Pensieve header: A unified verification notebook for the $sl_2$-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

## Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

## Initialization / Utilities

It is verification-risky to work with low $E!

TD

```
$p = 2; $k = 1; $E := {$k, $p};
$trim := {ħ^(p_.) /; p > $p → 0, ϵ^(k_.) /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ħ t_i), T → e^(ħ t)}; q_ħ = e^(γ ϵ ħ);
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. TRule], {ħ, 0, $p}],
   ħ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. TRule, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) :→ c ∂_{{x,m},{y,n}} λ]
```

```
HL[DP_{x→D_ξ, y→D_η}[x^2 y^3][e^(δ ξ η)] == 6 e^(δ η ξ) δ^3 ξ + 6 e^(δ η ξ) δ^4 η ξ^2 + e^(δ η ξ) δ^5 η^2 ξ^3]
```

```
True
```

CF

```
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CF[x]];
```

SeriesData

```
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

SP

```
SP_{}[P_] := P; SP_{ξ_→x_,ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^d_. ⧴ ∂_{x,d} f
```

$SP_{\{\xi\to x\}}\left[\left(\xi^2+\xi+3\right)\left(x^5 e^x+7x\right)+99a\right]$

$7+99a+21x+20e^x x^3+15e^x x^4+5e^x x^5$

$SP_{\{\xi\to x,\eta\to y\}}\left[\left(\xi^2+\xi+3+2\xi\eta\right)\left(x^5 e^x+7x\right)+99a+e^{\delta x y}\xi\eta\right]$

$7+99a+21x+20e^x x^3+15e^x x^4+5e^x x^5+e^{x y \delta}\delta+e^{x y \delta} x y \delta^2$

## DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
  U_i_[ℰ_] := ℰ /. {t : cp :> t_i, u_U :> (#_i &) /@ u};
  U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List :> l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ :> (l /. x_i_ :> x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
      ]] /. x_null :> x];
  pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) :> c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  m_{j_→k_}[c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _{j|k}]) **
     U @@ Cases[u, w_j :> w_k] ** U @@ Cases[u, _k]];
  U /: c_. * u_U * v_U := CE[c u ** v];
  S_i_[c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
     U_i[NCM @@ Reverse@Cases[u, x_i :> S@U@x]] ]
```

## DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
   {(g_ → img_) ⧴ (m[U[g]] = img), (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

## Meta-Operations

QLImplementation

```
m_{j_→j_} = Identity;
m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
m_{is___,i_,j_→k_}[ℰ_] := m_{j→k} @ m_{is,i→j} @ ℰ;
S_{i_}[ℰ_Plus] := Simp[S_i /@ ℰ];
```

## Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_{i_}[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.84375,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of $S$ on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t1 t3 CU[y1, a1, x2] + t1 t1 CU[y1, a2, x2] // m1→3
```

$\text{CU}[a_2, x_2, y_3]\, t_3^2 + \text{CU}[x_2, y_3, a_3]\, t_3^2$

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z⟦1⟧1, z⟦2⟧2, z⟦3⟧3]; z → HL[m1,3→3@m2,3→3@u == m2,3→3@m1,2→2@u],
  {z, Tuples[{y, a, x}, 3]}]]
```

```
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

## Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T e^(-2 ϵ a ℏ)) / ℏ, {a, 0, 3}]
```

Aside

$$\frac{1-T}{\hbar} + 2 T \epsilon a - 2 \left(T \epsilon^2 \hbar\right) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[aQU, yQU] = -γ yQU; B[xQU, aQU] = -γ QU@x;
B[xQU, yQU] := SS[qℏ - 1] QU@{y, x} + 𝕆QU[{a}, SS[(1 - T e^(-2 ϵ a ℏ)) / ℏ]];
(S@yQU := 𝕆QU[{a, y}, SS[-T^-1 e^(ℏ ϵ a) y]]; S@aQU = -aQU; S@xQU := 𝕆QU[{a, x}, SS[-e^(ℏ ϵ a) x]];)
Si_[QU, Centrals] = {ti → -ti, Ti → Ti^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$\{\{\{QU[y], QU[y]\} \to 0, \{QU[y], QU[a]\} \to \gamma\, QU[y],$
$\{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2\,T\,\epsilon\, QU[a] - \gamma\,\epsilon\,\hbar\, QU[y, x]\},$
$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y], \{QU[a], QU[a]\} \to 0, \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$
$\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2\,T\,\epsilon\, QU[a] + \gamma\,\epsilon\,\hbar\, QU[y, x],$
$\{QU[x], QU[a]\} \to -\gamma\, QU[x], \{QU[x], QU[x]\} \to 0\}\}$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

$\{4.67188, \{\left(\frac{28\gamma^4 - 56\,T\,\gamma^4 + 28\,T^2\gamma^4}{\hbar^2} + \frac{82\gamma^5\epsilon - 280\,T \ll 1 \gg \epsilon + 198\,T^2\gamma^5\epsilon}{\hbar}\right) QU[y, y, y, x, x] +$
$\ll 18 \gg + (1 + 8\gamma\,\epsilon\,\hbar)\, QU[y, \ll 11 \gg, x], 0\}\}$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

$\{\{\{QU[y_1], QU[y_1]\} \to 0, \{QU[y_1], QU[a_1]\} \to 0, \{QU[y_1], QU[x_1]\} \to 0\},$
$\{\{QU[a_1], QU[y_1]\} \to 0, \{QU[a_1], QU[a_1]\} \to 0, \{QU[a_1], QU[x_1]\} \to 0\},$
$\{\{QU[x_1], QU[y_1]\} \to 0, \{QU[x_1], QU[a_1]\} \to 0, \{QU[x_1], QU[x_1]\} \to 0\}\}$

Verifying that $\mathrm{Lim}_{\hbar\to 0}\, QU = CU$ using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{12.5625, \{28\,t^2\gamma^4\, CU[y, y, y, x, x] +$
$116\,t\,\gamma^5\epsilon\, CU[y, y, y, x, x] + \ll 44 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],$
$2\left(\frac{\gamma^4}{\hbar^2} - \frac{2\,T\,\gamma^4}{\hbar^2} + \frac{T^2\gamma^4}{\hbar^2} + \frac{\gamma^5\epsilon}{\hbar} - \frac{2\,T\,\gamma^5\epsilon}{\hbar} + \frac{T^2\gamma^5\epsilon}{\hbar}\right) QU[y, y, y, x, x] +$
$\ll 209 \gg + (1 + 8\gamma\,\epsilon\,\hbar)\, QU[y, y, y, \ll 7 \gg, x, x, x], 0\}\}$

## Implementing $\theta$

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y :→ 𝕆_QU[{a, x}, SS[-T^-1/2 ℯ^(ℏ ϵ a) x]],
  a → -a_QU, x :→ 𝕆_QU[{a, y}, SS[-T^-1/2 ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → −CU[x] → CU[y], CU[a] → −CU[a] → CU[a], CU[x] → −CU[y] → CU[x]}

Verifying that $\theta$ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$$
$$\left. QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\right\}$$

Verifying that $\theta$ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

## The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\text{A}\mathbb{D}\$f = \gamma\left(\left(\text{Cosh}\left[\hbar\left(a\epsilon + \frac{\gamma\epsilon}{2} - \frac{t}{2}\right)\right] - \text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\epsilon}{2}\right)^2 + \epsilon\omega}\right]\right)\Big/\left(\hbar\, e^{\hbar((a+\gamma)\epsilon - t/2)}\,\text{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(a^2\epsilon + a\gamma\epsilon - a t - \omega\right)\right)\right);$$

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

ADeq

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[A𝔻, QU → CU,
 {a → a_CU, x → CU@x, y :→ 𝕊_CU[SS[A𝔻$f], a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

## The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$S\mathbb{D}\$g = \sqrt{\left(\left(2\,\gamma\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\,\epsilon^2+4\,\epsilon\,\varpi}\right]-\mathrm{Cosh}\left[\frac{t-\epsilon\,\gamma-2\,\epsilon\,a}{2/\hbar}\right]\right)\right)\Big/\left(\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(t\,(2\,a+\gamma)-2\,a\,(a+\gamma)\,\epsilon+2\,\varpi\right)\hbar\right)\right)};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ℏ ((ϵ-t)/2 + ϵ a)] - Cosh[ℏ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) / (ℏ Sinh[-ϵ ℏ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a(t-\epsilon)-a^2\epsilon+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$

$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\epsilon+4\,\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True}\right\}$$

SDeq

```
SD$f = Simplify[ℯ^(ℏ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x :→ S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y :→ S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the $\theta$-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

## The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = (0 (1 - e^(-γ ϵ ℏ)) / (ϵ ℏ); 0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] := (ℰ /. TRule /. t → γ ϵ /. (U : CU | QU) [u___] ↦ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ϵ^(k_·) /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[e^(ξ U@x)].ρ[e^(α U@a)] == ρ[e^(α U@a)].ρ[e^(e^(-γ α) ξ U@x)]], {U, {CU, QU}}]
{True, True}
```

## Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

MultiplyingOEs

```
Œ_U_[s1__, Q1_, P1_] Œ_U_[s2__, Q2_, P2_] ^:= Œ_U[s1, s2, Q1 + Q2, P1 + P2];
```

CdsO

```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^(Q+P)]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^(Q+P)]];
```

Logos

```
O_(k_Integer) := O[ϵ]^(k+1);
Λ_(U_,k_)[{α_, β_}, {x_, x_}] := Œ_U[{x}, (α + β) x, O_k];
Λ_(U_,k_)[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-γ α) ξ x, O_k];
Λ_(U_,k_)[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-γ α) η y, O_k];
```

```
Table[
 {Λ_{U,1}[{α, β}, {u, u}],
  lhs = U@𝔈_U[{u_1, u_2}, ℏ (α u_1 + β u_2), θ_1], HL[lhs == U@Λ_{U,1}[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

$$\left\{\left\{\left\{\mathbb{E}_{CU}\left[\{y\},\ y\,(\alpha+\beta),\ O[\epsilon]^2\right],\ CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[y,y],\ \text{True}\right\},\right.\right.$$

$$\left\{\mathbb{E}_{CU}\left[\{a\},\ a\,(\alpha+\beta),\ O[\epsilon]^2\right],\ CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[a,a],\ \text{True}\right\},$$

$$\left.\left\{\mathbb{E}_{CU}\left[\{x\},\ x\,(\alpha+\beta),\ O[\epsilon]^2\right],\ CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[x,x],\ \text{True}\right\}\right\},$$

$$\left\{\left\{\mathbb{E}_{QU}\left[\{y\},\ y\,(\alpha+\beta),\ O[\epsilon]^2\right],\ QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[y,y],\ \text{True}\right\},\right.$$

$$\left\{\mathbb{E}_{QU}\left[\{a\},\ a\,(\alpha+\beta),\ O[\epsilon]^2\right],\ QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[a,a],\ \text{True}\right\},$$

$$\left.\left.\left\{\mathbb{E}_{QU}\left[\{x\},\ x\,(\alpha+\beta),\ O[\epsilon]^2\right],\ QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[x,x],\ \text{True}\right\}\right\}\right\}$$

```
{Λ_{#,1}[{ξ, α}, {x, a}], lhs = #@𝔈_#[{x, a}, ℏ (ξ x + α a), θ_1],
   HL[lhs == #@Λ_{#,1}[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$$\left\{\left\{\mathbb{E}_{CU}\left[\{a,x\},\ a\,\alpha+e^{-\alpha\gamma}\,x\,\xi,\ O[\epsilon]^2\right],\right.\right.$$
$$CU[\,]+\alpha\hbar\,CU[a]+\left(\xi\hbar-\alpha\gamma\xi\hbar^2\right)CU[x]+\frac{1}{2}\alpha^2\hbar^2\,CU[a,a]+\alpha\xi\hbar^2\,CU[a,x]+\frac{1}{2}\xi^2\hbar^2\,CU[x,x],$$
$$\left.\text{True}\right\},\ \left\{\mathbb{E}_{QU}\left[\{a,x\},\ a\,\alpha+e^{-\alpha\gamma}\,x\,\xi,\ O[\epsilon]^2\right],\ QU[\,]+\alpha\hbar\,QU[a]+\right.$$
$$\left.\left.\left(\xi\hbar-\alpha\gamma\xi\hbar^2\right)QU[x]+\frac{1}{2}\alpha^2\hbar^2\,QU[a,a]+\alpha\xi\hbar^2\,QU[a,x]+\frac{1}{2}\xi^2\hbar^2\,QU[x,x],\ \text{True}\right\}\right\}$$

```
{Λ_{#,2}[{α, η}, {a, y}], lhs = #@𝔈_#[{a, y}, ℏ (η y + α a), θ_2],
   HL[lhs == #@Λ_{#,2}[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$$\left\{\left\{\mathbb{E}_{CU}\left[\{y,a\},\ a\,\alpha+e^{-\alpha\gamma}\,y\,\eta,\ O[\epsilon]^3\right],\right.\right.$$
$$CU[\,]+\alpha\hbar\,CU[a]+\left(\eta\hbar-\alpha\gamma\eta\hbar^2\right)CU[y]+\frac{1}{2}\alpha^2\hbar^2\,CU[a,a]+\alpha\eta\hbar^2\,CU[y,a]+\frac{1}{2}\eta^2\hbar^2\,CU[y,y],$$
$$\left.\text{True}\right\},\ \left\{\mathbb{E}_{QU}\left[\{y,a\},\ a\,\alpha+e^{-\alpha\gamma}\,y\,\eta,\ O[\epsilon]^3\right],\ QU[\,]+\alpha\hbar\,QU[a]+\right.$$
$$\left.\left.\left(\eta\hbar-\alpha\gamma\eta\hbar^2\right)QU[y]+\frac{1}{2}\alpha^2\hbar^2\,QU[a,a]+\alpha\eta\hbar^2\,QU[y,a]+\frac{1}{2}\eta^2\hbar^2\,QU[y,y],\ \text{True}\right\}\right\}$$

Goal. In either $U$, compute $F=e^{-\eta y}e^{\xi x}e^{\eta y}e^{-\xi x}$. First compute $G = e^{\xi x}\,y\,e^{-\xi x}$, a finite sum. Now $F$ satisfies the ODE $\partial_\eta F=\partial_\eta\left(e^{-\eta y}e^{\eta G}\right)=-yF+FG$ with initial conditions $F(\eta=0)=1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {ⅇ- → 1, U → Times}];
   Collect[sol /. {ⅇ- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

```
-t ξ CU[] + 2 ϵ ξ CU[a] - γ ϵ ξ^2 CU[x] + CU[y]

{f_{0,0,0,0}[η], f_{1,0,0,0}[η], f_{1,0,0,1}[η], f_{1,0,1,0}[η],
 f_{1,0,1,1}[η], f_{1,1,0,0}[η], f_{1,1,0,1}[η], f_{1,1,1,0}[η], f_{1,1,1,1}[η]}

CU[] f_{0,0,0,0}[η] + ϵ CU[] f_{1,0,0,0}[η] + ϵ CU[x] f_{1,0,0,1}[η] + ϵ CU[a] f_{1,0,1,0}[η] + ϵ CU[a, x] f_{1,0,1,1}[η] +
 ϵ CU[y] f_{1,1,0,0}[η] + ϵ CU[y, x] f_{1,1,0,1}[η] + ϵ CU[y, a] f_{1,1,1,0}[η] + ϵ CU[y, a, x] f_{1,1,1,1}[η]
```

$$e^{-t\eta\xi} CU[] + \frac{1}{2} e^{-t\eta\xi} t\gamma\epsilon\eta^2\xi^2 CU[] + 2 e^{-t\eta\xi}\epsilon\eta\xi CU[a] - e^{-t\eta\xi}\gamma\epsilon\eta\xi^2 CU[x] - e^{-t\eta\xi}\gamma\epsilon\eta^2\xi CU[y]$$

$$1 + 2 a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2} t\gamma\epsilon\eta^2\xi^2$$

$$1 + \frac{1}{2}\epsilon\eta\xi\left(4a + \gamma\left(-2y\eta - 2x\xi + t\eta\xi\right)\right)$$

Logos

```
Λ_{U_,kk_}[{ξ1_, η1_}, {x, y}] :=
  Λ_U[{ξ1, η1}, {x, y}] = Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
     G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
     fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
     F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
     es = Flatten[
       Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
     F = F /. DSolve[es, fs, η][[1]];
     Œ_U[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
      Log[F + O_kk /. {ⅇ- → 1, U → Times}]
     ] /. {ξ → ξ1, η → η1}]];
```

```
{Λ_{CU,1}[{ξ, η}, {x, y}], lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y), O_1],
 HL[lhs == CU@Λ_{CU,1}[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{Œ_{CU}\left[\{y,a,x\},\ y\eta + x\xi - t\eta\xi,\ \left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right)\epsilon + O[\epsilon]^2\right],\right.$$
$$\left(1 - t\eta\xi\hbar^2\right) CU[] + 2\epsilon\eta\xi\hbar^2 CU[a] + \xi\hbar CU[x] + \eta\hbar CU[y] +$$
$$\left.\frac{1}{2}\xi^2\hbar^2 CU[x,x] + \eta\xi\hbar^2 CU[y,x] + \frac{1}{2}\eta^2\hbar^2 CU[y,y],\ \text{True}\right\}$$

```
{Λ_{QU,1}[{ξ, η}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y), θ₁],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{Œ_{QU}\left[\{y, a, x\}, y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar}, \frac{1}{4\hbar}\left(\gamma\,\eta^2\,\xi^2 - 4\,T\,\gamma\,\eta^2\,\xi^2 + 3\,T^2\,\gamma\,\eta^2\,\xi^2 + 8\,a\,T\,\eta\,\xi\,\hbar + 2\,y\,\gamma\,\eta^2\,\xi\,\hbar - 6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar + 2\,x\,\gamma\,\eta\,\xi^2\,\hbar - 6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2\right)\epsilon + O[\epsilon]^2\right],\right.$$

$$\left.(1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar)\,QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] + \eta\,\hbar\,QU[y] + \frac{1}{2}\xi^2\,\hbar^2\,QU[x,x] + \eta\,\xi\,\hbar^2\,QU[y,x] + \frac{1}{2}\eta^2\,\hbar^2\,QU[y,y], \text{True}\right\}$$

```
{tt = Last[Λ_{CU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand
```

$$\left\{1 + \left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(2\,a^2\,\eta^2\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 2\,a\,y\,\gamma\,\eta^3\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + \frac{1}{2}y^2\,\gamma^2\,\eta^4\,\xi^2 - 2\,a\,x\,\gamma\,\eta^2\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 + a\,t\,\gamma\,\eta^3\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3 + x\,y\,\gamma^2\,\eta^3\,\xi^3 - \frac{1}{2}t\,y\,\gamma^2\,\eta^4\,\xi^3 + \frac{1}{2}x^2\,\gamma^2\,\eta^2\,\xi^4 - \frac{1}{2}t\,x\,\gamma^2\,\eta^3\,\xi^4 + \frac{1}{8}t^2\,\gamma^2\,\eta^4\,\xi^4\right)\epsilon^2 + O[\epsilon]^3,\right.$$

$$\left.\left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(-a\,\gamma\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + x\,\gamma^2\,\eta^2\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3\right)\epsilon^2 + O[\epsilon]^3, 6\right\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{1+\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+$$

$$\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2+2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+$$

$$a\,T\,y\,\gamma\,\eta^3\,\xi^2-3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$

$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2+\frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2+a\,T\,x\,\gamma\,\eta^2\,\xi^3-3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3+3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3+\frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4-$$

$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}-\frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+\frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2}-\frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+$$

$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}+\frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}-\frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar}+\frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}+\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+$$

$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}+\frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+\frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+$$

$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}+\frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-$$

$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$

$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$

$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2+\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,$$

$$\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-$$

$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-$$

$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+$$

$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-$$

$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

Logos

```
(*SimpPQ[PQ_] := Expand@Collect[PQ, {x_,y_,a_},CC]  /. e^δ_ :> e^Expand@Together[δ] /.
   CC→ExpandDenominator@*ExpandNumerator@*Together;*)
SimpPQ[PQ_] := Simplify[PQ //. e^δ_ :> e^Simplify[δ]];
Simp[𝔼_U_[specs___, Q_, P_]] := 𝔼_U[specs, CF[Q], CF[P]];
```

Logos

```
Λ_{U_,k_}[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
    {yax, q, p} = List @@ Λ_{U,k}[{υ, ω}, {u, w}];
    𝔼_U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ), Log[
        Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[e^p][e^Q]] + θ_k]] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
 {Λ_{CU,$k}[ℏ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@𝔼_CU[{x, y}, ℏ (ξ x + η y + δ x y), θ_$k], 5],
   HL@Simp[lhs - CU@Λ_{CU,$k}[ℏ {ξ, η, δ}, {x, y}]]}
]
```

$$\Big\{\mathbb{E}_{CU}\Big[\{y, a, x\}, \frac{x y \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar},$$

$$\mathrm{Log}\Big[\frac{1}{1 + t\delta\hbar}\Big] + \Big(\Big(4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 - 4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 + 4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 - 6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 + 8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 + 4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 + 4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 + 4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4\Big) \epsilon\Big) \Big/ \Big(2 + 8 t \delta \hbar + 12 t^2 \delta^2 \hbar^2 + 8 t^3 \delta^3 \hbar^3 + 2 t^4 \delta^4 \hbar^4\Big) + O[\epsilon]^2\Big],$$

$$\Big(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 + 2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 - 9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4\Big) CU[] +$$

$$\Big(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 + 18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4\Big) CU[a] +$$

$$\ll 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 \, CU[y, y, y, y, x, x, x] +$$

$$\frac{1}{24} \delta^4 \hbar^4 \, CU[y, y, y, y, x, x, x, x], 0\Big\}$$

```
{ΛQU,2[{ξ, η, δ}, {x, y}], lhs = QU@ŒQU[{x, y}, ℏ (ξ x + η y + δ x y), θ2],
 HL@SimpT[lhs == QU@ΛQU,1[ℏ {ξ, η, δ}, {x, y}]]}
```

$\{\text{Œ}_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$
$\text{Log}\left[\frac{1}{1-\frac{(1-T)\,\delta}{\hbar}}\right] + \big(\big(-8\,a\,T\,\delta^4\,\hbar + 24\,a\,T^2\,\delta^4\,\hbar - 24\,a\,T^3\,\delta^4\,\hbar + 8\,a\,T^4\,\delta^4\,\hbar + \cdots 149 \cdots +$
$4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^5 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^5 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^5 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^5\big)\,\epsilon\big) /$
$\big(4\,\delta^4 - 16\,T\,\delta^4 + 24\,T^2\,\delta^4 - 16\,T^3\,\delta^4 + 4\,T^4\,\delta^4 - 16\,\delta^3\,\hbar + 48\,T\,\delta^3\,\hbar - 48\,T^2\,\delta^3\,\hbar +$
$16\,T^3\,\delta^3\,\hbar + 24\,\delta^2\,\hbar^2 - 48\,T\,\delta^2\,\hbar^2 + 24\,T^2\,\delta^2\,\hbar^2 - 16\,\delta\,\hbar^3 + 16\,T\,\delta\,\hbar^3 + 4\,\hbar^4\big) +$
$\frac{(\cdots 1 \cdots)\,\cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3],\ \cdots 1 \cdots,\ \text{True}\}$

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```
{tt = ComposeSeries[(1 + t δ) Last[ΛCU,2[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$\{\big(2\,a\,\delta + 6\,a\,t\,\delta^2 + 2\,a\,x\,y\,\delta^2 + t\,\gamma\,\delta^2 - 4\,x\,y\,\gamma\,\delta^2 + 6\,a\,t^2\,\delta^3 + 4\,a\,t\,x\,y\,\delta^3 + 2\,t^2\,\gamma\,\delta^3 - 6\,t\,x\,y\,\gamma\,\delta^3 -$
$2\,x^2\,y^2\,\gamma\,\delta^3 + 2\,a\,t^3\,\delta^4 + 2\,a\,t^2\,x\,y\,\delta^4 + t^3\,\gamma\,\delta^4 - 2\,t^2\,x\,y\,\gamma\,\delta^4 - \frac{3}{2}\,t\,x^2\,y^2\,\gamma\,\delta^4 + 2\,a\,y\,\delta\,\eta -$
$2\,y\,\gamma\,\delta\,\eta + 4\,a\,t\,y\,\delta^2\,\eta - 2\,t\,y\,\gamma\,\delta^2\,\eta - 3\,x\,y^2\,\gamma\,\delta^2\,\eta + 2\,a\,t^2\,y\,\delta^3\,\eta - 2\,t\,x\,y^2\,\gamma\,\delta^3\,\eta -$
$y^2\,\gamma\,\delta\,\eta^2 - \frac{1}{2}\,t\,y^2\,\gamma\,\delta^2\,\eta^2 + 2\,a\,x\,\delta\,\xi - 2\,x\,\gamma\,\delta\,\xi + 4\,a\,t\,x\,\delta^2\,\xi - 2\,t\,x\,\gamma\,\delta^2\,\xi - 3\,x^2\,y\,\gamma\,\delta^2\,\xi +$
$2\,a\,t^2\,x\,\delta^3\,\xi - 2\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 2\,a\,\eta\,\xi + 4\,a\,t\,\delta\,\eta\,\xi + 2\,t\,\gamma\,\delta\,\eta\,\xi - 4\,x\,y\,\gamma\,\delta\,\eta\,\xi + 2\,a\,t^2\,\delta^2\,\eta\,\xi +$
$2\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 2\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x^2\,\gamma\,\delta\,\xi^2 - \frac{1}{2}\,t\,x^2\,\gamma\,\delta^2\,\xi^2 - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\big)\,\epsilon +$
$\big(2\,a^2\,\delta^2 - 2\,a\,\gamma\,\delta^2 + 12\,a^2\,t\,\delta^3 + 4\,a^2\,x\,y\,\delta^3 - 8\,a\,t\,\gamma\,\delta^3 - 20\,a\,x\,y\,\gamma\,\delta^3 - 2\,t\,\gamma^2\,\delta^3 + 18\,x\,y\,\gamma^2\,\delta^3 +$
$30\,a^2\,t^2\,\delta^4 + 20\,a^2\,t\,x\,y\,\delta^4 - 10\,a\,t^2\,\gamma\,\delta^4 - 88\,a\,t\,x\,y\,\gamma\,\delta^4 - 13\,a\,x^2\,y^2\,\gamma\,\delta^4 - \frac{15}{2}\,t^2\,\gamma^2\,\delta^4 +$
$64\,t\,x\,y\,\gamma^2\,\delta^4 + 34\,x^2\,y^2\,\gamma^2\,\delta^4 + 40\,a^2\,t^3\,\delta^5 + 40\,a^2\,t^2\,x\,y\,\delta^5 - 152\,a\,t^2\,x\,y\,\gamma\,\delta^5 - 48\,a\,t\,x^2\,y^2\,\gamma\,\delta^5 -$
$10\,t^3\,\gamma^2\,\delta^5 + 86\,t^2\,x\,y\,\gamma^2\,\delta^5 + 107\,t\,x^2\,y^2\,\gamma^2\,\delta^5 + 11\,x^3\,y^3\,\gamma^2\,\delta^5 + 30\,a^2\,t^4\,\delta^6 + 40\,a^2\,t^3\,x\,y\,\delta^6 +$
$10\,a\,t^4\,\gamma\,\delta^6 - 128\,a\,t^3\,x\,y\,\gamma\,\delta^6 - 66\,a\,t^2\,x^2\,y^2\,\gamma\,\delta^6 - 5\,t^4\,\gamma^2\,\delta^6 + 54\,t^3\,x\,y\,\gamma^2\,\delta^6 + \frac{247}{2}\,t^2\,x^2\,y^2\,\gamma^2\,\delta^6 +$
$\frac{80}{3}\,t\,x^3\,y^3\,\gamma^2\,\delta^6 + 12\,a^2\,t^5\,\delta^7 + 20\,a^2\,t^4\,x\,y\,\delta^7 + 8\,a\,t^5\,\gamma\,\delta^7 - 52\,a\,t^4\,x\,y\,\gamma\,\delta^7 - 40\,a\,t^3\,x^2\,y^2\,\gamma\,\delta^7 +$
$16\,t^4\,x\,y\,\gamma^2\,\delta^7 + 62\,t^3\,x^2\,y^2\,\gamma^2\,\delta^7 + \frac{64}{3}\,t^2\,x^3\,y^3\,\gamma^2\,\delta^7 + 2\,a^2\,t^6\,\delta^8 + 4\,a^2\,t^5\,x\,y\,\delta^8 + 2\,a\,t^6\,\gamma\,\delta^8 -$
$8\,a\,t^5\,x\,y\,\gamma\,\delta^8 - 9\,a\,t^4\,x^2\,y^2\,\gamma\,\delta^8 + \frac{1}{2}\,t^6\,\gamma^2\,\delta^8 + 2\,t^5\,x\,y\,\gamma^2\,\delta^8 + \frac{23}{2}\,t^4\,x^2\,y^2\,\gamma^2\,\delta^8 + \frac{17}{3}\,t^3\,x^3\,y^3\,\gamma^2\,\delta^8 +$
$4\,a^2\,y\,\delta^2\,\eta - 12\,a\,y\,\gamma\,\delta^2\,\eta + 6\,y\,\gamma^2\,\delta^2\,\eta + 20\,a^2\,t\,y\,\delta^3\,\eta - 48\,a\,t\,y\,\gamma\,\delta^3\,\eta - 20\,a\,x\,y^2\,\gamma\,\delta^3\,\eta +$
$14\,t\,y\,\gamma^2\,\delta^3\,\eta + 40\,x\,y^2\,\gamma^2\,\delta^3\,\eta + 40\,a^2\,t^2\,y\,\delta^4\,\eta - 72\,a\,t^2\,y\,\gamma\,\delta^4\,\eta - 72\,a\,t\,x\,y^2\,\gamma\,\delta^4\,\eta + 6\,t^2\,y\,\gamma^2\,\delta^4\,\eta +$
$115\,t\,x\,y^2\,\gamma^2\,\delta^4\,\eta + 23\,x^2\,y^3\,\gamma^2\,\delta^4\,\eta + 40\,a^2\,t^3\,y\,\delta^5\,\eta - 48\,a\,t^3\,y\,\gamma\,\delta^5\,\eta - 96\,a\,t^2\,x\,y^2\,\gamma\,\delta^5\,\eta -$
$6\,t^3\,y\,\gamma^2\,\delta^5\,\eta + 118\,t^2\,x\,y^2\,\gamma^2\,\delta^5\,\eta + 53\,t\,x^2\,y^3\,\gamma^2\,\delta^5\,\eta + 20\,a^2\,t^4\,y\,\delta^6\,\eta - 12\,a\,t^4\,y\,\gamma\,\delta^6\,\eta -$

$$56\,a\,t^3\,x\,y^2\,\gamma\,\delta^6\,\eta - 4\,t^4\,y\,\gamma^2\,\delta^6\,\eta + 51\,t^3\,x\,y^2\,\gamma^2\,\delta^6\,\eta + 40\,t^2\,x^2\,y^3\,\gamma^2\,\delta^6\,\eta + 4\,a^2\,t^5\,y\,\delta^7\,\eta -$$
$$12\,a\,t^4\,x\,y^2\,\gamma\,\delta^7\,\eta + 8\,t^4\,x\,y^2\,\gamma^2\,\delta^7\,\eta + 10\,t^3\,x^2\,y^3\,\gamma^2\,\delta^7\,\eta - 7\,a\,y^2\,\gamma\,\delta^2\,\eta^2 + 10\,y^2\,\gamma^2\,\delta^2\,\eta^2 -$$
$$24\,a\,t\,y^2\,\gamma\,\delta^3\,\eta^2 + 24\,t\,y^2\,\gamma^2\,\delta^3\,\eta^2 + 15\,x\,y^3\,\gamma^2\,\delta^3\,\eta^2 - 30\,a\,t^2\,y^2\,\gamma\,\delta^4\,\eta^2 + \frac{37}{2}\,t^2\,y^2\,\gamma^2\,\delta^4\,\eta^2 +$$
$$32\,t\,x\,y^3\,\gamma^2\,\delta^4\,\eta^2 - 16\,a\,t^3\,y^2\,\gamma\,\delta^5\,\eta^2 + 5\,t^3\,y^2\,\gamma^2\,\delta^5\,\eta^2 + 22\,t^2\,x\,y^3\,\gamma^2\,\delta^5\,\eta^2 - 3\,a\,t^4\,y^2\,\gamma\,\delta^6\,\eta^2 +$$
$$\frac{1}{2}\,t^4\,y^2\,\gamma^2\,\delta^6\,\eta^2 + 5\,t^3\,x\,y^3\,\gamma^2\,\delta^6\,\eta^2 + 3\,y^3\,\gamma^2\,\delta^2\,\eta^3 + \frac{17}{3}\,t\,y^3\,\gamma^2\,\delta^3\,\eta^3 + \frac{10}{3}\,t^2\,y^3\,\gamma^2\,\delta^4\,\eta^3 +$$
$$\frac{2}{3}\,t^3\,y^3\,\gamma^2\,\delta^5\,\eta^3 + 4\,a^2\,x\,\delta^2\,\xi - 12\,a\,x\,\gamma\,\delta^2\,\xi + 6\,x\,\gamma^2\,\delta^2\,\xi + 20\,a^2\,t\,x\,\delta^3\,\xi - 48\,a\,t\,x\,\gamma\,\delta^3\,\xi -$$
$$20\,a\,x^2\,y\,\gamma\,\delta^3\,\xi + 14\,t\,x\,\gamma^2\,\delta^3\,\xi + 40\,x^2\,y\,\gamma^2\,\delta^3\,\xi + 40\,a^2\,t^2\,x\,\delta^4\,\xi - 72\,a\,t^2\,x\,\gamma\,\delta^4\,\xi - 72\,a\,t\,x^2\,y\,\gamma\,\delta^4\,\xi +$$
$$6\,t^2\,x\,\gamma^2\,\delta^4\,\xi + 115\,t\,x^2\,y\,\gamma^2\,\delta^4\,\xi + 23\,x^3\,y^2\,\gamma^2\,\delta^4\,\xi + 40\,a^2\,t^3\,x\,\delta^5\,\xi - 48\,a\,t^3\,x\,\gamma\,\delta^5\,\xi -$$
$$96\,a\,t^2\,x^2\,y\,\gamma\,\delta^5\,\xi - 6\,t^3\,x\,\gamma^2\,\delta^5\,\xi + 118\,t^2\,x^2\,y\,\gamma^2\,\delta^5\,\xi + 53\,t\,x^3\,y^2\,\gamma^2\,\delta^5\,\xi + 20\,a^2\,t^4\,x\,\delta^6\,\xi -$$
$$12\,a\,t^4\,x\,\gamma\,\delta^6\,\xi - 56\,a\,t^3\,x^2\,y\,\gamma\,\delta^6\,\xi - 4\,t^4\,x\,\gamma^2\,\delta^6\,\xi + 51\,t^3\,x^2\,y\,\gamma^2\,\delta^6\,\xi + 40\,t^2\,x^3\,y^2\,\gamma^2\,\delta^6\,\xi +$$
$$4\,a^2\,t^5\,x\,\delta^7\,\xi - 12\,a\,t^4\,x^2\,y\,\gamma\,\delta^7\,\xi + 8\,t^4\,x^2\,y\,\gamma^2\,\delta^7\,\xi + 10\,t^3\,x^3\,y^2\,\gamma^2\,\delta^7\,\xi + 4\,a^2\,\delta\,\eta\,\xi - 4\,a\,\gamma\,\delta\,\eta\,\xi +$$
$$20\,a^2\,t\,\delta^2\,\eta\,\xi - 8\,a\,t\,\gamma\,\delta^2\,\eta\,\xi - 28\,a\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 6\,t\,\gamma^2\,\delta^2\,\eta\,\xi + 38\,x\,y\,\gamma^2\,\delta^2\,\eta\,\xi + 40\,a^2\,t^2\,\delta^3\,\eta\,\xi +$$
$$8\,a\,t^2\,\gamma\,\delta^3\,\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\,\eta\,\xi - 14\,t^2\,\gamma^2\,\delta^3\,\eta\,\xi + 88\,t\,x\,y\,\gamma^2\,\delta^3\,\eta\,\xi + 44\,x^2\,y^2\,\gamma^2\,\delta^3\,\eta\,\xi +$$
$$40\,a^2\,t^3\,\delta^4\,\eta\,\xi + 32\,a\,t^3\,\gamma\,\delta^4\,\eta\,\xi - 120\,a\,t^2\,x\,y\,\gamma\,\delta^4\,\eta\,\xi - 6\,t^3\,\gamma^2\,\delta^4\,\eta\,\xi + 62\,t^2\,x\,y\,\gamma^2\,\delta^4\,\eta\,\xi +$$
$$93\,t\,x^2\,y^2\,\gamma^2\,\delta^4\,\eta\,\xi + 20\,a^2\,t^4\,\delta^5\,\eta\,\xi + 28\,a\,t^4\,\gamma\,\delta^5\,\eta\,\xi - 64\,a\,t^3\,x\,y\,\gamma\,\delta^5\,\eta\,\xi + 6\,t^4\,\gamma^2\,\delta^5\,\eta\,\xi +$$
$$12\,t^3\,x\,y\,\gamma^2\,\delta^5\,\eta\,\xi + 63\,t^2\,x^2\,y^2\,\gamma^2\,\delta^5\,\eta\,\xi + 4\,a^2\,t^5\,\delta^6\,\eta\,\xi + 8\,a\,t^5\,\gamma\,\delta^6\,\eta\,\xi - 12\,a\,t^4\,x\,y\,\gamma\,\delta^6\,\eta\,\xi +$$
$$4\,t^5\,\gamma^2\,\delta^6\,\eta\,\xi + 14\,t^3\,x^2\,y^2\,\gamma^2\,\delta^6\,\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\,\xi + 6\,y\,\gamma^2\,\delta\,\eta^2\,\xi - 24\,a\,t\,y\,\gamma\,\delta^2\,\eta^2\,\xi +$$
$$5\,t\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi + 25\,x\,y^2\,\gamma^2\,\delta^2\,\eta^2\,\xi - 24\,a\,t^2\,y\,\gamma\,\delta^3\,\eta^2\,\xi - 8\,t^2\,y\,\gamma^2\,\delta^3\,\eta^2\,\xi + 45\,t\,x\,y^2\,\gamma^2\,\delta^3\,\eta^2\,\xi -$$
$$8\,a\,t^3\,y\,\gamma\,\delta^4\,\eta^2\,\xi - 7\,t^3\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi + 24\,t^2\,x\,y^2\,\gamma^2\,\delta^4\,\eta^2\,\xi + 4\,t^3\,x\,y^2\,\gamma^2\,\delta^5\,\eta^2\,\xi + 4\,y^2\,\gamma^2\,\delta\,\eta^3\,\xi +$$
$$5\,t\,y^2\,\gamma^2\,\delta^2\,\eta^3\,\xi + t^2\,y^2\,\gamma^2\,\delta^3\,\eta^3\,\xi - 7\,a\,x^2\,\gamma\,\delta^2\,\xi^2 + 10\,x^2\,\gamma^2\,\delta^2\,\xi^2 - 24\,a\,t\,x^2\,\gamma\,\delta^3\,\xi^2 +$$
$$24\,t\,x^2\,\gamma^2\,\delta^3\,\xi^2 + 15\,x^3\,y\,\gamma^2\,\delta^3\,\xi^2 - 30\,a\,t^2\,x^2\,\gamma\,\delta^4\,\xi^2 + \frac{37}{2}\,t^2\,x^2\,\gamma^2\,\delta^4\,\xi^2 + 32\,t\,x^3\,y\,\gamma^2\,\delta^4\,\xi^2 -$$
$$16\,a\,t^3\,x^2\,\gamma\,\delta^5\,\xi^2 + 5\,t^3\,x^2\,\gamma^2\,\delta^5\,\xi^2 + 22\,t^2\,x^3\,y\,\gamma^2\,\delta^5\,\xi^2 - 3\,a\,t^4\,x^2\,\gamma\,\delta^6\,\xi^2 + \frac{1}{2}\,t^4\,x^2\,\gamma^2\,\delta^6\,\xi^2 +$$
$$5\,t^3\,x^3\,y\,\gamma^2\,\delta^6\,\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\,\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\,\eta\,\xi^2 + 5\,t\,x\,\gamma^2\,\delta^2\,\eta\,\xi^2 +$$
$$25\,x^2\,y\,\gamma^2\,\delta^2\,\eta\,\xi^2 - 24\,a\,t^2\,x\,\gamma\,\delta^3\,\eta\,\xi^2 - 8\,t^2\,x\,\gamma^2\,\delta^3\,\eta\,\xi^2 + 45\,t\,x^2\,y\,\gamma^2\,\delta^3\,\eta\,\xi^2 - 8\,a\,t^3\,x\,\gamma\,\delta^4\,\eta\,\xi^2 -$$
$$7\,t^3\,x\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 24\,t^2\,x^2\,y\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 4\,t^3\,x^2\,y\,\gamma^2\,\delta^5\,\eta\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 3\,t\,\gamma^2\,\delta\,\eta^2\,\xi^2 +$$
$$11\,x\,y\,\gamma^2\,\delta\,\eta^2\,\xi^2 + 6\,a\,t^2\,\gamma\,\delta^2\,\eta^2\,\xi^2 - \frac{5}{2}\,t^2\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 12\,t\,x\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 8\,a\,t^3\,\gamma\,\delta^3\,\eta^2\,\xi^2 +$$
$$4\,t^3\,\gamma^2\,\delta^3\,\eta^2\,\xi^2 + 3\,a\,t^4\,\gamma\,\delta^4\,\eta^2\,\xi^2 + \frac{7}{2}\,t^4\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 - t^3\,x\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 -$$
$$t\,y\,\gamma^2\,\delta\,\eta^3\,\xi^2 - 2\,t^2\,y\,\gamma^2\,\delta^2\,\eta^3\,\xi^2 + 3\,x^3\,\gamma^2\,\delta^2\,\xi^3 + \frac{17}{3}\,t\,x^3\,\gamma^2\,\delta^3\,\xi^3 + \frac{10}{3}\,t^2\,x^3\,\gamma^2\,\delta^4\,\xi^3 +$$
$$\frac{2}{3}\,t^3\,x^3\,\gamma^2\,\delta^5\,\xi^3 + 4\,x^2\,\gamma^2\,\delta\,\eta\,\xi^3 + 5\,t\,x^2\,\gamma^2\,\delta^2\,\eta\,\xi^3 + t^2\,x^2\,\gamma^2\,\delta^3\,\eta\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 - t\,x\,\gamma^2\,\delta\,\eta^2\,\xi^3 -$$
$$2\,t^2\,x\,\gamma^2\,\delta^2\,\eta^2\,\xi^3 - \frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{3}\,t^2\,\gamma^2\,\delta\,\eta^3\,\xi^3 + \frac{2}{3}\,t^3\,\gamma^2\,\delta^2\,\eta^3\,\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}$$

```
{tt = Last[ΛQU,2[{ξ, η, δ}, {x, y}]],
  Exponent[Normal@Together[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

{ ⋯ 1 ⋯ , 6}

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## Reorderings with Rord

Rord

```
Rord_{u_i_,w_j_→k_}[ℭ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
   {yax, q, p} = Echo[List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}],
        Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /. {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s}];
   ℭ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0),
     Log[DP_{u_i→D_υ, w_j→D_ω}[e^P][e^{q+p}]] - q] /. {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

Rord

```
Rord_{u_i_,w_j_→k_}[ℭ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, n, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
   {yax, q, p} = List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}], Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /.
     {y → y_n, a → a_n, x → x_n, t → t_s, T → T_s};
   (*Echo@{{{u_i,υ},{w_j,ω}},P,p e^q};*)
   ℭ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), Log[SP_{u_i→υ, w_j→ω}[e^{P+q+p}]] -
      q] /. {n → k, υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
With[{c𝕠 = ℭ_CU[{y_1, x_1}_1, {x_2, a_2, y_2}_2, ħ t_1 a_2 + ħ t_1^{-1} (e^{t_1} - 1) y_1 x_2, 𝟘_2 + ϵ x_1 y_2]},
 {Short[rhs = c𝕠 // Rord_{x_2,a_2→3}, 3], HL[CU[c𝕠] == CU[rhs]]}]
```

$$\left\{\mathfrak{C}_{CU}\left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\hbar a_3 t_1^2 - \ll 1\gg + e^{\ll 1\gg}\hbar x_3 y_1\right)}{t_1},\right.\right.$$

$$\left.\left.\frac{1}{t_1} e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\operatorname{Log}\left[e^{\frac{e^{\ll 1\gg}(\ll 1\gg)}{t_1}}\right] t_1 - e^{\gamma\hbar t_1}\hbar a_3 t_1^2 + \hbar x_3 y_1 - e^{t_1}\hbar x_3 y_1\right) + x_1 y_2 \epsilon + O[\epsilon]^3\right], \text{True}\right\}$$

```
With[{c𝕠 = ℭ_CU[{y_1, a_1, a_2}_1, {x_2, x_1, y_2}_2,
    ħ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    𝟘_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = c𝕠 // Rord_{a_1,a_2→3} // Rord_{x_2,x_1→4}, 3], HL[CU[c𝕠] == CU[rhs]]}]
```

$$\left\{\mathfrak{C}_{CU}\left[\{y_1, a_3\}_1, \{x_4, y_2\}_2, \hbar a_3 l_{11} t_1 + \ll 6\gg + \hbar x_4 y_2 \gamma_{22},\right.\right.$$

$$\left(\operatorname{Log}\left[e^{\hbar x_4 y_1 \gamma_{11} + \hbar x_4 y_2 \gamma_{12} + \hbar x_4 y_1 \gamma_{21} + \hbar x_4 y_2 \gamma_{22}}\right] - \hbar x_4 y_1 \gamma_{11} - \hbar x_4 y_2 \gamma_{12} - \hbar x_4 y_1 \gamma_{21} - \hbar x_4 y_2 \gamma_{22}\right) +$$

$$\left.\left.(a_3 l_1 + a_3 l_2 + p_{11} x_4 y_1 + p_{21} x_4 y_1 + p_{12} x_4 y_2 + p_{22} x_4 y_2)\,\epsilon + O[\epsilon]^3\right], \text{True}\right\}$$

```
With[{c𝕠 = ℭ_CU[{y_1, a_1, x_1}_1, {x_2, a_2, y_2}_2,
    ħ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    𝟘_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = c𝕠 // Rord_{x_2,a_2→3}, 3], HL[CU[c𝕠] == CU[rhs]]}]
```

$$\left\{\mathfrak{C}_{CU}\left[\{y_1, a_1, x_1\}_1, \{a_3, x_3, y_2\}_2,\right.\right.$$

$$e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\left(e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\hbar a_1 l_{11} t_1 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\hbar a_3 l_{12} t_1 +\right.$$

$$\left.\left.\left.\ll 4\gg + \hbar x_3 y_1 \gamma_{21} + \hbar x_3 y_2 \gamma_{22}\right), \ll 1\gg\right], \text{True}\right\}$$

```
With[{qφ = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    θ2 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = qφ // Rord_{x2,a2→3}, 3], HL[QU[qφ] == QU[rhs]]}]
```

$$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \{a_3, x_3, y_2\}_2, e^{-\gamma\hbar l_{12} t_1-\gamma\hbar l_{22} t_2}(e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\hbar a_1 l_{11} t_1 + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\hbar a_3 l_{12} t_1 + \ll 4\gg + \hbar x_3 y_1 \gamma_{21} + \hbar x_3 y_2 \gamma_{22}), \ll 1\gg], \text{True}\}$$

```
With[{qφ = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    θ2 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = qφ // Rord_{a2,y2→3}, 3], HL[QU[qφ] == QU[rhs]]}]
```

$$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \{x_2, y_3, a_3\}_2, e^{-\gamma\hbar l_{12} t_1-\gamma\hbar l_{22} t_2}(e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\hbar a_1 l_{11} t_1 + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\hbar a_3 l_{12} t_1 + \ll 4\gg + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\hbar x_2 y_1 \gamma_{21} + \hbar x_2 y_3 \gamma_{22}), \ll 1\gg], \text{True}\}$$

```
Timing@With[{qφ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
    ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    θ2 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qφ // Rord_{x1,y1→3}, 5]}]
```

$$\Big\{431.297, \Big\{\mathbb{E}_{QU}\Big[\{y_3, a_3, x_3\}_1, \{\ll 1\gg\}_2, \frac{\ll 1\gg}{\ll 1\gg}, \frac{1}{1-\gamma_{11}+T_1\gamma_{11}}\Big(\text{Log}\Big[\frac{e^{\frac{\ll 1\gg}{1-\gamma_{11}+T_1\gamma_{11}}}}{1-(1-T_1)\gamma_{11}}\Big] - \text{Log}\Big[\frac{e^{\frac{\ll 1\gg+\ll 5\gg}{\ll 1\gg}}}{1-(1-T_1)\gamma_{11}}\Big]\gamma_{11} + \text{Log}\Big[\frac{e^{\frac{\ll 1\gg}{\ll 1\gg}}}{1-(1-T_1)\gamma_{11}}\Big]T_1\gamma_{11} - \hbar x_3 y_3 \gamma_{11} - \hbar \ll 1\gg \ll 1\gg \gamma_{12} - \hbar x_2 y_3 \gamma_{21} - \hbar x_2 y_2 \gamma_{12}\gamma_{21} + \hbar T_1 x_2 y_2 \gamma_{12}\gamma_{21}\Big) + \Big(\big(4\hbar a_2 l_2 + \ll 346\gg + 3\gamma\hbar^4 T_1^2 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2\big)\epsilon\Big) \Big/ \Big(4\hbar - 16\hbar\gamma_{11} + 16\hbar T_1\gamma_{11} + \ll 13\gg + 24\hbar T_1^2\gamma_{11}^4 - 16\hbar T_1^3\gamma_{11}^4 + 4\hbar T_1^4\gamma_{11}^4\Big) + \frac{(36 \ll 1\gg + \ll 2674\gg)\ll 1\gg}{\ll 51\gg + 72 \ll 2\gg \ll 1\gg} + O[\epsilon]^3\Big]\Big\}\Big\}$$

```
Timing@With[{qφ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
    ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    θ2 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qφ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qφ] == QU[rhs]]}]
```

$$\Big\{688.656, \Big\{\mathbb{E}_{QU}\Big[\{y_3, a_3, x_3\}_1, \{\ll 1\gg\}_2, \frac{\ll 1\gg}{\ll 1\gg}, \text{Log}\Big[\frac{1}{1-(1-T_1)\gamma_{11}}\Big] + \Big(\big(4\hbar a_2 l_2 + 4 p_{11} - 4 p_{11} T_1 + 4\hbar p_{22} x_2 y_2 + \ll 339\gg + \gamma\hbar^4 x_2^2 y_2^2 \gamma_{12}^2\gamma_{21}^2 - 4\gamma\hbar^4 T_1 x_2^2 y_2^2 \gamma_{12}^2\gamma_{21}^2 + 3\gamma\hbar^4 T_1^2 x_2^2 y_2^2\gamma_{12}^2\gamma_{21}^2\big)\epsilon\Big) \Big/ \Big(4\hbar - 16\hbar\gamma_{11} + 16\hbar T_1\gamma_{11} + 24\hbar\gamma_{11}^2 - 48\hbar T_1\gamma_{11}^2 + 24\hbar T_1^2\gamma_{11}^2 - \ll 1\gg + \ll 1\gg - \ll 1\gg + 16\hbar T_1^3\gamma_{11}^3 + 4\hbar\gamma_{11}^4 - 16\hbar T_1\gamma_{11}^4 + 24\hbar T_1^2\gamma_{11}^4 - 16\hbar T_1^3\gamma_{11}^4 + 4\hbar T_1^4\gamma_{11}^4\Big) + \frac{(\ll 1\gg)\ll 1\gg}{\ll 1\gg} + O[\epsilon]^3\Big], \text{True}\Big\}\Big\}$$

```
Timing@With[{q⊙ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      θ2 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
   {Short[rhs = q⊙ // Rord_{x1,y1→1}, 5], HL@SimpT[QU[q⊙] == QU[rhs]]}]
```

{691.703, {𝔈QU[{y1, a1, x1}1, <<1>>2, <<1>>/<<1>>,

Log[1/(1 − (1 − T1) γ11)] + ((4 ℏ a2 l2 + 4 p11 − 4 p11 T1 + 4 ℏ p11 x1 y1 + <<339>> +
γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 − 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) /
(4 ℏ − 16 ℏ γ11 + 16 ℏ T1 γ11 + 24 ℏ γ11^2 − 48 ℏ T1 γ11^2 + 24 ℏ T1^2 γ11^2 − 16 ℏ γ11^3 + <<1>> −
48 ℏ <<1>> γ11^3 + 16 ℏ T1^3 γ11^3 + 4 ℏ γ11^4 − 16 ℏ T1 γ11^4 + 24 ℏ T1^2 γ11^4 − 16 ℏ T1^3 γ11^4 + 4 ℏ T1^4 γ11^4) +
(36 <<1>> + <<2673>> + <<1>>) <<1>> / (<<51>> + 72 <<2>> γ^7_<<2>>) + O[ϵ]^3], True}}

## Canonical ordering with Cord

Cord

```
Cord[𝔈U_[L___, {l___, u_i_, w_j_, r___}s_, R___, Q_, P_]] /;
    OrderedQ[{w, u} /. {y → 1, a → 2, x → 3}] :=
   Cord[Rord_{ui,wj→Unique[]}[𝔈U[L, {l, ui, wj, r}s, R, Q, P]]];
Cord[𝔈U_[specs__, Q_, P_]] := 𝔈U[Sequence @@ Sort@{specs}, Q, P] /.
   Flatten[{specs} /. {yax__}s_ :→ ({yax} /. u_i_ :→ (ui → us))]
```

```
Block[{$p = 4, c⊙ = 𝔈CU[{y1, a1, x1, x2, a2, y2}1,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     θ1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Cord[c⊙], HL[CU[c⊙] == CU[Cord[c⊙]]]}]
```

{𝔈CU[{y1, a1, x1}1, ⋯1⋯ / (e^{⋯1⋯} + ⋯1⋯ + ⋯1⋯),

Log[1/(1 + e^{−γ ℏ l12 t1 − γ ℏ l22 t2} t1 (ℏ γ12 + ℏ γ22))] + ((2 e^{2 γ ℏ l11 t1 + ⋯1⋯ + ⋯1⋯ + 6 γ ℏ l22 t2} a1 l1 + ⋯419⋯) ϵ) /
(2 e^{2 γ ℏ l11 t1 + 6 ⋯3⋯ t1 + ⋯1⋯ + 6 γ ℏ l22 t2} + 8 e^{⋯1⋯} ℏ t1 γ12 +
⋯11⋯ + 8 ⋯4⋯ γ22^3 + 2 e^{⋯1⋯} ℏ^4 t1^4 γ22^4) + O[ϵ]^2], True}

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```
With[{qⲫ = ŒQU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    θ0 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Cord[
  qⲫ]]
```

$$
\begin{aligned}
&\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1,\\
&\Big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{11} t_1 + e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{12} t_1 +\\
&\quad e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{21} t_2 + e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{22} t_2 +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, x_1 y_1 \gamma_{11} - e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{11} t_1 \gamma_{12} -\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{12} t_1 \gamma_{12} - e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{21} t_2 \gamma_{12} -\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{22} t_2 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{11} t_1 T_1 \gamma_{12} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{12} t_1 T_1 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{21} t_2 T_1 \gamma_{12} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{22} t_2 T_1 \gamma_{12} + \hbar\, x_1 y_1 \gamma_{12} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, x_1 y_1 \gamma_{21} - e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{11} t_1 \gamma_{22} -\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{12} t_1 \gamma_{22} - e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{21} t_2 \gamma_{22} -\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{22} t_2 \gamma_{22} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{11} t_1 T_1 \gamma_{22} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{12} t_1 T_1 \gamma_{22} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{21} t_2 T_1 \gamma_{22} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\hbar\, a_1 l_{22} t_2 T_1 \gamma_{22} + \hbar\, x_1 y_1 \gamma_{22}\Big) \Big/\\
&\Big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} - e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\gamma_{12} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,T_1 \gamma_{12} - e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,\gamma_{22} +\\
&\quad e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\,T_1 \gamma_{22}\Big),\ \mathrm{Log}\Big[\frac{1}{1-\frac{e^{-\gamma\hbar l_{12} t_1-\gamma\hbar l_{22} t_2}(1-T_1)(\hbar\gamma_{12}+\hbar\gamma_{22})}{\hbar}}\Big] + O[\epsilon]^1\Big]
\end{aligned}
$$

# Stitching Œ's.

StitchingOEs

```
m_{j_→k_}[Œ_{U_}[specs__, Q_, P_]] := Cord[Œ_U[Sequence @@ Append[DeleteCases[{specs}, {__}_{j|k}],
      Flatten[{Cases[{specs}, {us__}_j :→ {us}], Cases[{specs}, {us__}_k :→ {us}]}]_k],
    Q, P] /. {t_j → t_k, T_j → T_k}]
```

```
cⲫ = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l_{10 i+j} ti aj + γ_{10 i+j} yi xj, {i, 3}, {j, 3}], θ2];
{cⲫ // m3→4, HL@Simp[CU[m3→4[cⲫ]] - m3→4[CU[cⲫ]]]}
```

$$
\begin{aligned}
&\Big\{\text{Œ}_{CU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4,\\
&\quad \hbar\,(a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 +\\
&\qquad a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} +\\
&\qquad x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}),\ O[\epsilon]^3\Big],\ 0\Big\}
\end{aligned}
$$

Verifying that *m* commutes with evaluation, in CU:

```
c⊕ = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 02];
Timing@{c⊕ // m2→3, HL@Simp[CU[m2→3[c⊕]] - m2→3[CU[c⊕]]]}
```

```
{1815.8, {𝔈CU[{y1, a1, x1}1, {y3, a3, x3}3, «1»/«1», Log[1/(1+ℏ t3 γ32)] +
    ((«1») ϵ) / (2 e^(2 γ ℏ l12 t1+«4»+2 γ ℏ l33 t3) + 8 e^«1» ℏ t3 γ32 + 12 «3» γ32^2 +
      8 e^«1» ℏ^3 t3^3 γ32^3 + 2 e^«1» ℏ^4 t3^4 γ32^4) + ((-6 «7» γ31^2 + «871» + «1») ϵ^2)/(6 e^«1» + «6» + 6 e^«1» ℏ^7 t3^7 γ32^7) + O[ϵ]^3], 0}}
large output | show less | show more | show all | set size limit...
```

Verifying that *m* commutes with evaluation, in QU:

```
q⊕ = 𝔈QU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 02];
Timing@{q⊕ // m2→3, HL@SimpT[QU[m2→3[q⊕]] - m2→3[QU[q⊕]]]}
```

```
{17 843.1, {𝔈QU[{y1, a1, x1}1, {«1»}3, «1»,
    Log[1/(1-(1-T3) γ32)] + ((8 e^(2 γ ℏ l12 t1+«4»+2 «3» t3) ℏ^2 a3 T3 x1 y1 γ12 γ31 +
      4 «8» γ31 + «368» + «1» - 3 «7» γ33^2) ϵ) /
    (4 e^(2 γ ℏ l12 t1+«4»+2 γ ℏ l33 t3) - 16 e^«1» γ32 + «17» + 4 e^«1» T3^4 γ32^4) +
    ((«1») «1»)/«1» + O[ϵ]^3], 0}}
large output | show less | show more | show all | set size limit...
```

```
𝔈U_[sp1__, Q1_, P1_] ≡ 𝔈U_[sp2__, Q2_, P2_] :=
  Sort[{sp1}] == Sort[{sp2}] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
c⊕ = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 00];
Timing@HL[(lhs = c⊕ // m1,2→1 // m1,3→1) ≡ (rhs = c⊕ // m2,3→2 // m1,2→1)]
```

```
{48., True}
```

```
c⊕ = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 01];
Timing@HL[(lhs = c⊕ // m1,2→1 // m1,3→1) ≡ (rhs = c⊕ // m2,3→2 // m1,2→1)]
```

```
{19 914.9, True}
```

mexamples

```
cⅅ = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 2}, {j, 2}], 01];
Short[Simplify /@ (cexample = cⅅ // m1→2), 12]
Short[Simplify /@ (qexample = (qⅅ = cⅅ /. CU → QU) // m1→2), 12]
```

mexamples

$$\mathbb{E}_{CU}\left[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11}+l_{12}+l_{21}+l_{22}) t_2 + \frac{1}{1+\hbar t_2 \gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} \hbar x_2 y_2 \left(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+\hbar t_2\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\left(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar t_2\gamma_{22}\right)\right),\right.$$
$$\frac{1}{1+\hbar t_2\gamma_{21}} + \frac{1}{2(1+\hbar t_2\gamma_{21})^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\Big(4a_2(1+\hbar t_2\gamma_{21})^2 \left(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\left(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2 + x_2y_2\right)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\left(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2\left(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22}\right)\right)\right) -$$
$$\gamma\hbar\Big(-2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2\gamma_{21}^2(1+\hbar t_2\gamma_{21})^2 + 4e^{\ll 1\gg}\ll 3\gg(1+\ll 1\gg)\left(2\gamma_{21}+\hbar t_2\gamma_{21}^2 + e^{\gamma\hbar\ll 1\gg t_2}\gamma_{22} + \gamma_{11}\left(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{\ll 2\gg}+l_{22})t_2}\hbar t_2\gamma_{22}\right)\right) + \hbar x_2^2 y_2^2\left(e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}+\gamma_{21}\right)\left(\gamma_{21}+e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}\right)\left(3\hbar t_2\gamma_{21}^2 + 2e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{21}\left(4+e^{\gamma\hbar(l_{12}+l_{22})t_2}\hbar t_2\gamma_{22}\right) + e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}\left(2+\hbar t_2\left(\gamma_{21}-e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}\right)\right)\right)\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

mexamples

$$\mathbb{E}_{QU}\left[\{y_2, a_2, x_2\}_2, \hbar a_2(l_{11}+l_{12}+l_{21}+l_{22})t_2 + \left(e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\left(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+(-1+T_2)\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\left(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-1+T_2)\gamma_{22}\right)\right)\right) \Big/ (1+(-1+T_2)\gamma_{21}),\right.$$
$$\frac{1}{1+(-1+T_2)\gamma_{21}} + \frac{1}{4(1+(-1+T_2)\gamma_{21})^5}e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\Big(8a_2T_2(1+(-1+T_2)\gamma_{21})^2\left(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\left(-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}T_2 + \hbar x_2 y_2\right)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\left(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2\left(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22}\right)\right)\right) + \gamma(\ll 1\gg)\Big)\epsilon + O[\epsilon]^2\Big]$$

## *R* in QU.

The Faddeev-Quesne formula:

Faddeev

```
𝕖q_,k_[x_] := 𝕖^(Sum[(1 - q)^j x^j / (j (1 - q^j)), {j, 1, k+1}]); 𝕖q_[x_] := 𝕖q,$k[x]
```

$$\mathbb{e}_{q\_,k\_}[x\_] := \mathbb{e}^{\wedge}\left(\sum_{j=1}^{k+1}\frac{(1-q)^j x^j}{j(1-q^j)}\right);\quad \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
Table[Series[e_{qħ,k}[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$$\begin{array}{l}
e^x \\
e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384}\left(e^x x^2\left(-8+x^4\right)\gamma^3\hbar^3\right)\epsilon^3 + \frac{e^x x^4\left(-32+x^4\right)\gamma^4\hbar^4\epsilon^4}{6144} + O[\epsilon]^5 \\
e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3 \left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{e^x x^3\left(-4608-864x+1024x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165888} + O[\epsilon]^5 \\
e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3 \left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{e^x x^3\left(-4608-864x+3616x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165888} + O[\epsilon]^5 \\
e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3 \left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{1}{4147200} e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5 \\
e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3 \left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{1}{4147200} e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5
\end{array}$$

```
Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)\left(1+q+q^2\right)}, \frac{1}{(1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)}, 1\Big/\left((1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)\left(1+q+q^2+q^3+q^4\right)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

```
QU[R_{i_,j_}] := 𝕆_QU[{y1, a1}_i, {a2, x2}_j, SS[e^{ħ b1 a2} e_{qħ}[ħ y1 x2] /. b1 → γ^-1 (ϵ a1 - t_i)]];
QU[R^-1_{i_,j_}] := S_j@QU[R_{i,j}];
```

```
QU[R_{3,4}] // Short
```

$$QU[\,] + \frac{\epsilon\hbar\, QU[a_3, a_4]}{\gamma} + \hbar\, QU[y_3, x_4] + \frac{\epsilon\hbar^2 QU[y_3, a_3, a_4, x_4]}{\gamma} + \frac{1}{2}\hbar^2 QU[y_3, y_3, x_4, x_4] - \frac{\hbar\, QU[a_4]\, t_3}{\gamma} - \frac{\epsilon\hbar^2 QU[a_3, a_4, a_4]\, t_3}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4]\, t_3^2}{2\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^-1_{1,2}] // Simp // HL // Timing
```

{0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$$\left\{0.203125, \left\{QU[\,] + \frac{\epsilon\,\hbar\,QU[a_1, a_2]}{\gamma} + \frac{\epsilon\,\hbar\,QU[a_1, a_3]}{\gamma} + \frac{\epsilon\,\hbar\,QU[a_2, a_3]}{\gamma} + \hbar\,QU[y_1, x_2] + \hbar\,QU[y_1, x_3] + \ll 67 \gg + \frac{\hbar^2\,QU[a_2, a_3]\,t_1\,t_2}{\gamma^2} + \frac{\hbar^2\,QU[a_3, a_3]\,t_1\,t_2}{\gamma^2} + \frac{\hbar^2\,QU[a_3, a_3]\,t_2^2}{2\,\gamma^2} + 2\,\epsilon\,\hbar^2\,QU[y_1, a_2, x_3]\,T_2 + QU[y_1, x_3]\,(\hbar - \hbar\,T_2), 0\right\}\right\}$$

## *R* in $Œ_{QU}$.

RinOE

```
ŒQU,k_[Ri_,j_] := ŒQU[{yi, ai, xi}i, {yj, aj, xj}j, -ħ γ^-1 ti aj + ħ yi xj,
  Log@Series[e^(ħ γ^-1 ti aj - ħ yi xj) (e^(ħ bi aj) e_(qħ,k)[ħ yi xj] /. bi → γ^-1 (ϵ ai - ti)), {ϵ, 0, k}]]
```

```
{ŒQU,1[R1,2], ŒQU,2[R1,2]}
```

$$\left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1, 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + O[\epsilon]^2\right],\right.$$
$$\left.Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1, 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + \frac{1}{288\,\gamma^2}\left(144\,\hbar^2\,a_1^2\,a_2^2 - 72\,\gamma^2\,\hbar^4\,a_1\,a_2\,x_2^2\,y_1^2 + 32\,\gamma^4\,\hbar^5\,x_2^3\,y_1^3 + 9\,\gamma^4\,\hbar^6\,x_2^4\,y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$$

## The morphism $Œ_{U,k}$.

MorphismOE

```
ŒU_,k_[a_ * b_] := ŒU,k[a] ŒU,k[b];
ŒU_,k_[mis__[a_]] := mis[ŒU,k[a]];
```

```
ŒQU,1[R1,2 R3,4 R5,6]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar\,a_2\,t_1}{\gamma} - \frac{\hbar\,a_4\,t_3}{\gamma} - \frac{\hbar\,a_6\,t_5}{\gamma} + \hbar\,x_2\,y_1 + \hbar\,x_4\,y_3 + \hbar\,x_6\,y_5,\right.$$
$$\left.1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} + \frac{\hbar\,a_3\,a_4}{\gamma} + \frac{\hbar\,a_5\,a_6}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2 - \frac{1}{4}\,\gamma\,\hbar^3\,x_4^2\,y_3^2 - \frac{1}{4}\,\gamma\,\hbar^3\,x_6^2\,y_5^2\right)\epsilon + O[\epsilon]^2\right]$$

```
ŒQU,1[R1,2 R3,4 R5,6 // m1,3→1 // m2,5→2 // m4,6→4]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar\,a_2\,t_1 - \hbar\,a_4\,t_1 - \hbar\,a_4\,t_2 + \gamma\,\hbar\,x_2\,y_1 + \gamma\,\hbar\,x_4\,y_1 + e^{\hbar\,t_2}\,\gamma\,\hbar\,x_4\,y_1 - \gamma\,\hbar\,T_2\,x_4\,y_1 + e^{\hbar\,t_1}\,\gamma\,\hbar\,x_4\,y_2\right),\right.$$
$$1 + \frac{1}{4\,\gamma}\left(4\,\hbar\,a_1\,a_2 + 4\,\hbar\,a_1\,a_4 + 4\,\hbar\,a_2\,a_4 - 4\,\gamma\,\hbar^2\,a_4\,x_2\,y_1 - 8\,e^{\hbar\,t_2}\,\gamma\,\hbar^2\,a_2\,x_4\,y_1 + 8\,\gamma\,\hbar^2\,a_2\,T_2\,x_4\,y_1 - \gamma^2\,\hbar^3\,x_2^2\,y_1^2 + 4\,e^{\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1^2 - 4\,\gamma^2\,\hbar^3\,T_2\,x_2\,x_4\,y_1^2 - \gamma^2\,\hbar^3\,x_4^2\,y_1^2 - e^{2\,\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1^2 + \gamma^2\,\hbar^3\,T_2^2\,x_4^2\,y_1^2 - 4\,e^{\hbar\,t_1}\,\gamma\,\hbar^2\,a_1\,x_4\,y_2 + 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1\,y_2 + 4\,e^{\hbar\,t_1+\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1\,y_2 - 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,T_2\,x_4^2\,y_1\,y_2 - e^{2\,\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_4^2\,y_2^2\right)\epsilon + O[\epsilon]^2\left.\right]$$

```
ŒQU,1[R1,2 R3,4 R5,6 // m3,1→1 // m5,2→2 // m6,4→4]
```

$\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma\hbar x_2 y_1 + \gamma\hbar x_4 y_1 + e^{\hbar t_1}\gamma\hbar x_4 y_2\right),$
$1 + \frac{1}{4\gamma}\left(4\hbar a_1 a_2 + 4\hbar a_1 a_4 + 4\hbar a_2 a_4 - 4\gamma\hbar^2 a_4 x_2 y_1 - \gamma^2\hbar^3 x_2^2 y_1^2 - \gamma^2\hbar^3 x_4^2 y_1^2 - 4 e^{\hbar t_1}\gamma\hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1}\gamma^2\hbar^3 x_2 x_4 y_1 y_2 - e^{2\hbar t_1}\gamma^2\hbar^3 x_4^2 y_2^2\right)\epsilon + O[\epsilon]^2\Big]$

```
ŒQU,1[R1,2 R3,4 R5,6 // m1,3→1 // m2,5→2 // m4,6→4] ≡ ŒQU,1[R1,2 R3,4 R5,6 // m3,1→1 // m5,2→2 // m6,4→4]
```

$\hbar\left(e^{\hbar t_2} - T_2\right) x_4 y_1 == 0 \,\&\&\, \epsilon\hbar\left(e^{\hbar t_2} - T_2\right) x_4 y_1\left(8 a_2 + \gamma\hbar\left(-4 x_2 y_1 + x_4\left(\left(e^{\hbar t_2} + T_2\right) y_1 - 4 e^{\hbar t_1} y_2\right)\right)\right) == 0$

## The antipode on exponentials

If $S(e^{\xi x})\,\mathbb{O}(ax : fe^{-\xi x})$, then $f(\xi = 0) = 1$ and
$\mathbb{O}(ax : (\partial_\xi f - x)\, e^{-\xi x}) = \partial_\xi S(e^{\xi x}) = S(xe^{\xi x}) = S(e^{\xi x})\, S(x) = \mathbb{O}(ax : fe^{-\xi x})\,(-e^{\hbar\epsilon a}\, x) = \mathbb{O}(axa_2\, x_2 : -x_2\, fe^{-\xi x + \hbar\epsilon a_2})$,
and that's an ODE for $f$.

```
ŒQU[{a1, x1, a2, x2}1,
```

## Alternative Algorithms

AltLogos

```
λalt,k_[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ∊ aCU)).ρ@e^(b xCU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Series[-η y - ξ x + η ξ t + c t + d y - 2 ∊ c a + b x /. so, {∊, 0, k}]]];
```

```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[ΛCU,2[{ξ, η}, {x, y}]]]}
```

$\Big\{1 + \left(2 a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2} t\gamma\eta^2\xi^2\right)\epsilon + \frac{1}{2}\left(\left(2 a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2} t\gamma\eta^2\xi^2\right)^2 + 2\left(-a\gamma\eta^2\xi^2 + y\gamma^2\eta^3\xi^2 + x\gamma^2\eta^2\xi^3 - \frac{1}{3} t\gamma^2\eta^3\xi^3\right)\right)\epsilon^2 + O[\epsilon]^3,\ \text{True}\Big\}$