Pensieve header: A unified verification notebook for the $sl_2$-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

## Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

## Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$p = 2; $k = 0; $E := {$k, $p};
$trim := {ℏ^p_. /; p > $p → 0, ϵ^k_. /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ℏ t_i), T → e^(ℏ t)}; q_ℏ = e^(γ ϵ ℏ);
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. TRule], {ℏ, 0, $p}],
   ℏ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. TRule, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c ∂_{{x,m},{y,n}} λ]
```

```
HL[DP_{x→D_ξ, y→D_η}[x^2 y^3][e^(δ ξ η)] == 6 e^(δ η ξ) δ^3 ξ + 6 e^(δ η ξ) δ^4 η ξ^2 + e^(δ η ξ) δ^5 η^2 ξ^3]
```

True

Self-Pair (SP):

SeriesData

```
Unprotect[SeriesData];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
Protect[SeriesData];
```

SP

```
SP_{}[P_] := P; SP_{ξ_→x_, ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^d_. :→ ∂_{x,d} f
```

```
SP_{ξ→x}[(ξ^2 + ξ + 3) (x^5 ⅇ^x + 7 x) + 99 a]
```

$7 + 99\,a + 21\,x + 20\,e^{x}\,x^{3} + 15\,e^{x}\,x^{4} + 5\,e^{x}\,x^{5}$

```
SP_{ξ→x,η→y}[(ξ^2 + ξ + 3 + 2 ξ η) (x^5 ⅇ^x + 7 x) + 99 a + ⅇ^(δ x y) ξ η]
```

$7 + 99\,a + 21\,x + 20\,e^{x}\,x^{3} + 15\,e^{x}\,x^{4} + 5\,e^{x}\,x^{5} + e^{x\,y\,\delta}\,\delta + e^{x\,y\,\delta}\,x\,y\,\delta^{2}$

## DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
  U_i_[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ Replace[u, x_ ⧴ x_i, 1]};
  U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List ⧴ l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
      ]] /. x_null ⧴ x];
  pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  m_j_→k_[c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
     U @@ Cases[u, w_j ⧴ w_k] ** U @@ Cases[u, _k]];
  S_i_[c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
     U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]] ]
```

## DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
   {(g_ → img_) ⧴ (m[U[g]] = img), (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

## Meta-Operations

QLImplementation

```
m_{j_→j_} = Identity;
m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
S_{i_}[ℰ_Plus] := Simp[S_i /@ ℰ];
```

## Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_{i_}[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}]]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.01563, {28 t^2 γ^4 CU[y, y, y, x, x] + <<23>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of $S$ on products of triples:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
CU[a₂, x₂, y₃] t₃² + CU[x₂, y₃, a₃] t₃²
```

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z[[1]]₁, z[[2]]₂, z[[3]]₃]; z → HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}]]
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

## Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ, {a, 0, 3}]
```

Aside

$$\frac{1-T}{\hbar} + 2\,T\,\epsilon\,a - 2\left(T\,\epsilon^2\,\hbar\right)a^2 + \frac{4}{3}\,T\,\epsilon^3\,\hbar^2\,a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ℏ - 1] QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 ⅇ^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := 𝕆_QU[{a, x}, SS[-ⅇ^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y], {QU[y], QU[x]} → ((-1 + T) QU[])/ħ},
 {{QU[a], QU[y]} → -γ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} → ((1 - T) QU[])/ħ, {QU[x], QU[a]} → -γ QU[x], {QU[x], QU[x]} → 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

```
{1.60938, {((28 γ^4 - 56 T γ^4 + 28 T^2 γ^4) QU[y, y, y, x, x])/ħ^2 +
   <<17>> + QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y_1, a_1, x_1}},
 Table[{z1, z2} → HL@Simp[S_1[z1 ** z2] - S_1[z2] ** S_1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{{QU[y_1], QU[y_1]} → 0, {QU[y_1], QU[a_1]} → 0, {QU[y_1], QU[x_1]} → 0},
 {{QU[a_1], QU[y_1]} → 0, {QU[a_1], QU[a_1]} → 0, {QU[a_1], QU[x_1]} → 0},
 {{QU[x_1], QU[y_1]} → 0, {QU[x_1], QU[a_1]} → 0, {QU[x_1], QU[x_1]} → 0}}
```

Verifying that $\mathrm{Lim}_{\hbar\to 0}\,\mathrm{QU} = \mathrm{CU}$ using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ħ → 0] - lhs] // HL
  }] // Timing
```

```
{3.71875, {28 t^2 γ^4 CU[y, y, y, x, x] + <<23>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
  7 (4 γ^4/ħ^2 - 8 T γ^4/ħ^2 + 4 T^2 γ^4/ħ^2) QU[y, y, y, x, x] +
   <<41>> + QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

## Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y :→ O_QU[{a, x}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) x]],
  a → -a_QU, x :→ O_QU[{a, y}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\ QU[x] \to -\frac{QU[y]}{\sqrt{T}} \to QU[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

## The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$A\mathbb{D}\$f = \gamma\,\frac{\mathrm{Cosh}\left[\hbar\left(a\,\epsilon + \frac{\gamma\,\epsilon}{2} - \frac{t}{2}\right)\right] - \mathrm{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,\epsilon}{2}\right)^2 + \epsilon\,\omega}\right]}{\hbar\,e^{\hbar\,((a+\gamma)\,\epsilon - t/2)}\,\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(a^2\,\epsilon + a\,\gamma\,\epsilon - a\,t - \omega\right)};$$

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
True
```

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
True
```

ADeq

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[A𝔻, QU → CU,
 {a → a_CU, x → CU@x, y :→ 𝕊_CU[SS[A𝔻$f], a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

## The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$S\mathbb{D}\$g = \sqrt{\frac{2\,\gamma\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\,\epsilon^2+4\,\epsilon\,\varpi}\right]-\mathrm{Cosh}\left[\frac{t-\epsilon\,\gamma-2\,\epsilon\,a}{2/\hbar}\right]\right)}{\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(t\,\left(2\,a+\gamma\right)-2\,a\,\left(a+\gamma\right)\,\epsilon+2\,\varpi\right)\,\hbar}};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ħ ((ϵ-t)/2 + ϵ a)] - Cosh[ħ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) / (ħ Sinh[-ϵ ħ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}\,(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}\,(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$

$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}\,(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True}\right\}$$

SDeq

```
SD$f = Simplify[e^(ħ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x :→ S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y :→ S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the $\theta$-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

## The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = (0 0; ∈ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = (0 (1 - e^(-γ∈ħ)) / (∈ħ); 0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] := (ℰ /. TRule /. t → γ∈ /. (U : CU | QU)[u___] ⧴ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ∈^k_. /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[e^(ξ U@x)].ρ[e^(α U@a)] == ρ[e^(α U@a)].ρ[e^(e^(-γα) ξ U@x)]], {U, {CU, QU}}]
{True, True}
```

## Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^Q P]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^Q P]];
```

Logos

```
c_Integer_k_Integer := c + O[∈]^(k+1);
Λ_U_,k_[{α_, β_}, {x_, x_}] := Œ_U[{x}, (α + β) x, 1_k];
Λ_U_,k_[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-γα) ξ x, 1_k];
Λ_U_,k_[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-γα) η y, 1_k];
```

```
Table[
 {Λ_{U,1}[{α, β}, {u, u}],
  lhs = U@Œ_U[{u_1, u_2}, ℏ (α u_1 + β u_2), 1], HL[lhs == U@Λ_{U,1}[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

$\{\{\{Œ_{CU}[\{y\}, y(\alpha+\beta), 1+O[\epsilon]^2],$
$CU[] + (\alpha\hbar+\beta\hbar)\, CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[y, y], \text{True}\},$
$\{Œ_{CU}[\{a\}, a(\alpha+\beta), 1+O[\epsilon]^2], CU[] + (\alpha\hbar+\beta\hbar)\, CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[a, a],$
$\text{True}\}, \{Œ_{CU}[\{x\}, x(\alpha+\beta), 1+O[\epsilon]^2],$
$CU[] + (\alpha\hbar+\beta\hbar)\, CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[x, x], \text{True}\}\},$
$\{\{Œ_{QU}[\{y\}, y(\alpha+\beta), 1+O[\epsilon]^2], QU[] + (\alpha\hbar+\beta\hbar)\, QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[y, y],$
$\text{True}\}, \{Œ_{QU}[\{a\}, a(\alpha+\beta), 1+O[\epsilon]^2], QU[] + (\alpha\hbar+\beta\hbar)\, QU[a] +$
$\left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[a, a], \text{True}\}, \{Œ_{QU}[\{x\}, x(\alpha+\beta), 1+O[\epsilon]^2],$
$QU[] + (\alpha\hbar+\beta\hbar)\, QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[x, x], \text{True}\}\}\}$

```
{Λ_{#,1}[{ξ, α}, {x, a}], lhs = #@Œ_#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ_{#,1}[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x\xi, 1+O[\epsilon]^2],$
$CU[] + \alpha\hbar\, CU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\, CU[x] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + \alpha\xi\hbar^2 CU[a, x] + \frac{1}{2}\xi^2\hbar^2 CU[x, x],$
$\text{True}\}, \{Œ_{QU}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x\xi, 1+O[\epsilon]^2], QU[] + \alpha\hbar\, QU[a] +$
$(\xi\hbar - \alpha\gamma\xi\hbar^2)\, QU[x] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + \alpha\xi\hbar^2 QU[a, x] + \frac{1}{2}\xi^2\hbar^2 QU[x, x], \text{True}\}\}$

```
{Λ_{#,2}[{α, η}, {a, y}], lhs = #@Œ_#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ_{#,2}[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y\eta, 1+O[\epsilon]^3],$
$CU[] + \alpha\hbar\, CU[a] + (\eta\hbar - \alpha\gamma\eta\hbar^2)\, CU[y] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + \alpha\eta\hbar^2 CU[y, a] + \frac{1}{2}\eta^2\hbar^2 CU[y, y],$
$\text{True}\}, \{Œ_{QU}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y\eta, 1+O[\epsilon]^3], QU[] + \alpha\hbar\, QU[a] +$
$(\eta\hbar - \alpha\gamma\eta\hbar^2)\, QU[y] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + \alpha\eta\hbar^2 QU[y, a] + \frac{1}{2}\eta^2\hbar^2 QU[y, y], \text{True}\}\}$

Goal. In either $U$, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now $F$ satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {e- → 1, U → Times}];
   Collect[sol /. {e- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

Logos

```
Λ_{U_,kk_}[{ξ1_, η1_}, {x, y}] :=
  Λ_U[{ξ1, η1}, {x, y}] = Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
     G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
     fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
     F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ϵ^l U@{y^i, a^j, x^k});
     es = Flatten[
       Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
     F = F /. DSolve[es, fs, η][[1]];
     Œ_U[{y, a, x},
       ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
       F + 0_$k /. {e- → 1, U → Times}
      ] /. {ξ → ξ1, η → η1}]];
```

```
{Λ_{CU,1}[{ξ, η}, {x, y}], lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@Λ_{CU,1}[ℏ {ξ, η}, {x, y}]]}
```

$\{Œ_{CU}[\{y, a, x\}, y\eta + x\xi - t\eta\xi, 1 + \frac{1}{2}\eta\xi(4a - 2y\gamma\eta - 2x\gamma\xi + t\gamma\eta\xi)\epsilon + O[\epsilon]^2],$
$(1 - t\eta\xi\hbar^2)\,CU[\,] + \xi\hbar\,CU[x] + \eta\hbar\,CU[y] +$
$\frac{1}{2}\xi^2\hbar^2\,CU[x, x] + \eta\xi\hbar^2\,CU[y, x] + \frac{1}{2}\eta^2\hbar^2\,CU[y, y],$ True$\}$

```
{Λ_{QU,1}[{ξ, η}, {x, y}], lhs = QU@Œ_QU[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η}, {x, y}]]}
```

$\{Œ_{QU}[\{y, a, x\}, y\eta + x\xi + \frac{(1-T)\eta\xi}{\hbar}, 1 + \frac{1}{4\hbar}$
$\eta\xi(\gamma\eta\xi - 4T\gamma\eta\xi + 3T^2\gamma\eta\xi + 8aT\hbar + 2y\gamma\eta\hbar - 6Ty\gamma\eta\hbar + 2x\gamma\xi\hbar - 6Tx\gamma\xi\hbar + 4xy\gamma\hbar^2)\epsilon +$
$O[\epsilon]^2], (1 + \eta\xi\hbar - T\eta\xi\hbar)\,QU[\,] + \xi\hbar\,QU[x] + \eta\hbar\,QU[y] +$
$\frac{1}{2}\xi^2\hbar^2\,QU[x, x] + \eta\xi\hbar^2\,QU[y, x] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y],$ True$\}$

```
{tt = Last[Λ_CU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + \frac{1}{2}\eta\xi(4a - 2y\gamma\eta - 2x\gamma\xi + t\gamma\eta\xi)\epsilon + O[\epsilon]^2, \epsilon\left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right)\}$

```
{tt = Last[ΛQU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\left\{1+\frac{1}{4\hbar}\eta\,\xi\left(\gamma\,\eta\,\xi-4\,T\,\gamma\,\eta\,\xi+3\,T^2\,\gamma\,\eta\,\xi+8\,a\,T\,\hbar+2\,y\,\gamma\,\eta\,\hbar-6\,T\,y\,\gamma\,\eta\,\hbar+2\,x\,\gamma\,\xi\,\hbar-6\,T\,x\,\gamma\,\xi\,\hbar+4\,x\,y\,\gamma\,\hbar^2\right)\epsilon+O[\epsilon]^2,\ \frac{1}{4\hbar}\epsilon\left(\gamma\,\eta^2\,\xi^2-4\,T\,\gamma\,\eta^2\,\xi^2+3\,T^2\,\gamma\,\eta^2\,\xi^2+8\,a\,T\,\eta\,\xi\,\hbar+2\,y\,\gamma\,\eta^2\,\xi\,\hbar-6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+2\,x\,\gamma\,\eta\,\xi^2\,\hbar-6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2\right)\right\}$$

Logos

```
Simp[ŒU_[specs___, Q_, P_]] :=
  ŒU[specs, ExpandNumerator@Together[Q /. e^δ_ :> e^Expand@Together[δ]],
   MapAt[ExpandNumerator@*Together, P /. e^δ_ :> e^Expand@Together[δ], 3]];
```

Logos

```
ΛU_,k_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
    {yax, q, p} = List @@ ΛU,k[{υ, ω}, {u, w}];
   ŒU[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
     Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[p][e^Q]] + Θk] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}];
```

```
{ΛCU,2[{ξ, η, δ}, {x, y}], lhs = CU@ŒCU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL[lhs == CU@ΛCU,1[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\left\{Œ_{CU}\left[\{y,a,x\},\ \frac{x\,y\,\delta+y\,\eta+x\,\xi-t\,\eta\,\xi}{1+t\,\delta},\ \frac{1}{1+t\,\delta}+\frac{\left(4\,a\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\right)\epsilon}{2\,(1+t\,\delta)}+\frac{1}{24\,(1+t\,\delta)}\left(48\,a^2\,\eta^2\,\xi^2-24\,a\,\gamma\,\eta^2\,\xi^2-48\,a\,y\,\gamma\,\eta^3\,\xi^2+24\,y\,\gamma^2\,\eta^3\,\xi^2+12\,y^2\,\gamma^2\,\eta^4\,\xi^2-48\,a\,x\,\gamma\,\eta^2\,\xi^3+24\,x\,\gamma^2\,\eta^2\,\xi^3+24\,a\,t\,\gamma\,\eta^3\,\xi^3-8\,t\,\gamma^2\,\eta^3\,\xi^3+24\,x\,y\,\gamma^2\,\eta^3\,\xi^3-12\,t\,y\,\gamma^2\,\eta^4\,\xi^3+12\,x^2\,\gamma^2\,\eta^2\,\xi^4-12\,t\,x\,\gamma^2\,\eta^3\,\xi^4+3\,t^2\,\gamma^2\,\eta^4\,\xi^4\right)\epsilon^2+O[\epsilon]^3\right],\right.$$

$$\left(1-t\,\delta\,\hbar+t^2\,\delta^2\,\hbar^2-t\,\eta\,\xi\,\hbar^2\right)CU[\,]+\left(\xi\,\hbar-2\,t\,\delta\,\xi\,\hbar^2\right)CU[x]+\left(\eta\,\hbar-2\,t\,\delta\,\eta\,\hbar^2\right)CU[y]+\frac{1}{2}\xi^2\,\hbar^2\,CU[x,x]+\left(\delta\,\hbar-2\,t\,\delta^2\,\hbar^2+\eta\,\xi\,\hbar^2\right)CU[y,x]+\frac{1}{2}\eta^2\,\hbar^2\,CU[y,y]+\delta\,\xi\,\hbar^2\,CU[y,x,x]+\delta\,\eta\,\hbar^2\,CU[y,y,x]+\frac{1}{2}\delta^2\,\hbar^2\,CU[y,y,x,x],\ \text{True}\Big\}$$

```
{Λ_{QU,2}[{ξ, η, δ}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{\text{Œ}_{QU}\Big[\{y, a, x\}, \frac{\eta\,\xi - T\,\eta\,\xi + x\,y\,\delta\,\hbar + y\,\eta\,\hbar + x\,\xi\,\hbar}{-\delta + T\,\delta + \hbar},$$

$$\frac{\hbar}{-\delta + T\,\delta + \hbar} + \frac{1}{4\,(-\delta + T\,\delta + \hbar)}\big(\gamma\,\eta^2\,\xi^2 - 4\,T\,\gamma\,\eta^2\,\xi^2 + 3\,T^2\,\gamma\,\eta^2\,\xi^2 + 8\,a\,T\,\eta\,\xi\,\hbar + 2\,y\,\gamma\,\eta^2\,\xi\,\hbar - 6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar + 2\,x\,\gamma\,\eta\,\xi^2\,\hbar - 6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2\big)\,\epsilon + \frac{1}{288\,\hbar\,(-\delta + T\,\delta + \hbar)}$$

$$\big(9\,\gamma^2\,\eta^4\,\xi^4 - 72\,T\,\gamma^2\,\eta^4\,\xi^4 + 198\,T^2\,\gamma^2\,\eta^4\,\xi^4 - 216\,T^3\,\gamma^2\,\eta^4\,\xi^4 + 81\,T^4\,\gamma^2\,\eta^4\,\xi^4 + 144\,a\,T\,\gamma\,\eta^3\,\xi^3\,\hbar -$$
$$576\,a\,T^2\,\gamma\,\eta^3\,\xi^3\,\hbar + 432\,a\,T^3\,\gamma\,\eta^3\,\xi^3\,\hbar + 40\,\gamma^2\,\eta^3\,\xi^3\,\hbar - 312\,T\,\gamma^2\,\eta^3\,\xi^3\,\hbar + 600\,T^2\,\gamma^2\,\eta^3\,\xi^3\,\hbar -$$
$$328\,T^3\,\gamma^2\,\eta^3\,\xi^3\,\hbar + 36\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar - 252\,T\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar + 540\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar - 324\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar +$$
$$36\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar - 252\,T\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar + 540\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar - 324\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar + 576\,a^2\,T^2\,\eta^2\,\xi^2\,\hbar^2 +$$
$$576\,a\,T\,\gamma\,\eta^2\,\xi^2\,\hbar^2 - 864\,a\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar^2 + 36\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2 - 216\,T\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2 + 180\,T^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2 +$$
$$288\,a\,T\,y\,\gamma\,\eta^3\,\xi^2\,\hbar^2 - 864\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2\,\hbar^2 + 120\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2 - 816\,T\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2 +$$
$$984\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2 + 36\,y^2\,\gamma^2\,\eta^4\,\xi^2\,\hbar^2 - 216\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2\,\hbar^2 + 324\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2\,\hbar^2 +$$
$$288\,a\,T\,x\,\gamma\,\eta^2\,\xi^3\,\hbar^2 - 864\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3\,\hbar^2 + 120\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2 - 816\,T\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2 +$$
$$984\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2 + 144\,x\,y\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2 - 720\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2 + 864\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2 +$$
$$36\,x^2\,\gamma^2\,\eta^2\,\xi^4\,\hbar^2 - 216\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4\,\hbar^2 + 324\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4\,\hbar^2 - 576\,a^2\,T\,\eta\,\xi\,\hbar^3 +$$
$$864\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^3 + 72\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^3 - 360\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^3 + 48\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^3 - 336\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^3 +$$
$$864\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^3 + 72\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^3 - 360\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^3 + 576\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar^3 +$$
$$360\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^3 - 1512\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^3 + 144\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar^3 - 432\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar^3 +$$
$$48\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^3 - 336\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^3 + 144\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar^3 - 432\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar^3 +$$
$$144\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^4 + 144\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^4 + 144\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^4 + 144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^4\big)\,\epsilon^2 + O[\epsilon]^3\Big],$$

$$\big(1 + \delta - T\,\delta + \delta^2 - 2\,T\,\delta^2 + T^2\,\delta^2 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\big)\,QU[\,] +$$
$$\big(\xi\,\hbar + 2\,\delta\,\xi\,\hbar - 2\,T\,\delta\,\xi\,\hbar\big)\,QU[x] +$$
$$\big(\eta\,\hbar + 2\,\delta\,\eta\,\hbar - 2\,T\,\delta\,\eta\,\hbar\big)\,QU[y] + \frac{1}{2}\,\xi^2\,\hbar^2\,QU[x, x] +$$
$$\big(\delta\,\hbar + 2\,\delta^2\,\hbar - 2\,T\,\delta^2\,\hbar + \eta\,\xi\,\hbar^2\big)\,QU[y, x] +$$
$$\frac{1}{2}\,\eta^2\,\hbar^2\,QU[y, y] +$$
$$\delta\,\xi\,\hbar^2\,QU[y, x, x] +$$
$$\delta\,\eta\,\hbar^2\,QU[y, y, x] +$$
$$\frac{1}{2}\,\delta^2\,\hbar^2\,QU[y, y, x, x], \text{True}\Big\}$$

```
{tt = Last[Λ_{CU,2}[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, 2}]}
```

$$\Big\{\frac{1}{1+t\,\delta}+\frac{\left(4\,a\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\right)\,\epsilon}{2\,\left(1+t\,\delta\right)}+$$

$$\frac{1}{24\,\left(1+t\,\delta\right)}\,\big(48\,a^2\,\eta^2\,\xi^2-24\,a\,\gamma\,\eta^2\,\xi^2-48\,a\,y\,\gamma\,\eta^3\,\xi^2+24\,y\,\gamma^2\,\eta^3\,\xi^2+12\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$
$$48\,a\,x\,\gamma\,\eta^2\,\xi^3+24\,x\,\gamma^2\,\eta^2\,\xi^3+24\,a\,t\,\gamma\,\eta^3\,\xi^3-8\,t\,\gamma^2\,\eta^3\,\xi^3+24\,x\,y\,\gamma^2\,\eta^3\,\xi^3-$$
$$12\,t\,y\,\gamma^2\,\eta^4\,\xi^3+12\,x^2\,\gamma^2\,\eta^2\,\xi^4-12\,t\,x\,\gamma^2\,\eta^3\,\xi^4+3\,t^2\,\gamma^2\,\eta^4\,\xi^4\big)\,\epsilon^2+O[\epsilon]^3,$$

$$\epsilon\,\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)+\frac{1}{3}\,\epsilon^2\,\left(-3\,a\,\gamma\,\eta^2\,\xi^2+3\,y\,\gamma^2\,\eta^3\,\xi^2+3\,x\,\gamma^2\,\eta^2\,\xi^3-t\,\gamma^2\,\eta^3\,\xi^3\right)+$$
$$\mathrm{Log}\Big[\frac{1}{1+t\,\delta}\Big]\Big\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, 2}]}
```

$$\Big\{\frac{\hbar}{-\delta+T\,\delta+\hbar}+\frac{1}{4\,\left(-\delta+T\,\delta+\hbar\right)}$$
$$\big(\gamma\,\eta^2\,\xi^2-4\,T\,\gamma\,\eta^2\,\xi^2+3\,T^2\,\gamma\,\eta^2\,\xi^2+8\,a\,T\,\eta\,\xi\,\hbar+2\,y\,\gamma\,\eta^2\,\xi\,\hbar-6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$2\,x\,\gamma\,\eta\,\xi^2\,\hbar-6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2\big)\,\epsilon+$$

$$\frac{1}{288\,\hbar\,\left(-\delta+T\,\delta+\hbar\right)}\,\big(9\,\gamma^2\,\eta^4\,\xi^4-72\,T\,\gamma^2\,\eta^4\,\xi^4+198\,T^2\,\gamma^2\,\eta^4\,\xi^4-216\,T^3\,\gamma^2\,\eta^4\,\xi^4+81\,T^4\,\gamma^2\,\eta^4\,\xi^4+$$
$$144\,a\,T\,\gamma\,\eta^3\,\xi^3\,\hbar-576\,a\,T^2\,\gamma\,\eta^3\,\xi^3\,\hbar+432\,a\,T^3\,\gamma\,\eta^3\,\xi^3\,\hbar+40\,\gamma^2\,\eta^3\,\xi^3\,\hbar-312\,T\,\gamma^2\,\eta^3\,\xi^3\,\hbar+$$
$$600\,T^2\,\gamma^2\,\eta^3\,\xi^3\,\hbar-328\,T^3\,\gamma^2\,\eta^3\,\xi^3\,\hbar+36\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar-252\,T\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar+540\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar-$$
$$324\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3\,\hbar+36\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar-252\,T\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar+540\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar-324\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4\,\hbar+$$
$$576\,a^2\,T^2\,\eta^2\,\xi^2\,\hbar^2+576\,a\,T\,\gamma\,\eta^2\,\xi^2\,\hbar^2-864\,a\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar^2+36\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2-216\,T\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2+$$
$$180\,T^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2+288\,a\,T\,y\,\gamma\,\eta^3\,\xi^2\,\hbar^2-864\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2\,\hbar^2+120\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2-$$
$$816\,T\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2+984\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2+36\,y^2\,\gamma^2\,\eta^4\,\xi^2\,\hbar^2-216\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2\,\hbar^2+$$
$$324\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2\,\hbar^2+288\,a\,T\,x\,\gamma\,\eta^2\,\xi^3\,\hbar^2-864\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3\,\hbar^2+120\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2-$$
$$816\,T\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2+984\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2+144\,x\,y\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2-720\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2+$$
$$864\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2+36\,x^2\,\gamma^2\,\eta^2\,\xi^4\,\hbar^2-216\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4\,\hbar^2+324\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4\,\hbar^2-$$
$$576\,a^2\,T\,\eta\,\xi\,\hbar^3+864\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^3+72\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^3-360\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^3+48\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^3-$$
$$336\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^3+864\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^3+72\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^3-360\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^3+576\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar^3+$$
$$360\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^3-1512\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^3+144\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar^3-432\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar^3+$$
$$48\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^3-336\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^3+144\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar^3-432\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar^3+$$
$$144\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^4+144\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^4+144\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^4+144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^4\big)\,\epsilon^2+O[\epsilon]^3,$$

$$\frac{1}{4\,\hbar}\epsilon\,\big(\gamma\,\eta^2\,\xi^2-4\,T\,\gamma\,\eta^2\,\xi^2+3\,T^2\,\gamma\,\eta^2\,\xi^2+8\,a\,T\,\eta\,\xi\,\hbar+2\,y\,\gamma\,\eta^2\,\xi\,\hbar-6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$2\,x\,\gamma\,\eta\,\xi^2\,\hbar-6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2\big)+$$

$$\frac{1}{72\,\hbar}\,\epsilon^2\,\big(10\,\gamma^2\,\eta^3\,\xi^3-78\,T\,\gamma^2\,\eta^3\,\xi^3+150\,T^2\,\gamma^2\,\eta^3\,\xi^3-82\,T^3\,\gamma^2\,\eta^3\,\xi^3+144\,a\,T\,\gamma\,\eta^2\,\xi^2\,\hbar-$$
$$216\,a\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar+9\,\gamma^2\,\eta^2\,\xi^2\,\hbar-54\,T\,\gamma^2\,\eta^2\,\xi^2\,\hbar+45\,T^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar+30\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$
$$204\,T\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar+246\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar+30\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar-204\,T\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$
$$246\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar-144\,a^2\,T\,\eta\,\xi\,\hbar^2+216\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^2+18\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^2-90\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^2+$$
$$12\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^2-84\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^2+216\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^2+18\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^2-90\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^2+$$
$$90\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2-378\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2+12\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^2-84\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^2+$$
$$36\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^3+36\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^3+36\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^3\big)+\mathrm{Log}\Big[\frac{\hbar}{-\delta+T\,\delta+\hbar}\Big]\Big\}$$

## Reorderings with Rord

Rord

```
Rord_{u_i_,w_j_→k_}[ℭ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, kk = P[[5]], δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = Echo[List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}],
        Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /. {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s}];
   ℭ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} DP_{u_i→D_υ, w_j→D_ω}[P][p e^q]] /.
    {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

Rord

```
Rord_{u_i_,w_j_→k_}[ℭ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Simp@Module[{υ, ω, δ, Λ1, yax, q, p, n, kk = P[[5]], δ1 = ∂_{u_i,w_j} Q},
     {yax, q, p} = List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}], Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /.
       {y → y_n, a → a_n, x → x_n, t → t_s, T → T_s};
     (*Echo@{{{u_i,υ},{w_j,ω}},P,p e^q};*)
     ℭ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} SP_{u_i→υ, w_j→ω}[P p e^q]] /.
      {n → k, υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
With[{cℴ = ℭ_CU[{y_1, x_1}_1, {x_2, a_2, y_2}_2, ℏ t_1 a_2 + ℏ t_1^-1 (e^t_1 - 1) y_1 x_2, 1_2 + ϵ x_1 y_2]},
 {Short[rhs = cℴ // Rord_{x_2,a_2→3}, 3], HL[CU[cℴ] == CU[rhs]]}]
```

$$\left\{\mathfrak{C}_{CU}\left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1}\hbar x_3 y_1\right)}{t_1}, 1 + x_1 y_2 \in + O[\in]^3\right], \text{True}\right\}$$

```
With[{cℴ = ℭ_CU[{y_1, a_1, a_2}_1, {x_2, x_1, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = cℴ // Rord_{a_1,a_2→3} // Rord_{x_2,x_1→4}, 3], HL[CU[cℴ] == CU[rhs]]}]
```

$$\{\mathfrak{C}_{CU}[\{y_1, a_3\}_1, \{x_4, y_2\}_2, \hbar a_3 l_{11} t_1 + \hbar a_3 l_{12} t_1 + \hbar a_3 l_{21} t_2 + \hbar a_3 l_{22} t_2 + \hbar x_4 y_1 \gamma_{11} + \hbar x_4 y_2 \gamma_{12} + \hbar x_4 y_1 \gamma_{21} + \hbar x_4 y_2 \gamma_{22}, 1 + O[\in]^3], \text{True}\}$$

```
ℏ a_3 l_11 t_1 + ℏ a_3 l_12 t_1 + ℏ a_3 l_21 t_2 + ℏ a_3 l_22 t_2 +
  ℏ x_4 y_1 γ_11 + ℏ x_4 y_2 γ_12 + ℏ x_4 y_1 γ_21 + ℏ x_4 y_2 γ_22 // Simplify
```

$$\hbar\left(a_3\left(l_{11} t_1 + l_{12} t_1 + \left(l_{21} + l_{22}\right) t_2\right) + x_4\left(y_1\left(\gamma_{11} + \gamma_{21}\right) + y_2\left(\gamma_{12} + \gamma_{22}\right)\right)\right)$$

```
With[{cℴ = ℭ_CU[{y_1, a_1, x_1}_1, {x_2, a_2, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = cℴ // Rord_{x_2,a_2→3}, 3], HL[CU[cℴ] == CU[rhs]]}]
```

$$\{\mathfrak{C}_{CU}[\{y_1, a_1, x_1\}_1, \{a_3, x_3, y_2\}_2, e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\left(e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\hbar a_1 l_{11} t_1 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\hbar a_3 l_{12} t_1 + \ll 4 \gg + \hbar x_3 y_1 \gamma_{21} + \hbar x_3 y_2 \gamma_{22}\right), 1 + \left(a_1 l_1 + p_{11} x_1 y_1 + p_{12} x_1 y_2\right) \in + O[\in]^3], \text{True}\}$$

```
With[{qφ = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = qφ // Rord_{x2,a2→3}, 3], HL[QU[qφ] == QU[rhs]]}]
```

```
{𝔼QU[{y1, a1, x1}1, {a3, x3, y2}2,
  e^(-γ ℏ (l12 t1+l22 t2)) (e^(γ ℏ (l12 t1+l22 t2)) ℏ a1 l11 t1 + e^(γ ℏ (l12 t1+l22 t2)) ℏ a3 l12 t1 + e^(γ ℏ (l12 t1+l22 t2)) ℏ a1 l21 t2 +
     e^<<1>> ℏ a3 l22 t2 + <<1>> + e^(γ ℏ (<<1>>)) ℏ x1 y2 γ12 + ℏ x3 y1 γ21 + ℏ x3 y2 γ22),
  1 + (a1 l1 + p11 x1 y1 + p12 x1 y2) ϵ + O[ϵ]^3], True}
```

```
With[{qφ = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = qφ // Rord_{a2,y2→3}, 3], HL[QU[qφ] == QU[rhs]]}]
```

```
{𝔼QU[{y1, a1, x1}1, {x2, y3, a3}2,
  e^(-γ ℏ (l12 t1+l22 t2)) (e^(γ ℏ (l12 t1+l22 t2)) ℏ a1 l11 t1 + e^(γ ℏ (l12 t1+l22 t2)) ℏ a3 l12 t1 + e^(γ ℏ (l12 t1+l22 <<1>>)) ℏ a1 l21 t2 +
     e^<<1>> <<3>> t2 + <<1>> + ℏ x1 y3 γ12 + e^(γ ℏ (l12 t1+l22 t2)) ℏ x2 y1 γ21 + ℏ x2 y3 γ22),
  1 + (a1 l1 + p11 x1 y1 + p21 x2 y1) ϵ + O[ϵ]^3], True}
```

```
Timing@With[{qφ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
     ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qφ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qφ] == QU[rhs]]}]
```

```
{31.2031, {𝔼QU[{y3, a3, x3}1, {x2, a2, y2}2, (ℏ a2 l12 t1 + <<16>> + ℏ T1 x2 y2 γ11 γ22) / (1 - γ11 + T1 γ11),
   1/(1 - γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 - 4 p11 T1 + 4 ℏ p22 x2 y2 + <<111>> + 12 γ ℏ^4 T1^2 x2^2 y2^2 γ11^2 γ12^2 γ21^2 -
         10 γ ℏ^4 T1^3 x2^2 y2^2 γ11^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^4 x2^2 y2^2 γ11^2 γ12^2 γ21^2) ϵ) / (4 ℏ (1 - γ11 + T1 γ11)^3) +
    ((576 a3 p11 T1 + <<994>> + 81 γ^2 ℏ^6 T1^6 x2^4 y2^4 γ11^2 γ12^4 γ21^4) ϵ^2) / (288 (1 - γ11 + T1 γ11)^3) + O[ϵ]^3], True}}
```

```
Timing@With[{qφ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
     ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qφ // Rord_{x1,y1→1}, 5], HL@SimpT[QU[qφ] == QU[rhs]]}]
```

```
{52.8594, {𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2, (ℏ a2 l12 t1 + <<16>> + ℏ T1 x2 y2 γ11 γ22) / (1 - γ11 + T1 γ11),
   1/(1 - γ11 + T1 γ11) + ((4 a2 l2 + 4 p22 x2 y2 + 8 ℏ^2 a1 T1 x2 y2 γ12 γ21 + 4 γ ℏ^3 x1 x2 y1 y2 γ12 γ21 + <<6>> +
         γ ℏ^3 x2^2 y2^2 γ12^2 γ21^2 - 4 γ ℏ^3 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^3 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) / (4 (1 - γ11 + T1 γ11)) +
    ((-576 ℏ^3 a1^2 T1 x2 y2 γ12 γ21 + 576 ℏ^2 a1 a2 l2 T1 x2 y2 γ12 γ21 + <<113>> + 81 γ^2 ℏ^6 T1^4 x2^4 y2^4 γ12^4 γ21^4)
       ϵ^2) / (288 (1 - γ11 + T1 γ11)) + O[ϵ]^3], True}}
```

## Canonical ordering with Cord

Cord

```
Cord[𝔈U_[L___, {l___, u_i_, w_j_, r___}s_, R___, Q_, P_]] /;
   OrderedQ[{w, u} /. {y → 1, a → 2, x → 3}] :=
  Cord[Echo@Rord_{ui,wj→Unique[]}[𝔈U[L, {l, ui, wj, r}s, R, Q, P]]];
Cord[𝔈U_[specs__, Q_, P_]] := 𝔈U[Sequence @@ Sort@{specs}, Q, P] /.
  Flatten[{specs} /. {yax__}s_ :> ({yax} /. u_i_ :> (ui → us))]
```

```
Block[{$p = 4, cΦ = 𝔈CU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1ϵ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Cord[cΦ], HL[CU[cΦ] == CU[Cord[cΦ]]]}]
```

$$\Big\{\mathbb{E}_{CU}\Big[\{y_1, a_1, x_1\}_1, \frac{1}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} e^{-\gamma\hbar l_{11} t_1-\gamma\hbar l_{12} t_1-\gamma\hbar l_{21} t_2-\gamma\hbar l_{22} t_2}$$
$$\Big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} \hbar a_1 l_{12} t_1 +$$
$$e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} \hbar a_1 l_{21} t_2 + e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} \hbar a_1 l_{22} t_2 +$$
$$e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{11} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{11} t_1^2 \gamma_{12} +$$
$$e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{12} t_1^2 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{21} t_1 t_2 \gamma_{12} +$$
$$e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{22} t_1 t_2 \gamma_{12} + \hbar x_1 y_1 \gamma_{12} +$$
$$e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{21} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{11} t_1^2 \gamma_{22} +$$
$$e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{12} t_1^2 \gamma_{22} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{21} t_1 t_2 \gamma_{22} +$$
$$e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2} \hbar^2 a_1 l_{22} t_1 t_2 \gamma_{22} + \hbar x_1 y_1 \gamma_{22}\Big),$$
$$\frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + O[\epsilon]^1\Big], \text{True}\Big\}$$

```
With[{qΦ = ŒQU[{y1, a1, x1, x2, a2, y2}1,
     ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     1ϵ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Cord[
  qΦ]]
```

$$
\begin{aligned}
&\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \Big(e^{-\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\\
&\quad\Big(e^{\gamma\hbar\, l_{11} t_1+2\gamma\hbar\, l_{12} t_1+\gamma\hbar\, l_{21} t_2+2\gamma\hbar\, l_{22} t_2}\,\hbar\, a_1 l_{11} t_1 + e^{\gamma\hbar\, l_{11} t_1+2\gamma\hbar\, l_{12} t_1+\gamma\hbar\, l_{21} t_2+2\gamma\hbar\, l_{22} t_2}\,\hbar\, a_1 l_{12} t_1 +\\
&\quad e^{\gamma\hbar\, l_{11} t_1+2\gamma\hbar\, l_{12} t_1+\gamma\hbar\, l_{21} t_2+2\gamma\hbar\, l_{22} t_2}\,\hbar\, a_1 l_{21} t_2 + e^{\gamma\hbar\, l_{11} t_1+2\gamma\hbar\, l_{12} t_1+\gamma\hbar\, l_{21} t_2+2\gamma\hbar\, l_{22} t_2}\,\hbar\, a_1 l_{22} t_2 +\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, x_1 y_1 \gamma_{11} - e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{11} t_1 \gamma_{12} -\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{12} t_1 \gamma_{12} - e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{21} t_2 \gamma_{12} -\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{22} t_2 \gamma_{12} + e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{11} t_1 T_1 \gamma_{12} +\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{12} t_1 T_1 \gamma_{12} + e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{21} t_2 T_1 \gamma_{12} +\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{22} t_2 T_1 \gamma_{12} + \hbar\, x_1 y_1 \gamma_{12} +\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, x_1 y_1 \gamma_{21} - e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{11} t_1 \gamma_{22} -\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{12} t_1 \gamma_{22} - e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{21} t_2 \gamma_{22} -\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{22} t_2 \gamma_{22} + e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{11} t_1 T_1 \gamma_{22} +\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{12} t_1 T_1 \gamma_{22} + e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{21} t_2 T_1 \gamma_{22} +\\
&\quad e^{\gamma\,(\hbar\, l_{11} t_1+\hbar\, l_{12} t_1+\hbar\, l_{21} t_2+\hbar\, l_{22} t_2)}\,\hbar\, a_1 l_{22} t_2 T_1 \gamma_{22} + \hbar\, x_1 y_1 \gamma_{22}\Big)\Big) \Big/\\
&\quad \Big(e^{\gamma\,(\hbar\, l_{12} t_1+\hbar\, l_{22} t_2)} - \gamma_{12} + T_1 \gamma_{12} - \gamma_{22} + T_1 \gamma_{22}\Big),\\
&\frac{e^{\gamma\,(\hbar\, l_{12} t_1+\hbar\, l_{22} t_2)}}{e^{\gamma\,(\hbar\, l_{12} t_1+\hbar\, l_{22} t_2)} - \gamma_{12} + T_1 \gamma_{12} - \gamma_{22} + T_1 \gamma_{22}} +\\
&\quad O[\epsilon]^1\Big]
\end{aligned}
$$

## Stitching Œ's.

StitchingOEs

```
m_{j_→k_}[Œ_{U_}[specs__, Q_, P_]] := Cord[Œ_U[Sequence @@ Append[DeleteCases[{specs}, {__}_{j|k}],
       Flatten[{Cases[{specs}, {us__}_j :→ {us}], Cases[{specs}, {us__}_k :→ {us}]}]_k],
     Q, P] /. {t_j → t_k, T_j → T_k}]
```

```
cΦ = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ħ Sum[l_{10 i+j} ti aj + γ_{10 i+j} yi xj, {i, 3}, {j, 3}], 1ϵ];
{cΦ // m3→4, HL@Simp[CU[m3→4[cΦ]] - m3→4[CU[cΦ]]]}
```

$$
\begin{aligned}
&\Big\{\text{Œ}_{CU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4,\\
&\quad \hbar\,\big(a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 +\\
&\qquad a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} +\\
&\qquad x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}\big),\ 1 + O[\epsilon]^3\Big],\ 0\Big\}
\end{aligned}
$$

Verifying that *m* commutes with evaluation, in CU:

```
c⊙ = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{c⊙ // m2→3, HL@Simp[CU[m2→3[c⊙]] - m2→3[CU[c⊙]]]}
```

$\{27.5938,$

$$\Big\{\mathbb{E}_{CU}\Big[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{1}{1+\hbar t_3 \gamma_{32}} e^{-2\gamma\hbar l_{12} t_1-\gamma\hbar l_{13} t_1-2\gamma\hbar l_{22} t_3-\gamma\hbar l_{23} t_3-2\gamma\hbar l_{32} t_3-\gamma\hbar l_{33} t_3}$$

$$\Big(e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_1 l_{11} t_1 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_3 l_{12} t_1 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_3 l_{13} t_1 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_1 l_{21} t_3 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_3 l_{22} t_3 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_3 l_{23} t_3 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_1 l_{31} t_3 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_3 l_{32} t_3 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, a_3 l_{33} t_3 +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, x_1 y_1 \gamma_{11} +$$
$$e^{2\gamma(\hbar l_{12} t_1+\hbar l_{22} t_3+\hbar l_{32} t_3)}\,\hbar\, x_3 y_1 \gamma_{12} + e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}$$
$$\hbar\, x_3 y_1 \gamma_{13} + e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, x_1 y_3 \gamma_{21} +$$
$$e^{2\gamma(\hbar l_{12} t_1+\hbar l_{22} t_3+\hbar l_{32} t_3)}\,\hbar\, x_3 y_3 \gamma_{22} + e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}$$
$$\hbar\, x_3 y_3 \gamma_{23} + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, x_1 y_3 \gamma_{31} -$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_1 y_1 \gamma_{12} \gamma_{31} -$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_1 y_3 \gamma_{22} \gamma_{31} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_1 l_{11} t_1 t_3 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_3 l_{12} t_1 t_3 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_3 l_{13} t_1 t_3 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_1 l_{21} t_3^2 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_3 l_{22} t_3^2 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_3 l_{23} t_3^2 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_1 l_{31} t_3^2 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_3 l_{32} t_3^2 \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 a_3 l_{33} t_3^2 \gamma_{32} +$$
$$e^{\gamma(\hbar l_{12} t_1+\hbar l_{22} t_3+\hbar l_{32} t_3)}\,\hbar\, x_3 y_3 \gamma_{32} + e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2$$
$$t_3 x_1 y_1 \gamma_{11} \gamma_{32} + e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_3 y_1 \gamma_{13} \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_1 y_3 \gamma_{21} \gamma_{32} +$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_3 y_3 \gamma_{23} \gamma_{32} +$$
$$e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar\, x_3 y_3 \gamma_{33} -$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_3 y_1 \gamma_{12} \gamma_{33} -$$
$$e^{2\gamma\hbar l_{12} t_1+\gamma\hbar l_{13} t_1+2\gamma\hbar l_{22} t_3+\gamma\hbar l_{23} t_3+2\gamma\hbar l_{32} t_3+\gamma\hbar l_{33} t_3}\,\hbar^2 t_3 x_3 y_3 \gamma_{22} \gamma_{33}\Big),$$

$$\frac{1}{1+\hbar t_3\gamma_{32}} + \frac{\gamma\hbar^4 t_3 x_1^2 y_1^2 \gamma_{12}^2 \gamma_{31}^2 \epsilon}{2\left(1+\hbar t_3\gamma_{32}\right)} + \frac{\left(-8\gamma^2\hbar^6 t_3 x_1^3 y_1^3 \gamma_{12}^3 \gamma_{31}^3 + 3\gamma^2\hbar^8 t_3^2 x_1^4 y_1^4 \gamma_{12}^4 \gamma_{31}^4\right)\epsilon^2}{24\left(1+\hbar t_3\gamma_{32}\right)} +$$
$$O[\epsilon]^3\Big], 0\Big\}\Big\}$$

Verifying that *m* commutes with evaluation, in QU:

```
q⊙ = 𝔈QU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{q⊙ // m2→3, HL@SimpT[QU[m2→3[q⊙]] - m2→3[QU[q⊙]]]}
```

$\{62.5313,$

$$\Big\{\mathbb{E}_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{1}{1-\gamma_{32}+T_3\gamma_{32}} e^{-2\gamma\hbar l_{12} t_1-\gamma\hbar l_{13} t_1-2\gamma\hbar l_{22} t_3-\gamma\hbar l_{23} t_3-2\gamma\hbar l_{32} t_3-\gamma\hbar l_{33} t_3}$$

$\big(e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{11}\,t_1\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{12}\,t_1\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{13}\,t_1\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{21}\,t_3\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{22}\,t_3\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{23}\,t_3\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{31}\,t_3\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{32}\,t_3\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{33}\,t_3\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_1\,y_1\,\Upsilon_{11}\,+$
$e^{2\gamma(\hbar l_{12}t_1+\hbar l_{22}t_3+\hbar l_{32}t_3)}\,\hbar\,x_3\,y_1\,\Upsilon_{12}+e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}$
$\hbar\,x_3\,y_1\,\Upsilon_{13}+e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_1\,y_3\,\Upsilon_{21}\,+$
$e^{2\gamma(\hbar l_{12}t_1+\hbar l_{22}t_3+\hbar l_{32}t_3)}\,\hbar\,x_3\,y_3\,\Upsilon_{22}+e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}$
$\hbar\,x_3\,y_3\,\Upsilon_{23}+e^{\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_1\,y_3\,\Upsilon_{31}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_1\,y_1\,\Upsilon_{12}\,\Upsilon_{31}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_1\,y_1\,\Upsilon_{12}\,\Upsilon_{31}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_1\,y_3\,\Upsilon_{22}\,\Upsilon_{31}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_1\,y_3\,\Upsilon_{22}\,\Upsilon_{31}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{11}\,t_1\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{12}\,t_1\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{13}\,t_1\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{21}\,t_3\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{22}\,t_3\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{23}\,t_3\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{31}\,t_3\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{32}\,t_3\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{33}\,t_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{11}\,t_1\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{12}\,t_1\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{13}\,t_1\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{21}\,t_3\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{22}\,t_3\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{23}\,t_3\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_1\,l_{31}\,t_3\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{32}\,t_3\,T_3\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,a_3\,l_{33}\,t_3\,T_3\,\Upsilon_{32}\,+$
$e^{\gamma(\hbar l_{12}t_1+\hbar l_{22}t_3+\hbar l_{32}t_3)}\,\hbar\,x_3\,y_3\,\Upsilon_{32}-e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}$
$\hbar\,x_1\,y_1\,\Upsilon_{11}\,\Upsilon_{32}+e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_1\,y_1\,\Upsilon_{11}\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_3\,y_1\,\Upsilon_{13}\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_3\,y_1\,\Upsilon_{13}\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_1\,y_3\,\Upsilon_{21}\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_1\,y_3\,\Upsilon_{21}\,\Upsilon_{32}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_3\,y_3\,\Upsilon_{23}\,\Upsilon_{32}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_3\,y_3\,\Upsilon_{23}\,\Upsilon_{32}\,+$
$e^{\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_3\,y_3\,\Upsilon_{33}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_3\,y_1\,\Upsilon_{12}\,\Upsilon_{33}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_3\,y_1\,\Upsilon_{12}\,\Upsilon_{33}\,+$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,x_3\,y_3\,\Upsilon_{22}\,\Upsilon_{33}\,-$
$e^{2\gamma\hbar l_{12}t_1+\gamma\hbar l_{13}t_1+2\gamma\hbar l_{22}t_3+\gamma\hbar l_{23}t_3+2\gamma\hbar l_{32}t_3+\gamma\hbar l_{33}t_3}\,\hbar\,T_3\,x_3\,y_3\,\Upsilon_{22}\,\Upsilon_{33}\big),$

$$\frac{1}{1-\Upsilon_{32}+T_3\,\Upsilon_{32}}+\Big(\big(\gamma\,\hbar^3\,x_1^2\,y_1^2\,\Upsilon_{12}^2\,\Upsilon_{31}^2-4\,\gamma\,\hbar^3\,T_3\,x_1^2\,y_1^2\,\Upsilon_{12}^2\,\Upsilon_{31}^2+3\,\gamma\,\hbar^3\,T_3^2\,x_1^2\,y_1^2\,\Upsilon_{12}^2\,\Upsilon_{31}^2\big)\,\epsilon\Big)\,\Big/$$

$$\left(4\left(1-\gamma_{32}+T_3\gamma_{32}\right)\right)+\frac{1}{288\left(1-\gamma_{32}+T_3\gamma_{32}\right)}\left(36\gamma^2\hbar^4x_1^2y_1^2\gamma_{12}^2\gamma_{31}^2-216\gamma^2\hbar^4T_3x_1^2y_1^2\gamma_{12}^2\gamma_{31}^2+180\gamma^2\hbar^4T_3^2x_1^2y_1^2\gamma_{12}^2\gamma_{31}^2+40\gamma^2\hbar^5x_1^3y_1^3\gamma_{12}^3\gamma_{31}^3-312\gamma^2\hbar^5T_3x_1^3y_1^3\gamma_{12}^3\gamma_{31}^3+600\gamma^2\hbar^5T_3^2x_1^3y_1^3\gamma_{12}^3\gamma_{31}^3-328\gamma^2\hbar^5T_3^3x_1^3y_1^3\gamma_{12}^3\gamma_{31}^3+9\gamma^2\hbar^6x_1^4y_1^4\gamma_{12}^4\gamma_{31}^4-72\gamma^2\hbar^6T_3x_1^4y_1^4\gamma_{12}^4\gamma_{31}^4+198\gamma^2\hbar^6T_3^2x_1^4y_1^4\gamma_{12}^4\gamma_{31}^4-216\gamma^2\hbar^6T_3^3x_1^4y_1^4\gamma_{12}^4\gamma_{31}^4+81\gamma^2\hbar^6T_3^4x_1^4y_1^4\gamma_{12}^4\gamma_{31}^4\right)\epsilon^2+O[\epsilon]^3\Big],\ 0\}\}$$

Verifying meta-associativity in CU:

```
cⵚ = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 10];
Timing@HL[(lhs = cⵚ // m1,2→1 // m1,3→1) == (rhs = cⵚ // m2,3→2 // m1,2→1)]
```

{9.64063, True}

```
cⵚ = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 11];
Timing@HL[(lhs = cⵚ // m1,2→1 // m1,3→1) == (rhs = cⵚ // m2,3→2 // m1,2→1)]
```

```
cⵚ = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@HL[(lhs = cⵚ // m1,2→1 // m1,3→1) == (rhs = cⵚ // m2,3→2 // m1,2→1)]
```

## *R* in QU.

Faddeev-Quesne's formula:

Faddeev

```
eq_,k_[x_] := e^(Sum_{j=1}^{k+1} (1 - q)^j x^j / (j (1 - q^j))); eq_[x_] := eq,$k[x]
```

```
Table[Series[eqℏ,k[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$e^x$

$e^x-\frac{1}{4}\left(e^x x^2\gamma\hbar\right)\epsilon+\frac{1}{32}e^x x^4\gamma^2\hbar^2\epsilon^2-\frac{1}{384}\left(e^x x^2\left(-8+x^4\right)\gamma^3\hbar^3\right)\epsilon^3+\frac{e^x x^4\left(-32+x^4\right)\gamma^4\hbar^4\epsilon^4}{6144}+O[\epsilon]^5$

$e^x-\frac{1}{4}\left(e^x x^2\gamma\hbar\right)\epsilon+\frac{1}{288}e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2-\frac{\left(e^x x^2\left(-24+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152}+\frac{e^x x^3\left(-4608-864x+1024x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165888}+O[\epsilon]^5$

$e^x-\frac{1}{4}\left(e^x x^2\gamma\hbar\right)\epsilon+\frac{1}{288}e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2-\frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152}+\frac{e^x x^3\left(-4608-864x+3616x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165888}+O[\epsilon]^5$

$e^x-\frac{1}{4}\left(e^x x^2\gamma\hbar\right)\epsilon+\frac{1}{288}e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2-\frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152}+\frac{e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4}{4147200}+O[\epsilon]^5$

$e^x-\frac{1}{4}\left(e^x x^2\gamma\hbar\right)\epsilon+\frac{1}{288}e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2-\frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152}+\frac{e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4}{4147200}+O[\epsilon]^5$

```
Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)}\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

```
QU[R_{i_,j_}] := O_QU[{y1, a1}_i, {a2, x2}_j, SS[e^{ħ b1 a2} e_{q_ħ}[ħ y1 x2] /. b1 → γ^-1 (ϵ a1 − t_i)]];
QU[R^{-1}_{i_,j_}] := S_j@QU[R_{i,j}];
```

```
QU[R_{3,4}] // Short
```

$$QU[\,] + \hbar\, QU[y_3, x_4] + \frac{1}{2}\hbar^2 QU[y_3, y_3, x_4, x_4] - \frac{\hbar\, QU[a_4]\, t_3}{\gamma} - \frac{\hbar^2 QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4]\, t_3^2}{2\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^{-1}_{1,2}] // Simp // HL // Timing
```

{0.09375, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]], HL@SimpT[lhs − QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]} // Timing
```

$$\left\{0.1875, \left\{QU[\,] + \hbar\, QU[y_1, x_2] + \hbar\, QU[y_1, x_3] + \ll 38 \gg + \frac{\hbar^2 QU[a_3, a_3]\, t_2^2}{2\gamma^2} + QU[y_1, x_3](\hbar - \hbar T_2), 0\right\}\right\}$$

## *R* in Œ$_{QU}$.

RinOE

```
Œ_{QU,k_}[R_{i_,j_}] := Œ_QU[{y_i, a_i, x_i}_i, {y_j, a_j, x_j}_j, −ħ γ^-1 t_i a_j + ħ y_i x_j,
   Series[e^{ħ γ^-1 t_i a_j − ħ y_i x_j} (e^{ħ b_i a_j} e_{q_ħ,k}[ħ y_i x_j] /. b_i → γ^-1 (ϵ a_i − t_i)), {ϵ, 0, k}]]
```

```
{Œ_{QU,1}[R_{1,2}], Œ_{QU,2}[R_{1,2}]}
```

$$\left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\, a_2\, t_1}{\gamma} + \hbar\, x_2\, y_1, 1 + \left(\frac{\hbar\, a_1\, a_2}{\gamma} - \frac{1}{4}\gamma\,\hbar^3 x_2^2 y_1^2\right)\epsilon + O[\epsilon]^2\right],\right.$$

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\, a_2\, t_1}{\gamma} + \hbar\, x_2\, y_1, 1 + \left(\frac{\hbar\, a_1\, a_2}{\gamma} - \frac{1}{4}\gamma\,\hbar^3 x_2^2 y_1^2\right)\epsilon + \right.$$

$$\left.\left.\frac{1}{288\gamma^2}\left(144\,\hbar^2 a_1^2 a_2^2 - 72\gamma^2\hbar^4 a_1 a_2 x_2^2 y_1^2 + 32\gamma^4\hbar^5 x_2^3 y_1^3 + 9\gamma^4\hbar^6 x_2^4 y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$$

## Alternative Algorithms

AltLogos

```
λalt,k_[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Series[e^(-η y-ξ x+η ξ t+c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, k}]]];
```

```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[ΛCU,2[{ξ, η}, {x, y}]]]}
```

$$\left\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\frac{1}{2}\left(\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)^2+2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\right)\epsilon^2+O[\epsilon]^3,\ \text{True}\right\}$$