} \left[\{ y_1, a_1, x_1 \} 1, \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{11} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 T_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 T_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 T_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 T_1 \gamma_{12} + \hbar x_1 y_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{21} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \gamma_{22} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \gamma_{22} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \gamma_{22} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 T_1 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 T_1 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 T_1 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 T_1 \gamma_{22} + \hbar x_1 y_1 \gamma_{22} \right) / \left(e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} t_2 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} T_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} t_2 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} T_1 \gamma_{22} \right), \frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} - \gamma_{12} + T_1 \gamma_{12} - \gamma_{22} + T_1 \gamma_{22}} + O[\epsilon]^1 \right\}$$

Stitching \mathbb{C} 's.

StitchingOEs

```
In[ ]:= mj→k[ $\mathbb{C}_U$ [specs__, Q_, P_]] := Cord[ $\mathbb{C}_U$ [Sequence@@Append[DeleteCases[{specs}, {__}_j|k], Flatten[{Cases[{specs}, {us__}_j :> {us}], Cases[{specs}, {us__}_k :> {us}]}}]_k], Q, P] /. {tj → tk, Tj → Tk}
```

```
co =  $\mathbb{C}_{CU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y3, a3, x3}3,  $\hbar$  Sum[l10i+j ti aj +  $\gamma$ 10i+j yi xj, {i, 3}, {j, 3}], 12];
co // m3→4, HL@Simp[CU[m3→4[co]] - m3→4[CU[co]]]
{ $\mathbb{C}_{CU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y4, a4, x4}4,  $\hbar$  ( a1 l11 t1 + a2 l12 t1 + a4 l13 t1 + a1 l21 t2 + a2 l22 t2 + a4 l23 t2 + a1 l31 t4 + a2 l32 t4 + a4 l33 t4 + x1 y1  $\gamma$ 11 + x2 y1  $\gamma$ 12 + x4 y1  $\gamma$ 13 + x1 y2  $\gamma$ 21 + x2 y2  $\gamma$ 22 + x4 y2  $\gamma$ 23 + x1 y4  $\gamma$ 31 + x2 y4  $\gamma$ 32 + x4 y4  $\gamma$ 33 ), 1 + O[ $\epsilon$ ]3], 0 }
```

Verifying that *m* commutes with evaluation, in CU:

```
co =  $\mathbb{C}_{CU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y3, a3, x3}3,  $\hbar$  Sum[l10i+j ti aj +  $\gamma$ 10i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{co // m2→3, HL@Simp[CU[m2→3[co]] - m2→3[CU[co]]]}
```

$$\left\{ 513.453, \left\{ \mathbb{C}_{CU} \left[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\dots 1 \dots}{e^{\dots 1 \dots} \dots 1 \dots}, \frac{1}{1 + \hbar t_3 \gamma_{32}} + \left(\left(4 e^{2 \gamma \hbar l_{12} t_1 + \dots 4 \dots} + 2 \gamma \hbar l_{33} t_3 \hbar^2 a_3 x_1 y_1 \gamma_{12} \gamma_{31} - 2 \dots 7 \dots \gamma_{31} + \dots 154 \dots \right) \epsilon \right) / \left(2 e^{2 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{13} t_1 + \dots 3 \dots} + 2 \gamma \hbar l_{33} t_3 + 10 e^{\dots 1 \dots} \hbar t_3 \gamma_{32} + \dots 2 \dots + \dots 1 \dots + 2 e^{2 \gamma \hbar l_{12} t_1 + \dots 4 \dots} + \dots 1 \dots \hbar^5 t_3^5 \gamma_{32}^5 \right) + \frac{(\dots 1 \dots)^2}{\dots 1 \dots} + O[\epsilon]^3 \right\}, \mathbf{0} \right\}$$

large output show less show more show all set size limit...

Verifying that *m* commutes with evaluation, in QU:

```
qo =  $\mathbb{C}_{QU}$ [{y1, a1, x1}1, {y2, a2, x2}2, {y3, a3, x3}3,  $\hbar$  Sum[l10i+j ti aj +  $\gamma$ 10i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{qo // m2→3, HL@SimpT[QU[m2→3[qo]] - m2→3[QU[qo]]]}
```

$$\left\{ 7831.47, \left\{ \mathbb{C}_{QU} \left[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\dots 1 \dots}{\dots 1 \dots}, \frac{1}{1 - \gamma_{32} + T_3 \gamma_{32}} + \left(\left(8 e^{2 \gamma \hbar l_{12} t_1 + \dots 4 \dots} + 2 \gamma \hbar l_{33} t_3 \hbar^2 a_3 T_3 x_1 y_1 \gamma_{12} \gamma_{31} + 4 \dots 8 \dots \gamma_{31} + \dots 371 \dots \right) \epsilon \right) / \left(4 e^{2 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{13} t_1 + \dots 3 \dots} + 2 \gamma \hbar l_{33} t_3 - 20 e^{\dots 1 \dots} \gamma_{32} + \dots 26 \dots + 4 e^{2 \gamma \hbar l_{12} t_1 + \dots 4 \dots} + \dots 1 \dots T_3^5 \gamma_{32}^5 \right) + \frac{(\dots 1 \dots)^2}{\dots 79 \dots + \dots 1 \dots} + O[\epsilon]^3 \right\}, \mathbf{0} \right\}$$

large output show less show more show all set size limit...


```
In[*]:=  $\mathbb{C}_{U_1}[sp1\_ , Q1\_ , P1\_ ] \equiv \mathbb{C}_{U_2}[sp2\_ , Q2\_ , P2\_ ] :=$   
Sort[{sp1}] == Sort[{sp2}]  $\wedge$  Simplify[Q1 == Q2]  $\wedge$  Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
co =  $\mathbb{C}_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2,$   
 $\{y_3, a_3, x_3\}_3, \hbar \text{Sum}[\lambda_{10 i+j} t_i a_j + \gamma_{10 i+j} y_i x_j, \{i, 3\}, \{j, 3\}], 1_0];$   
Timing@HL[(lhs = co // m1,2→1 // m1,3→1)  $\equiv$  (rhs = co // m2,3→2 // m1,2→1)]  
{41.9219, True}
```

```
co =  $\mathbb{C}_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2,$   
 $\{y_3, a_3, x_3\}_3, \hbar \text{Sum}[\lambda_{10 i+j} t_i a_j + \gamma_{10 i+j} y_i x_j, \{i, 3\}, \{j, 3\}], 1_1];$   
Timing@HL[(lhs = co // m1,2→1 // m1,3→1)  $\equiv$  (rhs = co // m2,3→2 // m1,2→1)]  
{30119.8, True}
```

mexamples

```
co =  $\mathbb{C}_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \hbar \text{Sum}[l_{10 i+j} t_i a_j + \gamma_{10 i+j} y_i x_j, \{i, 2\}, \{j, 2\}], 1_1];$   
Short[Simplify /@ (cexample = co // m1→2), 12]  
Short[Simplify /@ (qexample = (qo = co /. CU → QU) // m1→2), 12]
```

mexamples

$$\mathbb{C}_{CU}[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11} + l_{12} + l_{21} + l_{22}) t_2 + \frac{1}{1 + \hbar t_2 \gamma_{21}}$$

$$e^{-\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 (\gamma_{21} + e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \gamma_{12} (1 + \hbar t_2 \gamma_{21}) +$$

$$e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{11} (e^{\gamma \hbar (l_{11} + l_{21}) t_2} - e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar t_2 \gamma_{22})),$$

$$\frac{1}{1 + \hbar t_2 \gamma_{21}} + \frac{1}{2 (1 + \hbar t_2 \gamma_{21})^5} e^{-2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar (4 a_2 (1 + \hbar t_2 \gamma_{21})^2$$

$$(e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} t_2 + x_2 y_2) \gamma_{21}^2 + e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{11} \gamma_{22} +$$

$$\gamma_{21} (e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \hbar x_2 y_2 (e^{\gamma \hbar (2 l_{11} + l_{12} + 2 l_{21} + l_{22}) t_2} \gamma_{11} + e^{\gamma \hbar (l_{11} + 2 l_{12} + l_{21} + 2 l_{22}) t_2} \gamma_{22}))) -$$

$$\gamma \hbar (-2 e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} t_2 \gamma_{21}^2 (1 + \hbar t_2 \gamma_{21})^2 + 4 \ll 5 \gg (\ll 1 \gg) +$$

$$\hbar \ll 4 \gg (3 \hbar t_2 \gamma_{21}^2 + 2 e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{21} (4 + e^{\gamma \ll 3 \gg} \hbar t_2 \gamma_{22}) +$$

$$e^{\gamma \hbar (l_{11} + l_{21}) t_2} \gamma_{11} (2 + \hbar t_2 (\gamma_{21} - e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22})))) \in + O[\epsilon]^2]$$

mexamples

$$\mathbb{C}_{QU}[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11} + l_{12} + l_{21} + l_{22}) t_2 +$$

$$\frac{1}{1 + (-1 + T_2) \gamma_{21}} e^{-\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 (\gamma_{21} + e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \gamma_{12} (1 + (-1 + T_2) \gamma_{21}) +$$

$$e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{11} (e^{\gamma \hbar (l_{11} + l_{21}) t_2} - e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (-1 + T_2) \gamma_{22})),$$

$$\frac{1}{1 + (-1 + T_2) \gamma_{21}} + \frac{1}{4 (1 + (-1 + T_2) \gamma_{21})^5} e^{-2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar$$

$$(8 a_2 T_2 (1 + (-1 + T_2) \gamma_{21})^2 (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (-e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} +$$

$$e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} T_2 + \hbar x_2 y_2) \gamma_{21}^2 + e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{11} \gamma_{22} +$$

$$\gamma_{21} (e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \hbar x_2 y_2 (e^{\gamma \hbar (2 l_{11} + l_{12} + 2 l_{21} + l_{22}) t_2} \gamma_{11} + e^{\gamma \hbar (l_{11} + 2 l_{12} + l_{21} + 2 l_{22}) t_2} \gamma_{22}))) +$$

$$\gamma (2 e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (1 - 4 T_2 + 3 T_2^2) \gamma_{21}^2 (1 + (-1 + T_2) \gamma_{21})^2 +$$

$$4 e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{21} (1 + (-1 + T_2) \gamma_{21}) (\ll 1 \gg) - \ll 1 \gg)) \in + O[\epsilon]^2]$$

R in QU.

The Faddeev-Quesne formula:

Faddeev

$$\text{In[*]:= } e_{q_-,k_-}[X_-] := e^{\sum_{j=1}^{k+1} \frac{(1-q)^j x^j}{j(1-q^j)}}; e_{q,-}[X_-] := e_{q,\$k}[X]$$

Table[Series[e_{q_n,k}[X], {ε, 0, 4}], {k, 0, 5}] // Column

$$e^x$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384} (e^x x^2 (-8 + x^4) \gamma^3 \hbar^3) \epsilon^3 + \frac{e^x x^4 (-32+x^4) \gamma^4 \hbar^4 \epsilon^4}{6144} + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+32x^3+3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{e^x x^3 (-4608-864x+1024x^3+576x^4+27x^5) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+72x^2+32x^3+3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{e^x x^3 (-4608-864x+3616x^3+576x^4+27x^5) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+72x^2+32x^3+3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{1}{4147200} \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

$$e^x x^3 (-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

$$e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \epsilon + \frac{1}{288} e^x x^3 (32 + 9x) \gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+72x^2+32x^3+3x^4) \gamma^3 \hbar^3) \epsilon^3}{1152} + \frac{1}{4147200} \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

$$e^x x^3 (-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

Table[Together@SeriesCoefficient[e_{q₅}[X], {x, 0, n}], {n, 0, 5}]

$$\left\{ 1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2+q^3+q^4)} \right\}$$

Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q₅}[X], {x, 0, n}]], {n, 0, 5}]

{1, 1, 1, 1, 1, 1}

R

$$\text{In[*]:= } QU[R_{i,j}] := O_{QU}[\{y_1, a_1\}_i, \{a_2, x_2\}_j, SS[e^{\hbar b_1 a_2} e_{q_h}[\hbar y_1 x_2] /. b_1 \rightarrow \gamma^{-1} (\epsilon a_1 - t_i)]]; QU[R_{i,j}^{-1}] := S_j @ QU[R_{i,j}];$$

QU[R_{3,4}] // Short

$$QU[] + \frac{\epsilon \hbar QU[a_3, a_4]}{\gamma} + \hbar QU[y_3, x_4] + \frac{\epsilon \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{\gamma} + \frac{1}{2} \frac{\langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{\gamma} - \frac{\langle\langle 1 \rangle\rangle}{\gamma} - \frac{\epsilon \hbar^2 \langle\langle 1 \rangle\rangle t_3}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4] t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ \$p=4, \$k=2):

QU[R_{1,2} ** R_{1,2}⁻¹] // Simp // HL // Timing

{0.078125, QU[]}

Verifying R3 (~156 secs @ \$p=4, \$k=2):

`{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing`

$$\left\{0.203125, \left\{QU\left[\right] + \frac{\epsilon \hbar QU[a_1, a_2]}{\gamma} + \frac{\epsilon \hbar QU[a_1, a_3]}{\gamma} + \ll 73 \gg + 2 \epsilon \hbar^2 QU[y_1, a_2, x_3] T_2 + QU[y_1, x_3] (\hbar - \hbar T_2), \mathbf{0}\right\}\right\}$$

R in \mathbb{C}_{QU} .

RinOE

$$\text{In[*]:= } \mathbb{C}_{QU,k}[R_{i,j}] := \mathbb{C}_{QU}\left[\{y_i, a_i, x_i\}_i, \{y_j, a_j, x_j\}_j, -\hbar \gamma^{-1} t_i a_j + \hbar y_i x_j, \text{Series}\left[e^{\hbar \gamma^{-1} t_i a_j - \hbar y_i x_j} \left(e^{\hbar b_i a_j} e_{q_n, k}[\hbar y_i x_j] / . b_i \rightarrow \gamma^{-1} (\epsilon a_i - t_i)\right), \{\epsilon, \mathbf{0}, k\}\right]\right]$$

$\{\mathbb{C}_{QU,1}[R_{1,2}], \mathbb{C}_{QU,2}[R_{1,2}]\}$

$$\left\{\mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + O[\epsilon]^2\right], \mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + \frac{1}{288 \gamma^2} (144 \hbar^2 a_1^2 a_2^2 - 72 \gamma^2 \hbar^4 a_1 a_2 x_2^2 y_1^2 + 32 \gamma^4 \hbar^5 x_2^3 y_1^3 + 9 \gamma^4 \hbar^6 x_2^4 y_1^4) \epsilon^2 + O[\epsilon]^3\right]\right\}$$

The morphism $\mathbb{C}_{U,k}$.

MorphismOE

$$\text{In[*]:= } \mathbb{C}_{U,k}[a_* b_*] := \mathbb{C}_{U,k}[a] \mathbb{C}_{U,k}[b]; \mathbb{C}_{U,k}[m_{iS}[a_*]] := m_{iS}[\mathbb{C}_{U,k}[a_*]];$$

$\mathbb{C}_{QU,1}[R_{1,2} R_{3,4} R_{5,6}]$

$$\mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar a_2 t_1}{\gamma} - \frac{\hbar a_4 t_3}{\gamma} - \frac{\hbar a_6 t_5}{\gamma} + \hbar x_2 y_1 + \hbar x_4 y_3 + \hbar x_6 y_5, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} + \frac{\hbar a_3 a_4}{\gamma} + \frac{\hbar a_5 a_6}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2 - \frac{1}{4} \gamma \hbar^3 x_4^2 y_3^2 - \frac{1}{4} \gamma \hbar^3 x_6^2 y_5^2\right) \epsilon + O[\epsilon]^2\right]$$

$\mathbb{C}_{QU,1}[R_{1,2} R_{3,4} R_{5,6} // m_{1,3 \rightarrow 1} // m_{2,5 \rightarrow 2} // m_{4,6 \rightarrow 4}]$

$$\mathbb{C}_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma} \left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_2} \gamma \hbar x_4 y_1 - \gamma \hbar T_2 x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2\right), 1 + \frac{1}{4 \gamma} \left(4 \hbar a_1 a_2 + 4 \hbar a_1 a_4 + 4 \hbar a_2 a_4 - 4 \gamma \hbar^2 a_4 x_2 y_1 - 8 e^{\hbar t_2} \gamma \hbar^2 a_2 x_4 y_1 + 8 \gamma \hbar^2 a_2 T_2 x_4 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 + 4 e^{\hbar t_2} \gamma^2 \hbar^3 x_2 x_4 y_1^2 - 4 \gamma^2 \hbar^3 T_2 x_2 x_4 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - e^{2 \hbar t_2} \gamma^2 \hbar^3 x_4^2 y_1^2 + \gamma^2 \hbar^3 T_2^2 x_4^2 y_1^2 - 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 + 4 e^{\hbar t_1 + \hbar t_2} \gamma^2 \hbar^3 x_4^2 y_1 y_2 - 4 e^{\hbar t_1} \gamma^2 \hbar^3 T_2 x_4^2 y_1 y_2 - e^{2 \hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2\right) \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{3,1 \rightarrow 1} // \mathbf{m}_{5,2 \rightarrow 2} // \mathbf{m}_{6,4 \rightarrow 4}]$$

$$\mathbb{C}_{\text{QU}} \left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma} \right. \\ \left. \left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2 \right), \right. \\ \left. 1 + \frac{1}{4\gamma} \left(4 \hbar a_1 a_2 + 4 \hbar a_1 a_4 + 4 \hbar a_2 a_4 - 4 \gamma \hbar^2 a_4 x_2 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - \right. \right. \\ \left. \left. 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 - e^{2\hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2 \right) \in + \mathbf{O}[\epsilon]^2 \right]$$

$$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{1,3 \rightarrow 1} // \mathbf{m}_{2,5 \rightarrow 2} // \mathbf{m}_{4,6 \rightarrow 4}] \equiv \mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{3,1 \rightarrow 1} // \mathbf{m}_{5,2 \rightarrow 2} // \mathbf{m}_{6,4 \rightarrow 4}]$$

$$\hbar \left(e^{\hbar t_2} - T_2 \right) x_4 y_1 = \mathbf{0} \&\& \in \hbar \left(e^{\hbar t_2} - T_2 \right) x_4 y_1 \left(8 a_2 + \gamma \hbar \left(-4 x_2 y_1 + x_4 \left(\left(e^{\hbar t_2} + T_2 \right) y_1 - 4 e^{\hbar t_1} y_2 \right) \right) \right) = \mathbf{0}$$

Exponentials as needed.

ReqExp

```
In[ ]:= Block[{$p = 2, $k = 2}, TableForm[StringSplit[
  "y | a | x | C@y_CU | C@a_CU | C@x_CU | Q@y_QU | Q@a_QU | Q@x_QU | AD@y_QU | AD@a_QU | AD@x_QU | SD@y_QU | SD@a_QU | SD
  @x_QU | S@y_CU | S@a_CU | S@x_CU | S@y_QU | S@a_QU | S@x_QU | Δ@y_CU | Δ@a_CU | Δ@x_CU | Δ@y_QU | Δ@a_QU | Δ@x_QU",
  "|"] /. s_String =>
  {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ε, 0, $k}]]]]
```

Out[]//TableForm=
 ReqExp

y	y
a	a
x	x
C@y_CU	-x
C@a_CU	-a
C@x_CU	-y
Q@y_QU	$-\frac{x}{\sqrt{T}} - \frac{a x \epsilon \hbar}{\sqrt{T}} - \frac{a^2 x \epsilon^2 \hbar^2}{2\sqrt{T}}$
Q@a_QU	-a
Q@x_QU	$-\frac{y}{\sqrt{T}} + \frac{y(-a+\gamma)\epsilon\hbar}{\sqrt{T}} - \frac{y(a-\gamma)^2\epsilon^2\hbar^2}{2\sqrt{T}}$
AD@y_QU	$\frac{2}{3} a^2 y \epsilon^2 \hbar^2 + \frac{1}{6} y (6 + 3 t \hbar + t^2 \hbar^2) + \frac{1}{12} y \epsilon \hbar (x y \gamma \hbar - 4 a (3 + 2 t \hbar))$
AD@a_QU	a
AD@x_QU	x
SD@y_QU	$y + \frac{1}{48} t^2 y \hbar^2 + \frac{1}{24} y (-2 a t + x y \gamma) \epsilon \hbar^2 + \frac{1}{12} a^2 y \epsilon^2 \hbar^2$
SD@a_QU	a
SD@x_QU	$\frac{7}{12} a^2 x \epsilon^2 \hbar^2 + x \left(1 + \frac{t \hbar}{2} + \frac{7 t^2 \hbar^2}{48}\right) + \frac{1}{24} x \epsilon \hbar (x y \gamma \hbar - 2 a (12 + 7 t \hbar))$
S@y_CU	-y
S@a_CU	-a
S@x_CU	-x
S@y_QU	$-\frac{y}{T} + \frac{y(-a+\gamma)\epsilon\hbar}{T} - \frac{y(a-\gamma)^2\epsilon^2\hbar^2}{2T}$
S@a_QU	-a
S@x_QU	$-x - a x \epsilon \hbar - \frac{1}{2} a^2 x \epsilon^2 \hbar^2$
Δ@y_CU	y ₁ + T ₁ y ₂
Δ@a_CU	a ₁ + a ₂
Δ@x_CU	x ₁ + x ₂
Δ@y_QU	y ₁ + T ₁ y ₂ - ε ħ a ₁ T ₁ y ₂ + $\frac{1}{2} \epsilon^2 \hbar^2 a_1^2 T_1 y_2$
Δ@a_QU	a ₁ + a ₂
Δ@x_QU	x ₁ + x ₂ - ε ħ a ₁ x ₂ + $\frac{1}{2} \epsilon^2 \hbar^2 a_1^2 x_2$

Exp

Task. Define $\text{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi Q(P)}$ to ϵ^k in the algebra U_i , where ξ is a scalar, X is x_i or y_i , and P is an ϵ -dependent near-docile element, giving the answer in $\mathbb{C}\epsilon$ -form. Should satisfy $U @ \text{Exp}_{U_i,k}[\xi, P] == \mathbf{S}_U[e^{\xi X}, X \rightarrow Q(P)]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi Q(P)} = Q(e^{\xi P_0} F(\xi))$, then $F(\xi = 0) = 1$ and we have:
 $Q(e^{\xi P_0} (P_0 F(\xi) + \partial_\xi F)) = Q(\partial_\xi e^{\xi P_0} F(\xi)) = \partial_\xi Q(e^{\xi P_0} F(\xi)) = \partial_\xi e^{\xi Q(P)} = e^{\xi Q(P)} Q(P) = Q(e^{\xi P_0} F(\xi)) Q(P)$.
 This is an ODE for F . Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ .

Exp

In[*]:=

```
(* Bug: The first line is valid only if 0 (e^{P_0}) == e^{0 (P_0)}. *)
Exp_{U_i,0}[\xi_, P_] := \mathbb{C}_U[\{y_i, a_i, x_i\}_i, Normal@P /. \epsilon \to 0, 1 + \theta_0];
Exp_{U_i,k}[\xi_, P_] := Module[\{yax = \{y_i, a_i, x_i\}, P0, \varphi, \varphi_S, F, j, rhs, at0, at\xi\},
  P0 = Normal@P /. \epsilon \to 0;
  \varphi_S =
    Flatten@Table[\varphi_{j1,j2,j3}[\xi], \{j2, 0, k\}, \{j1, 0, 2k+1-j2\}, \{j3, 0, 2k+1-j2-j1\}];
  F = Normal@Last@Exp_{U_i,k-1}[\xi, P] + \epsilon^k \varphi_S. (\varphi_S /. \varphi_{j_S}[\xi] \to Times@@yax^{(j_S)});
  rhs = Normal@Last@m_{i,j \to i}[\mathbb{C}_U[yax_i, \xi P0, F + \theta_k] m_{i \to j} @ \mathbb{C}_U[\{y_i, a_i, x_i\}_i, 0, P + \theta_k]];
  at0 = (\# == 0) & /@ Flatten@CoefficientList[F - 1 /. \xi \to 0, yax];
  at\xi = (\# == 0) & /@ Flatten@CoefficientList[(\partial_\xi F) + P0 F - rhs, yax];
  \mathbb{C}_U[yax_i, \xi P0, F + \theta_k] /. DSolve[And@@(at0 \cup at\xi), \varphi_S, \xi] [[1]] ]
```

In[*]:=

```
Timing@Block[\{\$p = 4, \$k = 2\}, {
  s = S_1[QU[y_1]] /. QU \to Times,
  exps = Exp_{QU_i, \$k}[\eta, s], (* Warning: wrong unless \$p \ge \$k+1! *)
  HL@Simp[S_1@O_{QU}[\{y_1\}_1, SS[e^{\hbar \eta y_1}]] - QU@(exps /. \eta \to \hbar \eta)]
}]
```

$$\text{Out[*]} = \left\{ 35.8281, \left\{ a_1 \left(-\frac{\epsilon \hbar}{T_1} + \frac{\gamma \epsilon^2 \hbar^2}{T_1} \right) y_1 + \left(-\frac{1}{T_1} + \frac{\gamma \epsilon \hbar}{T_1} - \frac{\gamma^2 \epsilon^2 \hbar^2}{2 T_1} \right) y_1 - \frac{\epsilon^2 \hbar^2 a_1^2 y_1}{2 T_1}, \right. \right.$$

$$\mathbb{C}_{QU}[\{y_1, a_1, x_1\}_1, -\frac{\eta y_1}{T_1}, 1 + \frac{(2 \gamma \eta \hbar T_1 y_1 - 2 \eta \hbar a_1 T_1 y_1 - \gamma \eta^2 \hbar y_1^2) \epsilon}{2 T_1^2} +$$

$$\left(-\frac{\gamma^2 \eta \hbar^2 y_1}{2 T_1} + \frac{\gamma \eta \hbar^2 a_1 y_1}{T_1} - \frac{\eta \hbar^2 a_1^2 y_1}{2 T_1} + \frac{7 \gamma^2 \eta^2 \hbar^2 y_1^2}{4 T_1^2} - \frac{2 \gamma \eta^2 \hbar^2 a_1 y_1^2}{T_1^2} + \right.$$

$$\left. \left. \frac{\eta^2 \hbar^2 a_1^2 y_1^2}{2 T_1^2} - \frac{\gamma^2 \eta^3 \hbar^2 y_1^3}{T_1^3} + \frac{\gamma \eta^3 \hbar^2 a_1 y_1^3}{2 T_1^3} + \frac{\gamma^2 \eta^4 \hbar^2 y_1^4}{8 T_1^4} \right) \epsilon^2 + O[\epsilon]^3, \mathbf{0} \right\}$$

In[*]:=

```
Timing@Block[\{\$p = 4, \$k = 2\}, {
  s = S_1[QU[a_1]] /. QU \to Times,
  exps = Exp_{QU_i, \$k}[\alpha, s], (* Warning: wrong unless \$p \ge \$k+1! *)
  HL@Simp[S_1@O_{QU}[\{a_1\}_1, SS[e^{\hbar \alpha a_1}]] - QU@(exps /. \alpha \to \hbar \alpha)]
}]
```

$$\text{Out[*]} = \left\{ 33.5938, \left\{ -a_1, \mathbb{C}_{QU}[\{y_1, a_1, x_1\}_1, -\alpha a_1, 1 + O[\epsilon]^3], \mathbf{0} \right\} \right\}$$

In[*]:=

```
Timing@Block[\{\$p = 4, \$k = 2\}, {
  s = S_1[QU[x_1]] /. QU \to Times,
  exps = Exp_{QU_i, \$k}[\xi, s], (* Warning: wrong unless \$p \ge \$k+1! *)
  HL@Simp[S_1@O_{QU}[\{x_1\}_1, SS[e^{\hbar \xi x_1}]] - QU@(exps /. \xi \to \hbar \xi)]
}]
```

$$\text{Out[*]} = \left\{ 34.0625, \left\{ -x_1 - \epsilon \hbar a_1 x_1 - \frac{1}{2} \epsilon^2 \hbar^2 a_1^2 x_1, \right. \right.$$

$$\mathbb{C}_{QU}[\{y_1, a_1, x_1\}_1, -\xi x_1, 1 + \left(-\xi \hbar a_1 x_1 - \frac{1}{2} \gamma \xi^2 \hbar x_1^2 \right) \epsilon + \left(-\frac{1}{2} \xi \hbar^2 a_1^2 x_1 + \frac{1}{4} \gamma^2 \xi^2 \hbar^2 x_1^2 - \right.$$

$$\left. \left. \gamma \xi^2 \hbar^2 a_1 x_1^2 + \frac{1}{2} \xi^2 \hbar^2 a_1^2 x_1^2 - \frac{1}{2} \gamma^2 \xi^3 \hbar^2 x_1^3 + \frac{1}{2} \gamma \xi^3 \hbar^2 a_1 x_1^3 + \frac{1}{8} \gamma^2 \xi^4 \hbar^2 x_1^4 \right) \epsilon^2 + O[\epsilon]^3, \mathbf{0} \right\}$$

Alternative Algorithms

AltLogos

In[]:=

```

λalt,k[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
  eq = ρ@eξxcu.ρ@eηycu == ρ@edycu.ρ@ec(t1cu - 2εacu).ρ@ebxcu;
  {so} = Solve[Thread[Flatten/@eq], {d, b, c}] /. C@1 → 0;
  Series[e-ηy-ξx+ηξt+c t+d y-2εca+bx /. so, {ε, 0, k}]]];

```

```

{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[Δcu,2[{ξ, η}, {x, y}]]]}

```

$$\left\{ 1 + \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right) \epsilon + \frac{1}{2} \left(\left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right)^2 + 2 \left(-a \gamma \eta^2 \xi^2 + y \gamma^2 \eta^3 \xi^2 + x \gamma^2 \eta^2 \xi^3 - \frac{1}{3} t \gamma^2 \eta^3 \xi^3 \right) \right) \epsilon^2 + O[\epsilon]^3, \text{True} \right\}$$
