

Pensieve header: A unified verification notebook for the \$sl_2\$-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.
 Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

Go;

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ε_] := Style[ε, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is \hbar , and all “coupling constants” are proportional to it.

TD

```
$p = 2; $k = 1; $E := {$k, $p};
$trim := {ħp /; p > $p → 0, ek /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {Ti → eħ ti, T → eħ t}; qħ = eγ ε ħ;
SS[ε_, op_] := Collect[
    Normal@Series[If[$p > 0, ε, ε /. TRule], {ħ, 0, $p}],
    ħ, op];
SS[ε_] := SS[ε, Together];
SST[ε_, op___] := SS[ε /. TRule, op];
Simp[ε_, op_] := Collect[ε, _CU | _QU, op];
Simp[ε_] := Simp[ε, SS[#, Expand] &];
SimpT[ε_] := Collect[ε, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

```
DPα→Dx, β→Dy[P_] [λ_] :=
    Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) ⇒ c ∂{x,m}, {y,n} λ]
```

$$HL[DP_{x \rightarrow D_\epsilon, y \rightarrow D_\eta}[x^2 y^3][e^{\delta \epsilon \eta}] == 6 e^{\delta \eta \epsilon} \delta^3 \epsilon + 6 e^{\delta \eta \epsilon} \delta^4 \eta \epsilon^2 + e^{\delta \eta \epsilon} \delta^5 \eta^2 \epsilon^3]$$

True

Self-Pair (SP):

SP

```
SP{}[P_] := P; SP{ε→x, ps___}[P_] := Expand[P // SP{ps}] /. f- . εd ⇒ ∂{x,d} f
```

$$SP_{\{\epsilon \rightarrow x\}}[(\epsilon^2 + \epsilon + 3)(x^5 e^x + 7x) + 99a]$$

$$7 + 99a + 21x + 20e^x x^3 + 15e^x x^4 + 5e^x x^5$$

$$\text{SP}_{\{\xi \rightarrow x, \eta \rightarrow y\}} \left[(\xi^2 + \xi + 3 + 2 \xi \eta) (x^5 e^x + 7 x) + 99 a + e^{\delta x y} \xi \eta \right]$$

$$7 + 99 a + 21 x + 20 e^x x^3 + 15 e^x x^4 + 5 e^x x^5 + e^{xy \delta} \delta + e^{xy \delta} x y \delta^2$$

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

```

DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
  gs = Generators /. {opts},
  cs = Centrals /. {opts} /. Centrals -> {}},
  (#U = U@#) & /@gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g -> ++k, gi_ -> {i, k}}, {g, gs}]; (* sorting -> *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[_] := Collect[_] /. $trim;
  Ui[_] := # /. {t : cp -> ti, u_U -> Replace[u, x_ -> xi, 1]};
  Ui[NCM[]] = pow[_] /. 0 = U@{} = 1U = U[];
  B[U@(x_)i_, U@(y_)i_] := Ui@B[U@x, U@y];
  B[U@(x_)i_, U@(y_)j_] /; i != j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1U) := CE[c x]; (c_. 1U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy];
  U@{c_. * (L : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[L, {n}] ** U@{r}];
  U@{c_. * L : gp, r___} := CE[c U[L] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
  U@{L_, r___} := U@{Expand[L], r};
  U[_NonCommutativeMultiply] := U /@ #;
  Ou[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, L_List -> Lnull, {1}];
    vs = Join@@(First /@ sp);
    us = Join@@(sp /. L_s_ -> (L /. x_i_ -> xs));
    CE[Total[
      CoefficientRules[poly, vs] /. (p_ -> c_) -> c U@(us^p)
    ] / . x_nnull -> x];
  pow[_] := pow[_] ** #;
  Su[_] := CE@Total[
    CoefficientRules[_] /. {ss} / .
    (p_ -> c_) -> c NCM@@MapThread[pow, {Last /@ {ss}, p}]];
  mj_r_ [c_. * u_U] := CE[ ((c /. (t : cp)_j -> tk) DeleteCases[u, _j|k]) **
    U@@Cases[u, w_j -> wk] ** U@@Cases[u, _r] ];
  Si_ [c_. * u_U] := CE[ ((c /. Si[U, Centrals]) DeleteCases[u, _i]) **
    Ui[NCM@@Reverse@Cases[u, x_i -> S@U@x] ] ]

```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ -> V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
    {(g_ -> img_) -> (m[U[g]] = img), (g_ -> img_) -> (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs_]] := NCM@@(m/@U/@{vs});
  m[E_] := Simp[E /. oncs /. u_U -> m[u]] /. $trim;
```

Meta-Operations

QLImplementation

```
m_j -> j_ = Identity;
m_j -> k_ [E_Plus] := Simp[m_j -> k_ /@ E];
m_i s_ i, j -> k_ [E_] := m_j -> k_ @ m_i s, i -> j @ E;
S_i [E_Plus] := Simp[S_i /@ E];
```

Implementing $CU = \mathcal{U}(sl_2^{\vee \epsilon})$

CU

```
DeclareAlgebra[CU, Generators -> {y, a, x}, Centrals -> {t}];
B[a_CU, y_CU] = -y y_CU; B[x_CU, a_CU] = -x x_CU;
B[x_CU, y_CU] = 2 e a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU);
S_i [CU, Centrals] = {t_i -> -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a "random" triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp // HL
}] // Timing
{1.67188,
  {(28 t^2 y^4 + 116 t y^5 e) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
    {z1, bas}, {z2, bas} ] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t1 t3 CU[y1, a1, x2] + t1 t1 CU[y1, a2, x2] // m1->3
CU[a2, x2, y3] t3^2 + CU[x2, y3, a3] t3^2
```

Verifying meta-associativity:

```
Module[{z, u},
  Table[u = CU[z[[1]]1, z[[2]]2, z[[3]]3]; z -> HL[m1,3->3@m2,3->3@u == m2,3->3@m1,2->2@u],
    {z, Tuples[{y, a, x}, 3]}]]
{{y, y, y} -> True, {y, y, a} -> True, {y, y, x} -> True, {y, a, y} -> True,
 {y, a, a} -> True, {y, a, x} -> True, {y, x, y} -> True, {y, x, a} -> True,
 {y, x, x} -> True, {a, y, y} -> True, {a, y, a} -> True, {a, y, x} -> True, {a, a, y} -> True,
 {a, a, a} -> True, {a, a, x} -> True, {a, x, y} -> True, {a, x, a} -> True, {a, x, x} -> True,
 {x, y, y} -> True, {x, y, a} -> True, {x, y, x} -> True, {x, a, y} -> True, {x, a, a} -> True,
 {x, a, x} -> True, {x, x, y} -> True, {x, x, a} -> True, {x, x, x} -> True}}
```

Implementing QU = $\mathcal{U}_q(\mathfrak{sl}_2^{\vee \epsilon})$

Aside

```
Series[(1 - T e^{-2 \epsilon a \hbar}) / \hbar, {a, 0, 3}]
```

Aside

$$\frac{1 - T}{\hbar} + 2 T \epsilon a - 2 (T \epsilon^2 \hbar) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators -> {y, a, x}, Centrals -> {t, T}];
B[aQU, yQU] = -\gamma yQU; B[xQU, aQU] = -\gamma QU@x;
B[xQU, yQU] := SS[qh - 1] QU@{y, x} + OQU[{a}, SS[(1 - T e^{-2 \epsilon a \hbar}) / \hbar]];
(S@yQU := OQU[{a, y}, SS[-T^{-1} e^{\hbar \epsilon a} y]]; S@aQU = -aQU; S@xQU := OQU[{a, x}, SS[-e^{\hbar \epsilon a} x]);
S_i[QU, Centrals] = {t_i -> -t_i, T_i -> T_i^{-1}};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas} ] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
 {QU[y], QU[x]} →  $\frac{(-1 + T) QU[]}{\hbar} - 2 T \in QU[a] - \gamma \in \hbar QU[y, x]$ },
 {{QU[a], QU[y]} → -γ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} →  $\frac{(1 - T) QU[]}{\hbar} + 2 T \in QU[a] + \gamma \in \hbar QU[y, x]$ ,
 {QU[x], QU[a]} → -γ QU[x], {QU[x], QU[x]} → 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
 {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ \$p=5, \$k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
 (rhs = (z1 ** z2) ** z3 // Simp) // Short,
 HL[z1 ** (z2 ** z3) - rhs // Simp ]
}] // Timing
{3.51563, {  $\left( \frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \in - 280 \ll 3 \gg + 198 T^2 \gamma^5 \in}{\hbar} \right) QU[y, y, y, x, x] +$ 
  $\ll 18 \gg + (1 + 8 \gamma \in \hbar) QU[\ll 1 \gg, 0]$ }}
```

Verifying that S is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
 {z1, bas}, {z2, bas} ] ]
{{{QU[y1], QU[y1]} → 0, {QU[y1], QU[a1]} → 0, {QU[y1], QU[x1]} → 0},
 {{QU[a1], QU[y1]} → 0, {QU[a1], QU[a1]} → 0, {QU[a1], QU[x1]} → 0},
 {{QU[x1], QU[y1]} → 0, {QU[x1], QU[a1]} → 0, {QU[x1], QU[x1]} → 0}}
```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs @ \$p=5, \$k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
 Short[lhs = z1 ** (z2 ** z3)],
 Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
 Expand[Limit[rhs /. TRule[QU → CU], ħ → 0] - lhs] // HL
}] // Timing
{9.9375, {28 t^2 γ^4 CU[y, y, y, x, x] +
 116 t γ^5 ∈ CU[y, y, y, x, x] +  $\ll 44 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x]$ ,
 2  $\left( \frac{\gamma^4}{\hbar^2} - \frac{2 T \gamma^4}{\hbar^2} + \frac{T^2 \gamma^4}{\hbar^2} + \frac{\gamma^5 \in}{\hbar} - \frac{2 T \gamma^5 \in}{\hbar} + \frac{T^2 \gamma^5 \in}{\hbar} \right) QU[y, y, y, x, x] +$ 
  $\ll 209 \gg + (1 + 8 \gamma \in \hbar) QU[y, \ll 11 \gg, x], 0}}$ 
```

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -xCU, a → -aCU, x → -yCU}, {t → -t, T → T-1}]];
DeclareMorphism[Qθ, QU → QU, {y := QQU[{a, x}, SS[-T-1/2 eħεa x}],
  a → -aQU, x := QQU[{a, y}, SS[-T-1/2 eħεa y}]]}, {t → -t, T → T-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
{QU[y] → - $\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon \hbar QU[a, x]}{\sqrt{T}}$  → QU[y], QU[a] → -QU[a] → QU[a],
  QU[x] →  $\left(-\frac{1}{\sqrt{T}} + \frac{\gamma \epsilon \hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon \hbar QU[y, a]}{\sqrt{T}}$  → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$ADef = \gamma \left(\left(\text{Cosh} \left[\hbar \left(a \epsilon + \frac{\gamma \epsilon}{2} - \frac{t}{2} \right) \right] - \text{Cosh} \left[\hbar \sqrt{\left(\frac{t - \gamma \epsilon}{2} \right)^2 + \epsilon \omega} \right] \right) / \right. \\ \left. \left(\hbar e^{\hbar ((a+\gamma) \epsilon - t/2)} \text{Sinh} \left[\frac{\gamma \epsilon \hbar}{2} \right] (a^2 \epsilon + a \gamma \epsilon - a t - \omega) \right) \right);$$

Scaling behaviour of AD\$f:

```
HL@Simplify[AD$f == ((AD$f /.  $\gamma \rightarrow 1$ ) /. { $\epsilon \rightarrow \gamma \epsilon$ ,  $a \rightarrow \gamma^{-1} a$ ,  $\omega \rightarrow \gamma^{-1} \omega$ })]
```

True

```
HL@FullSimplify[
  AD$f == ((AD$f /.  $\gamma \rightarrow 1$ ) /. { $\hbar \rightarrow \gamma^2 \hbar$ ,  $\epsilon \rightarrow \epsilon / \gamma$ ,  $a \rightarrow a / \gamma$ ,  $t \rightarrow \gamma^{-2} t$ ,  $\omega \rightarrow \gamma^{-3} \omega$ })]
```

True

ADeq

```
AD$ $\omega$  =  $\gamma$  CU[y, x] +  $\epsilon$  CU[a, a] - (t -  $\gamma \epsilon$ ) CU[a];
```

ADeq

```
DeclareMorphism[AD, QU  $\rightarrow$  CU,
  {a  $\rightarrow$  aCU, x  $\rightarrow$  CU@x, y  $\mapsto$  SCU[SS[AD$f], a  $\rightarrow$  aCU,  $\omega \rightarrow$  AD$ $\omega$ ] ** yCU}]
```

Verifying that the asymmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2}  $\rightarrow$  HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]}  $\rightarrow$  0, {QU[y], QU[a]}  $\rightarrow$  0, {QU[y], QU[x]}  $\rightarrow$  0},
 {{QU[a], QU[y]}  $\rightarrow$  0, {QU[a], QU[a]}  $\rightarrow$  0, {QU[a], QU[x]}  $\rightarrow$  0},
 {{QU[x], QU[y]}  $\rightarrow$  0, {QU[x], QU[a]}  $\rightarrow$  0, {QU[x], QU[x]}  $\rightarrow$  0}}
```

The Symmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$SD$g = \sqrt{\left(\left(2\gamma \left(\cosh\left[\frac{\hbar}{2} \sqrt{t^2 + \gamma^2 \epsilon^2 + 4\epsilon\omega}\right] - \cosh\left[\frac{t - \epsilon\gamma - 2\epsilon a}{2/\hbar}\right] \right) \right) / \left(\sinh\left[\frac{\gamma\epsilon\hbar}{2}\right] (t(2a + \gamma) - 2a(a + \gamma)\epsilon + 2\omega)\hbar \right) \right)}$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

$$\{SD\$P = \frac{\text{Cosh}[\hbar \left(\frac{\epsilon-t}{2} + \epsilon a \right)] - \text{Cosh}[\hbar \sqrt{\frac{t^2+\epsilon^2}{4} + \epsilon w}]}{\hbar \text{Sinh}[\frac{-\epsilon \hbar}{2}] (w - \epsilon a^2 + (t-\epsilon) a + t/2)},$$

$$\text{Simplify}[SD\$P == (SD\$P /. \{a \to -a-1, t \to -t\})] // HL,$$

$$\text{PowerExpand@Simplify}[(SD\$P /. \{\hbar \to \gamma^2 \hbar, \epsilon \to \epsilon/\gamma, a \to a/\gamma, t \to \gamma^{-2} t, w \to \gamma^{-3} w\}) ==$$

$$SD\$g (SD\$g /. \{a \to -a-\gamma, t \to -t\})] // HL,$$

$$SD\$Q = \text{Simplify}[SD\$P /. \{a \to c-1/2\}],$$

$$\text{Simplify}[SD\$Q == (SD\$Q /. \{c \to -c, t \to -t\})] // HL,$$

$$\text{FullSimplify}[SD\$g == \text{FullSimplify}[\sqrt{SD\$Q} /. c \to a+1/2 /. \{\hbar \to \gamma^2 \hbar, \epsilon \to \epsilon/\gamma, a \to a/\gamma, t \to \gamma^{-2} t, w \to \gamma^{-3} w\}]] // HL$$

$$\left\{ - \left(\left(\left(\text{Cosh} \left[\left(a \epsilon + \frac{1}{2} (-t + \epsilon) \right) \hbar \right] - \text{Cosh} \left[\sqrt{\frac{1}{4} (t^2 + \epsilon^2) + \epsilon w} \hbar \right] \right) \text{Csch} \left[\frac{\epsilon \hbar}{2} \right] \right) / \right.$$

$$\left. \left(\left(\frac{t}{2} + a (t - \epsilon) - a^2 \epsilon + w \right) \hbar \right) \right\}, \text{True, True},$$

$$- \left(\left(4 \left(\text{Cosh} \left[\frac{1}{2} (t - 2 c \epsilon) \hbar \right] - \text{Cosh} \left[\frac{1}{2} \sqrt{t^2 + \epsilon^2 + 4 \epsilon w} \hbar \right] \right) \text{Csch} \left[\frac{\epsilon \hbar}{2} \right] \right) / \right.$$

$$\left. \left((4 c t + \epsilon - 4 c^2 \epsilon + 4 w) \hbar \right) \right\}, \text{True, True}$$

SDeq

```
SD$f = Simplify[ $e^{\hbar (t/2 - \epsilon a)}$  (SD$g /. \{a \to -a, t \to -t\})];
```

SDeq

```
SD$w =  $\gamma$  CU[y, x] +  $\epsilon$  CU[a, a] - (t -  $\gamma \epsilon$ ) CU[a] - t  $\gamma$  1cu/2;
```

SDeq

```
DeclareMorphism[SD, QU  $\to$  CU, \{a  $\to$  aCU,  
x  $\to$  SCU[SS[SD$f], a  $\to$  aCU, w  $\to$  SD$w] ** xCU,  
y  $\to$  SCU[SS[SD$g], a  $\to$  aCU, w  $\to$  SD$w] ** yCU }]
```

Verifying the θ -symmetry:

```
Table[HL@SimpT[C $\theta$ [SD[z]] == SD[Q $\theta$ [z]]], \{z, QU /@ \{y, a, x\}\}  
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[\{bas = QU /@ \{y, a, x\}\},  
Table[\{z1, z2\}  $\to$  HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], \{z1, bas\}, \{z2, bas\}]]  
{\{QU[y], QU[y]\}  $\to$  0, \{QU[y], QU[a]\}  $\to$  0, \{QU[y], QU[x]\}  $\to$  0},  
{\{QU[a], QU[y]\}  $\to$  0, \{QU[a], QU[a]\}  $\to$  0, \{QU[a], QU[x]\}  $\to$  0},  
{\{QU[x], QU[y]\}  $\to$  0, \{QU[x], QU[a]\}  $\to$  0, \{QU[x], QU[x]\}  $\to$  0}}
```

The representation ρ

rho

```

rho@yCU = rho@yQU =  $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$ ; rho@aCU = rho@aQU =  $\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ ;
rho@xCU =  $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ ; rho@xQU =  $\begin{pmatrix} 0 & (1 - e^{-\gamma \epsilon \hbar}) / (\epsilon \hbar) \\ 0 & 0 \end{pmatrix}$ ;
rho[e^xi_] := MatrixExp[rho[xi]];
rho[xi_] := (xi /. TRule /. t -> gamma epsilon /. (U : CU | QU) [u___] => Fold[Dot,  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , rho /@ U /@ {u}])

```

Verifying that ρ represents CU and QU:

```

Table[HL[SS[rho[z1]**rho[z2]] /. e^k_ /; k > $k -> 0],
{U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}}]
{{{True, True, True}, {True, True, True}, {True, True, True}},
{{True, True, True}, {True, True, True}, {True, True, True}}}

```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```

Table[HL[rho[e^xi U ex].rho[e^alpha U ea] == rho[e^alpha U ea].rho[e^{-gamma alpha xi U ex}]], {U, {CU, QU}}]
{True, True}

```

\mathbb{C} and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```

CU@C[Cu][specs___, Q_, P_] := OCU[specs, SS[e^Q P]];
QU@C[Qu][specs___, Q_, P_] := OQU[specs, SS[e^Q P]];

```

Logos

```

Lambda_U_[{xi_, alpha_}, {x, a}] := CU[{a, x}, alpha a + e^{-gamma alpha xi} x, 1];
Lambda_U_[{alpha_, eta_}, {a, y}] := CU[{y, a}, alpha a + e^{-gamma alpha eta} y, 1];

{Lambda_H_[{xi, alpha}, {x, a}], lhs = #@C_H_[{x, a}, hbar (xi x + alpha a), 1],
HL[lhs == #@Lambda_H_[{xi, alpha}, {x, a}]] & /@ {CU, QU}
{ {C_CU[{a, x}, a alpha + e^{-alpha gamma} x xi, 1],
CU[] + alpha hbar CU[a] + (xi hbar - alpha gamma xi hbar^2) CU[x] + 1/2 alpha^2 hbar^2 CU[a, a] + alpha xi hbar^2 CU[a, x] + 1/2 xi^2 hbar^2 CU[x, x],
True}, {C_Qu[{a, x}, a alpha + e^{-alpha gamma} x xi, 1], QU[] + alpha hbar QU[a] + (xi hbar - alpha gamma xi hbar^2) QU[x] +
1/2 alpha^2 hbar^2 QU[a, a] + alpha xi hbar^2 QU[a, x] + 1/2 xi^2 hbar^2 QU[x, x], True} }

```

```
{Λ#[{α, η}, {a, y}], lhs = #@E#[{a, y}, ħ (η y + α a), 1],
  HL[lhs = #@Λ#[ħ {α, η}, {a, y}]] & /@ {CU, QU}
{{ECU[{y, a}, a α + e-α γ y η, 1],
  CU[] + α ħ CU[a] + (η ħ - α γ η ħ2) CU[y] +  $\frac{1}{2}$  α2 ħ2 CU[a, a] + α η ħ2 CU[y, a] +  $\frac{1}{2}$  η2 ħ2 CU[y, y],
  True}, {EQU[{y, a}, a α + e-α γ y η, 1], QU[] + α ħ QU[a] + (η ħ - α γ η ħ2) QU[y] +
   $\frac{1}{2}$  α2 ħ2 QU[a, a] + α η ħ2 QU[y, a] +  $\frac{1}{2}$  η2 ħ2 QU[y, y], True}}
```

Goal. In either U , compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta=0) = 1$. So we set it up and solve:

```
If[ $\$k > 0$ , With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
    G = Echo@Simp[Table[ξk/k!, {k, 0, $k + 1}].NestList[Simp[B[xU, #]] &, yU, $k + 1]];
    fs = Echo@Flatten@Table[f1,i,j,k[η], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = Echo[fs.(bs = fs /. fL,i,j,k[η] => eL U@{yi, aj, xk})];
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1U /. η → 0, F ** G - yU ** F - ∂ηF}}, {b, bs}]];
    sol = Echo@First[F /. DSolve[es, fs, η]];
    Echo[sol /. {e- → 1, U → Times}];
    Collect[sol /. {e- → 1, U → Times}, e, Simplify]
  ]]
```

```
“ -t ξ CU[] + 2 e ξ CU[a] - γ e ξ2 CU[x] + CU[y]
“ {f0,0,0,0[η], f1,0,0,0[η], f1,0,0,1[η], f1,0,1,0[η],
  f1,0,1,1[η], f1,1,0,0[η], f1,1,0,1[η], f1,1,1,0[η], f1,1,1,1[η]}
“ CU[] f0,0,0,0[η] + e CU[] f1,0,0,0[η] + e CU[x] f1,0,0,1[η] + e CU[a] f1,0,1,0[η] + e CU[a, x] f1,0,1,1[η] +
  e CU[y] f1,1,0,0[η] + e CU[y, x] f1,1,0,1[η] + e CU[y, a] f1,1,1,0[η] + e CU[y, a, x] f1,1,1,1[η]
» e-t η ξ CU[] +  $\frac{1}{2}$  e-t η ξ t γ e η2 ξ2 CU[] + 2 e-t η ξ e η ξ CU[a] - e-t η ξ γ e η ξ2 CU[x] - e-t η ξ γ e η2 ξ CU[y]
» 1 + 2 a e η ξ - y γ e η2 ξ - x γ e η ξ2 +  $\frac{1}{2}$  t γ e η2 ξ2
1 +  $\frac{1}{2}$  e η ξ (4 a + γ (-2 y η - 2 x ξ + t η ξ))
```

Logos

```
ΛU [{ξ1_, η1_}, {x, y}] := ΛU [{ξ1, η1}, {x, y}] = Module[{ξ, η, G, F, fs, f, bs, e, b, es},
  G = Simp[Table[ξk/k!, {k, 0, $k + 1}].NestList[Simp[B[xU, #]] &, yU, $k + 1]];
  fs = Flatten@Table[f1,i,j,k[η], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
  F = fs.(bs = fs /. fL,i,j,k[η] => eL U@{yi, aj, xk});
  es = Flatten[
    Table[Coefficient[e, b] == 0, {e, {F - 1U /. η → 0, F ** G - yU ** F - ∂ηF}}, {b, bs}]];
  F = F /. DSolve[es, fs, η][[1]];
  EU[{y, a, x},
    ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ħ}),
    F /. {e- → 1, U → Times}
  ] /. {ξ → ξ1, η → η1}];
```

$\{\Delta_{\text{CU}}[\{\xi, \eta\}, \{x, y\}], \text{lhs} = \text{CU}@\mathbb{C}_{\text{CU}}[\{x, y\}, \hbar(\xi x + \eta y), 1],$
 $\text{HL}[\text{lhs} = \text{CU}@\Delta_{\text{CU}}[\hbar\{\xi, \eta\}, \{x, y\}]]\}$

$\{\mathbb{C}_{\text{CU}}[\{y, a, x\}, y\eta + x\xi - t\eta\xi, 1 + 2a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2}t\gamma\epsilon\eta^2\xi^2],$
 $(1 - t\eta\xi\hbar^2) \text{CU}[] + 2\epsilon\eta\xi\hbar^2 \text{CU}[a] + \xi\hbar \text{CU}[x] + \eta\hbar \text{CU}[y] +$
 $\frac{1}{2}\xi^2\hbar^2 \text{CU}[x, x] + \eta\xi\hbar^2 \text{CU}[y, x] + \frac{1}{2}\eta^2\hbar^2 \text{CU}[y, y], \text{True}\}$

$\{\Delta_{\text{QU}}[\{\xi, \eta\}, \{x, y\}], \text{lhs} = \text{QU}@\mathbb{C}_{\text{QU}}[\{x, y\}, \hbar(\xi x + \eta y), 1],$
 $\text{HL}@\text{SimpT}[\text{lhs} = \text{QU}@\Delta_{\text{QU}}[\hbar\{\xi, \eta\}, \{x, y\}]]\}$

$\{\mathbb{C}_{\text{QU}}[\{y, a, x\}, y\eta + x\xi + \frac{(1-T)\eta\xi}{\hbar}, 1 + 2aT\epsilon\eta\xi - \frac{1}{2}(-1+3T)y\gamma\epsilon\eta^2\xi -$
 $\frac{1}{2}(-1+3T)x\gamma\epsilon\eta\xi^2 + \frac{(-1+T)(-1+3T)\gamma\epsilon\eta^2\xi^2}{4\hbar} + xy\gamma\epsilon\eta\xi\hbar],$
 $(1 + \eta\xi\hbar - T\eta\xi\hbar) \text{QU}[] + 2T\epsilon\eta\xi\hbar^2 \text{QU}[a] + \xi\hbar \text{QU}[x] + \eta\hbar \text{QU}[y] +$
 $\frac{1}{2}\xi^2\hbar^2 \text{QU}[x, x] + \eta\xi\hbar^2 \text{QU}[y, x] + \frac{1}{2}\eta^2\hbar^2 \text{QU}[y, y], \text{True}\}$

$\{\text{tt} = \text{Last}[\Delta_{\text{CU}}[\{\xi, \eta\}, \{x, y\}]], \text{Normal}@\text{Series}[\text{Log}[\text{tt}], \{\epsilon, \theta, \$k\}]\}$

$\{1 + 2a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2}t\gamma\epsilon\eta^2\xi^2, \epsilon \left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2 \right)\}$

$\{\text{tt} = \text{Last}[\Delta_{\text{QU}}[\{\xi, \eta\}, \{x, y\}]], \text{Normal}@\text{Series}[\text{Log}[\text{tt}], \{\epsilon, \theta, \$k\}]\}$

$\{1 + 2aT\epsilon\eta\xi - \frac{1}{2}(-1+3T)y\gamma\epsilon\eta^2\xi - \frac{1}{2}(-1+3T)x\gamma\epsilon\eta\xi^2 +$
 $\frac{(-1+T)(-1+3T)\gamma\epsilon\eta^2\xi^2}{4\hbar} + xy\gamma\epsilon\eta\xi\hbar, \frac{1}{4\hbar}\epsilon(\gamma\eta^2\xi^2 - 4T\gamma\eta^2\xi^2 + 3T^2\gamma\eta^2\xi^2 +$
 $8aT\eta\xi\hbar + 2y\gamma\eta^2\xi\hbar - 6T y\gamma\eta^2\xi\hbar + 2x\gamma\eta\xi^2\hbar - 6Tx\gamma\eta\xi^2\hbar + 4xy\gamma\eta\xi\hbar^2)\}$

Logos

```
Simp[C_U[specs___, Q_, P_] ] :=
  C_U[specs, ExpandNumerator@Together[Q], Collect[P, \epsilon, ExpandNumerator@*Together]];
```

Logos

```
\Delta_U[\{\nu1_, \omega1_, \delta_}, \{u_, w_}] := Simp@Module[\{v, \omega, yax, q, p, Q, d\},
  \{yax, q, p\} = List@@\Delta_U[\{v, \omega\}, \{u, w\}];
  C_U[yax, Q = (v u + \omega w + \delta u w + d v \omega) / (1 - d \delta),
  Expand[(1 - d \delta)^{-1} e^{-Q} DP_{v \to D_v, \omega \to D_\omega}[p][e^Q]]] /. \{d \to \partial_{v, \omega} q\} /. \{v \to \nu1, \omega \to \omega1\};
```

$\{\Delta_{\text{CU}}[\{\xi, \eta, \delta\}, \{x, y\}], \text{lhs} = \text{CU}@\Delta_{\text{CU}}[\{x, y\}, \hbar(\xi x + \eta y + \delta xy), 1],$
 $\text{HL}[\text{lhs} = \text{CU}@\Delta_{\text{CU}}[\hbar\{\xi, \eta, \delta\}, \{x, y\}]]\}$

$$\{\text{CU}[\{y, a, x\}, \frac{xy\delta + y\eta + x\xi - t\eta\xi}{1+t\delta}, \frac{1}{1+t\delta}] +$$

$$\frac{1}{2(1+t\delta)^5} \in (4a\delta + 12at\delta^2 + 4axy\delta^2 + 2t\gamma\delta^2 - 8xy\gamma\delta^2 + 12at^2\delta^3 + 8atxy\delta^3 + 4t^2\gamma\delta^3 -$$

$$12txy\gamma\delta^3 - 4x^2y^2\gamma\delta^3 + 4at^3\delta^4 + 4at^2xy\delta^4 + 2t^3\gamma\delta^4 - 4t^2xy\gamma\delta^4 - 3tx^2y^2\gamma\delta^4 +$$

$$4ay\delta\eta - 4y\gamma\delta\eta + 8aty\delta^2\eta - 4ty\gamma\delta^2\eta - 6xy^2\gamma\delta^2\eta + 4at^2y\delta^3\eta - 4txy^2\gamma\delta^3\eta -$$

$$2y^2\gamma\delta\eta^2 - ty^2\gamma\delta^2\eta^2 + 4ax\delta\xi - 4x\gamma\delta\xi + 8atx\delta^2\xi - 4tx\gamma\delta^2\xi - 6x^2y\gamma\delta^2\xi +$$

$$4at^2x\delta^3\xi - 4tx^2y\gamma\delta^3\xi + 4a\eta\xi + 8at\delta\eta\xi + 4t\gamma\delta\eta\xi - 8xy\gamma\delta\eta\xi + 4at^2\delta^2\eta\xi +$$

$$4t^2\gamma\delta^2\eta\xi - 4txy\gamma\delta^2\eta\xi - 2y\gamma\eta^2\xi - 2x^2\gamma\delta\xi^2 - tx^2\gamma\delta^2\xi^2 - 2x\gamma\eta\xi^2 + t\gamma\eta^2\xi^2),$$

$$(1-t\delta\hbar + t^2\delta^2\hbar^2 + t\gamma\delta^2\in\hbar^2 - t\eta\xi\hbar^2) \text{CU}[] + (2\delta\in\hbar - 4t\delta^2\in\hbar^2 + 2\in\eta\xi\hbar^2)$$

$$\text{CU}[a] +$$

$$(\xi\hbar - 2t\delta\xi\hbar^2 - 2\gamma\delta\in\xi\hbar^2) \text{CU}[x] +$$

$$(\eta\hbar - 2t\delta\eta\hbar^2 - 2\gamma\delta\in\eta\hbar^2) \text{CU}[y] +$$

$$4\delta\in\xi\hbar^2 \text{CU}[a, x] +$$

$$\frac{1}{2}\xi^2\hbar^2 \text{CU}[x, x] +$$

$$4\delta\in\eta\hbar^2 \text{CU}[y, a] +$$

$$(\delta\hbar - 2t\delta^2\hbar^2 - 4\gamma\delta^2\in\hbar^2 + \eta\xi\hbar^2) \text{CU}[y, x] +$$

$$\frac{1}{2}\eta^2\hbar^2 \text{CU}[y, y] +$$

$$4\delta^2\in\hbar^2 \text{CU}[y, a, x] +$$

$$\delta\xi\hbar^2 \text{CU}[y, x, x] +$$

$$\delta\eta\hbar^2 \text{CU}[y, y, x] +$$

$$\frac{1}{2}\delta^2\hbar^2 \text{CU}[y, y, x, x], \text{True}\}$$

{ $\Delta_{QU}[\{\xi, \eta, \delta\}, \{x, y\}]$, lhs = QU@ $\mathbb{C}_{QU}[\{x, y\}, \hbar(\xi x + \eta y + \delta x y), 1]$,
 HL@SimpT[lhs == QU@ $\Delta_{QU}[\hbar\{\xi, \eta, \delta\}, \{x, y\}]]$ }

{ $\mathbb{C}_{QU}[\{y, a, x\}, \frac{\eta \xi - T \eta \xi + x y \delta \hbar + y \eta \hbar + x \xi \hbar}{-\delta + T \delta + \hbar}]$,

$$\frac{\hbar}{-\delta + T \delta + \hbar} + \frac{1}{4(-\delta + T \delta + \hbar)^5} \in \left(-8 a T \delta^4 \hbar^2 + 24 a T^2 \delta^4 \hbar^2 - 24 a T^3 \delta^4 \hbar^2 + 8 a T^4 \delta^4 \hbar^2 + 2 \gamma \delta^4 \hbar^2 - \right.$$

$$\begin{aligned} & 12 T \gamma \delta^4 \hbar^2 + 24 T^2 \gamma \delta^4 \hbar^2 - 20 T^3 \gamma \delta^4 \hbar^2 + 6 T^4 \gamma \delta^4 \hbar^2 + 24 a T \delta^3 \hbar^3 - 48 a T^2 \delta^3 \hbar^3 + \\ & 24 a T^3 \delta^3 \hbar^3 - 4 \gamma \delta^3 \hbar^3 + 20 T \gamma \delta^3 \hbar^3 - 28 T^2 \gamma \delta^3 \hbar^3 + 12 T^3 \gamma \delta^3 \hbar^3 + 8 a T x y \delta^4 \hbar^3 - \\ & 16 a T^2 x y \delta^4 \hbar^3 + 8 a T^3 x y \delta^4 \hbar^3 - 8 T x y \gamma \delta^4 \hbar^3 + 16 T^2 x y \gamma \delta^4 \hbar^3 - 8 T^3 x y \gamma \delta^4 \hbar^3 + \\ & 8 a T y \delta^3 \eta \hbar^3 - 16 a T^2 y \delta^3 \eta \hbar^3 + 8 a T^3 y \delta^3 \eta \hbar^3 + 8 a T x \delta^3 \xi \hbar^3 - 16 a T^2 x \delta^3 \xi \hbar^3 + \\ & 8 a T^3 x \delta^3 \xi \hbar^3 + 8 a T \delta^2 \eta \xi \hbar^3 - 16 a T^2 \delta^2 \eta \xi \hbar^3 + 8 a T^3 \delta^2 \eta \xi \hbar^3 - 4 \gamma \delta^2 \eta \xi \hbar^3 + \\ & 20 T \gamma \delta^2 \eta \xi \hbar^3 - 28 T^2 \gamma \delta^2 \eta \xi \hbar^3 + 12 T^3 \gamma \delta^2 \eta \xi \hbar^3 - 24 a T \delta^2 \hbar^4 + 24 a T^2 \delta^2 \hbar^4 + \\ & 2 \gamma \delta^2 \hbar^4 - 8 T \gamma \delta^2 \hbar^4 + 6 T^2 \gamma \delta^2 \hbar^4 - 16 a T x y \delta^3 \hbar^4 + 16 a T^2 x y \delta^3 \hbar^4 + 24 T x y \gamma \delta^3 \hbar^4 - \\ & 24 T^2 x y \gamma \delta^3 \hbar^4 + x^2 y^2 \gamma \delta^4 \hbar^4 + 4 T x^2 y^2 \gamma \delta^4 \hbar^4 - 5 T^2 x^2 y^2 \gamma \delta^4 \hbar^4 - 16 a T y \delta^2 \eta \hbar^4 + \\ & 16 a T^2 y \delta^2 \eta \hbar^4 - 4 y \gamma \delta^2 \eta \hbar^4 + 16 T y \gamma \delta^2 \eta \hbar^4 - 12 T^2 y \gamma \delta^2 \eta \hbar^4 + 8 T x y^2 \gamma \delta^3 \eta \hbar^4 - \\ & 8 T^2 x y^2 \gamma \delta^3 \eta \hbar^4 - y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 T y^2 \gamma \delta^2 \eta^2 \hbar^4 - 3 T^2 y^2 \gamma \delta^2 \eta^2 \hbar^4 - 16 a T x \delta^2 \xi \hbar^4 + \\ & 16 a T^2 x \delta^2 \xi \hbar^4 - 4 x \gamma \delta^2 \xi \hbar^4 + 16 T x \gamma \delta^2 \xi \hbar^4 - 12 T^2 x \gamma \delta^2 \xi \hbar^4 + 8 T x^2 y \gamma \delta^3 \xi \hbar^4 - \\ & 8 T^2 x^2 y \gamma \delta^3 \xi \hbar^4 - 16 a T \delta \eta \xi \hbar^4 + 16 a T^2 \delta \eta \xi \hbar^4 + 4 \gamma \delta \eta \xi \hbar^4 - 16 T \gamma \delta \eta \xi \hbar^4 + \\ & 12 T^2 \gamma \delta \eta \xi \hbar^4 + 8 T x y \gamma \delta^2 \eta \xi \hbar^4 - 8 T^2 x y \gamma \delta^2 \eta \xi \hbar^4 - x^2 \gamma \delta^2 \xi^2 \hbar^4 + 4 T x^2 \gamma \delta^2 \xi^2 \hbar^4 - \\ & 3 T^2 x^2 \gamma \delta^2 \xi^2 \hbar^4 + \gamma \eta^2 \xi^2 \hbar^4 - 4 T \gamma \eta^2 \xi^2 \hbar^4 + 3 T^2 \gamma \eta^2 \xi^2 \hbar^4 + 8 a T \delta \hbar^5 + 8 a T x y \delta^2 \hbar^5 - \\ & 4 x y \gamma \delta^2 \hbar^5 - 12 T x y \gamma \delta^2 \hbar^5 - 4 x^2 y^2 \gamma \delta^3 \hbar^5 - 4 T x^2 y^2 \gamma \delta^3 \hbar^5 + 8 a T y \delta \eta \hbar^5 + 4 y \gamma \delta \eta \hbar^5 - \\ & 12 T y \gamma \delta \eta \hbar^5 - 2 x y^2 \gamma \delta^2 \eta \hbar^5 - 10 T x y^2 \gamma \delta^2 \eta \hbar^5 + 2 y^2 \gamma \delta \eta^2 \hbar^5 - 6 T y^2 \gamma \delta \eta^2 \hbar^5 + \\ & 8 a T x \delta \xi \hbar^5 + 4 x \gamma \delta \xi \hbar^5 - 12 T x \gamma \delta \xi \hbar^5 - 2 x^2 y \gamma \delta^2 \xi \hbar^5 - 10 T x^2 y \gamma \delta^2 \xi \hbar^5 + 8 a T \eta \xi \hbar^5 - \\ & 16 T x y \gamma \delta \eta \xi \hbar^5 + 2 y \gamma \eta^2 \xi \hbar^5 - 6 T y \gamma \eta^2 \xi \hbar^5 + 2 x^2 \gamma \delta \xi^2 \hbar^5 - 6 T x^2 \gamma \delta \xi^2 \hbar^5 + 2 x \gamma \eta \xi^2 \hbar^5 - \\ & 6 T x \gamma \eta \xi^2 \hbar^5 + 4 x y \gamma \delta \hbar^6 + 4 x^2 y^2 \gamma \delta^2 \hbar^6 + 4 x y^2 \gamma \delta \eta \hbar^6 + 4 x^2 y \gamma \delta \xi \hbar^6 + 4 x y \gamma \eta \xi \hbar^6 \left. \right)], \end{aligned}$$

$$\left(1 + \delta - T \delta + \delta^2 - 2 T \delta^2 + T^2 \delta^2 + \frac{1}{2} \gamma \delta^2 \in \hbar - 2 T \gamma \delta^2 \in \hbar + \frac{3}{2} T^2 \gamma \delta^2 \in \hbar + \eta \xi \hbar - T \eta \xi \hbar \right)$$

QU[] +

$$(2 T \delta \in \hbar + 4 T \delta^2 \in \hbar - 4 T^2 \delta^2 \in \hbar + 2 T \in \eta \xi \hbar^2)$$

QU[a] +

$$(\xi \hbar + 2 \delta \xi \hbar - 2 T \delta \xi \hbar + \gamma \delta \in \xi \hbar^2 - 3 T \gamma \delta \in \xi \hbar^2)$$

QU[x] +

$$(\eta \hbar + 2 \delta \eta \hbar - 2 T \delta \eta \hbar + \gamma \delta \in \eta \hbar^2 - 3 T \gamma \delta \in \eta \hbar^2)$$

QU[y] +

$$4 T \delta \in \xi \hbar^2 QU[a, x] + \frac{1}{2} \xi^2 \hbar^2 QU[x, x] +$$

4 T \delta \in

$\eta \hbar^2 QU[y, a] +$

$$(\delta \hbar + 2 \delta^2 \hbar - 2 T \delta^2 \hbar + \gamma \delta \in \hbar^2 + 4 \gamma \delta^2 \in \hbar^2 - 8 T \gamma \delta^2 \in \hbar^2 + \eta \xi \hbar^2)$$

QU[y, x] +

$$\frac{1}{2} \eta^2 \hbar^2 QU[y, y] + 4 T \delta^2 \in \hbar^2 QU[y, a, x] +$$

$\delta \xi \hbar^2 QU[y, x, x] +$

$\delta \eta \hbar^2 QU[y, y, x] +$

$$\frac{1}{2} \delta^2 \hbar^2 QU[y, y, x, x], \text{ True}$$

{tt = Last[DeltaCU[{xi, eta, delta}, {x, y}], Normal@Series[Log[tt], {epsilon, 0, \$k}]]}

$$\left\{ \frac{1}{1+t\delta} + \frac{1}{2(1+t\delta)^5} \left(4a\delta + 12at\delta^2 + 4axy\delta^2 + 2t\gamma\delta^2 - 8xy\gamma\delta^2 + 12at^2\delta^3 + 8atxy\delta^3 + 4t^2\gamma\delta^3 - 12txy\gamma\delta^3 - 4x^2y^2\gamma\delta^3 + 4at^3\delta^4 + 4at^2xy\delta^4 + 2t^3\gamma\delta^4 - 4t^2xy\gamma\delta^4 - 3tx^2y^2\gamma\delta^4 + 4ay\delta\eta - 4y\gamma\delta\eta + 8aty\delta^2\eta - 4ty\gamma\delta^2\eta - 6xy^2\gamma\delta^2\eta + 4at^2y\delta^3\eta - 4txy^2\gamma\delta^3\eta - 2y^2\gamma\delta\eta^2 - ty^2\gamma\delta^2\eta^2 + 4ax\delta\xi - 4x\gamma\delta\xi + 8atx\delta^2\xi - 4tx\gamma\delta^2\xi - 6x^2y\gamma\delta^2\xi + 4at^2x\delta^3\xi - 4tx^2y\gamma\delta^3\xi + 4a\eta\xi + 8at\delta\eta\xi + 4t\gamma\delta\eta\xi - 8xy\gamma\delta\eta\xi + 4at^2\delta^2\eta\xi + 4t^2\gamma\delta^2\eta\xi - 4txy\gamma\delta^2\eta\xi - 2y\gamma\eta^2\xi - 2x^2\gamma\delta\xi^2 - tx^2\gamma\delta^2\xi^2 - 2x\gamma\eta\xi^2 + t\gamma\eta^2\xi^2 \right), \frac{1}{2(1+t\delta)^4} \left(4a\delta + 12at\delta^2 + 4axy\delta^2 + 2t\gamma\delta^2 - 8xy\gamma\delta^2 + 12at^2\delta^3 + 8atxy\delta^3 + 4t^2\gamma\delta^3 - 12txy\gamma\delta^3 - 4x^2y^2\gamma\delta^3 + 4at^3\delta^4 + 4at^2xy\delta^4 + 2t^3\gamma\delta^4 - 4t^2xy\gamma\delta^4 - 3tx^2y^2\gamma\delta^4 + 4ay\delta\eta - 4y\gamma\delta\eta + 8aty\delta^2\eta - 4ty\gamma\delta^2\eta - 6xy^2\gamma\delta^2\eta + 4at^2y\delta^3\eta - 4txy^2\gamma\delta^3\eta - 2y^2\gamma\delta\eta^2 - ty^2\gamma\delta^2\eta^2 + 4ax\delta\xi - 4x\gamma\delta\xi + 8atx\delta^2\xi - 4tx\gamma\delta^2\xi - 6x^2y\gamma\delta^2\xi + 4at^2x\delta^3\xi - 4tx^2y\gamma\delta^3\xi + 4a\eta\xi + 8at\delta\eta\xi + 4t\gamma\delta\eta\xi - 8xy\gamma\delta\eta\xi + 4at^2\delta^2\eta\xi + 4t^2\gamma\delta^2\eta\xi - 4txy\gamma\delta^2\eta\xi - 2y\gamma\eta^2\xi - 2x^2\gamma\delta\xi^2 - tx^2\gamma\delta^2\xi^2 - 2x\gamma\eta\xi^2 + t\gamma\eta^2\xi^2 \right) + \text{Log}\left[\frac{1}{1+t\delta}\right] \right\}$$

{tt = Last[Aqu[{{ξ, η, δ}, {x, y}}], Normal@Series[Log[tt], {ε, 0, \$k}]]

$$\left\{ \frac{\hbar}{-\delta + T\delta + \hbar} + \frac{1}{4(-\delta + T\delta + \hbar)^5} \in \left(-8aT\delta^4\hbar^2 + 24aT^2\delta^4\hbar^2 - 24aT^3\delta^4\hbar^2 + 8aT^4\delta^4\hbar^2 + 2\gamma\delta^4\hbar^2 - 12T\gamma\delta^4\hbar^2 + 24T^2\gamma\delta^4\hbar^2 - 20T^3\gamma\delta^4\hbar^2 + 6T^4\gamma\delta^4\hbar^2 + 24aT\delta^3\hbar^3 - 48aT^2\delta^3\hbar^3 + 24aT^3\delta^3\hbar^3 - 4\gamma\delta^3\hbar^3 + 20T\gamma\delta^3\hbar^3 - 28T^2\gamma\delta^3\hbar^3 + 12T^3\gamma\delta^3\hbar^3 + 8aTx\gamma\delta^4\hbar^3 - 16aT^2xy\delta^4\hbar^3 + 8aT^3xy\delta^4\hbar^3 - 8Tx\gamma\delta^4\hbar^3 + 16T^2xy\gamma\delta^4\hbar^3 - 8T^3xy\gamma\delta^4\hbar^3 + 8aTy\delta^3\eta\hbar^3 - 16aT^2y\delta^3\eta\hbar^3 + 8aT^3y\delta^3\eta\hbar^3 + 8aTx\delta^3\xi\hbar^3 - 16aT^2x\delta^3\xi\hbar^3 + 8aT^3x\delta^3\xi\hbar^3 + 8aT\delta^2\eta\xi\hbar^3 - 16aT^2\delta^2\eta\xi\hbar^3 + 8aT^3\delta^2\eta\xi\hbar^3 - 4\gamma\delta^2\eta\xi\hbar^3 + 20T\gamma\delta^2\eta\xi\hbar^3 - 28T^2\gamma\delta^2\eta\xi\hbar^3 + 12T^3\gamma\delta^2\eta\xi\hbar^3 - 24aT\delta^2\hbar^4 + 24aT^2\delta^2\hbar^4 + 2\gamma\delta^2\hbar^4 - 8T\gamma\delta^2\hbar^4 + 6T^2\gamma\delta^2\hbar^4 - 16aTx\gamma\delta^3\hbar^4 + 16aT^2xy\delta^3\hbar^4 + 24Tx\gamma\delta^3\hbar^4 - 24T^2xy\delta^3\hbar^4 + x^2y^2\gamma\delta^4\hbar^4 + 4Tx^2y^2\gamma\delta^4\hbar^4 - 5T^2x^2y^2\gamma\delta^4\hbar^4 - 16aTy\delta^2\eta\hbar^4 + 16aT^2y\delta^2\eta\hbar^4 - 4y\gamma\delta^2\eta\hbar^4 + 16Ty\gamma\delta^2\eta\hbar^4 - 12T^2y\gamma\delta^2\eta\hbar^4 + 8Tx\gamma^2\delta^3\eta\hbar^4 - 8T^2xy^2\delta^3\eta\hbar^4 - y^2\gamma\delta^2\eta^2\hbar^4 + 4Ty^2\gamma\delta^2\eta^2\hbar^4 - 3T^2y^2\gamma\delta^2\eta^2\hbar^4 - 16aTx\delta^2\xi\hbar^4 + 16aT^2x\delta^2\xi\hbar^4 - 4x\gamma\delta^2\xi\hbar^4 + 16Tx\gamma\delta^2\xi\hbar^4 - 12T^2x\gamma\delta^2\xi\hbar^4 + 8Tx^2y\delta^3\xi\hbar^4 - 8T^2x^2y\delta^3\xi\hbar^4 - 16aT\delta\eta\xi\hbar^4 + 16aT^2\delta\eta\xi\hbar^4 + 4\gamma\delta\eta\xi\hbar^4 - 16T\gamma\delta\eta\xi\hbar^4 + 12T^2\gamma\delta\eta\xi\hbar^4 + 8Tx\gamma\delta^2\eta\xi\hbar^4 - 8T^2xy\gamma\delta^2\eta\xi\hbar^4 - x^2\gamma\delta^2\xi^2\hbar^4 + 4Tx^2\gamma\delta^2\xi^2\hbar^4 - 3T^2x^2\gamma\delta^2\xi^2\hbar^4 + \gamma\eta^2\xi^2\hbar^4 - 4T\gamma\eta^2\xi^2\hbar^4 + 3T^2\gamma\eta^2\xi^2\hbar^4 + 8aT\delta\hbar^5 + 8aTx\gamma\delta^2\hbar^5 - 4xy\gamma\delta^2\hbar^5 - 12Tx\gamma\delta^2\hbar^5 - 4x^2y^2\gamma\delta^3\hbar^5 - 4Tx^2y^2\gamma\delta^3\hbar^5 + 8aTy\delta\eta\hbar^5 + 4y\gamma\delta\eta\hbar^5 - 12Ty\gamma\delta\eta\hbar^5 - 2xy^2\gamma\delta^2\eta\hbar^5 - 10Tx\gamma^2\delta^2\eta\hbar^5 + 2y^2\gamma\delta\eta^2\hbar^5 - 6Ty^2\gamma\delta\eta^2\hbar^5 + 8aTx\delta\xi\hbar^5 + 4x\gamma\delta\xi\hbar^5 - 12Tx\gamma\delta\xi\hbar^5 - 2x^2y\gamma\delta^2\xi\hbar^5 - 10Tx^2y\gamma\delta^2\xi\hbar^5 + 8aT\eta\xi\hbar^5 - 16Tx\gamma\delta\eta\xi\hbar^5 + 2y\gamma\eta^2\xi\hbar^5 - 6Ty\gamma\eta^2\xi\hbar^5 + 2x^2\gamma\delta\xi^2\hbar^5 - 6Tx^2\gamma\delta\xi^2\hbar^5 + 2x\gamma\eta\xi^2\hbar^5 - 6Tx\gamma\eta\xi^2\hbar^5 + 4xy\gamma\delta\hbar^6 + 4x^2y^2\gamma\delta^2\hbar^6 + 4xy^2\gamma\delta\eta\hbar^6 + 4x^2y\gamma\delta\xi\hbar^6 + 4xy\gamma\eta\xi\hbar^6 \right), \frac{1}{4(-\delta + T\delta + \hbar)^4} \in \left(-8aT\delta^4\hbar + 24aT^2\delta^4\hbar - 24aT^3\delta^4\hbar + 8aT^4\delta^4\hbar + 2\gamma\delta^4\hbar - 12T\gamma\delta^4\hbar + 24T^2\gamma\delta^4\hbar - 20T^3\gamma\delta^4\hbar + 6T^4\gamma\delta^4\hbar + 24aT\delta^3\hbar^2 - 48aT^2\delta^3\hbar^2 + 24aT^3\delta^3\hbar^2 - 4\gamma\delta^3\hbar^2 + 20T\gamma\delta^3\hbar^2 - 28T^2\gamma\delta^3\hbar^2 + 12T^3\gamma\delta^3\hbar^2 + 8aTx\gamma\delta^4\hbar^2 - 16aT^2xy\delta^4\hbar^2 + 8aT^3xy\delta^4\hbar^2 - 8Tx\gamma\delta^4\hbar^2 + 16T^2xy\gamma\delta^4\hbar^2 - 8T^3xy\gamma\delta^4\hbar^2 + 8aTy\delta^3\eta\hbar^2 - 16aT^2y\delta^3\eta\hbar^2 + 8aT^3y\delta^3\eta\hbar^2 + 8aTx\delta^3\xi\hbar^2 - 16aT^2x\delta^3\xi\hbar^2 + 8aT^3x\delta^3\xi\hbar^2 + 8aT\delta^2\eta\xi\hbar^2 - 16aT^2\delta^2\eta\xi\hbar^2 + 8aT^3\delta^2\eta\xi\hbar^2 - 4\gamma\delta^2\eta\xi\hbar^2 + 20T\gamma\delta^2\eta\xi\hbar^2 - 28T^2\gamma\delta^2\eta\xi\hbar^2 + 12T^3\gamma\delta^2\eta\xi\hbar^2 - 24aT\delta^2\hbar^3 + 24aT^2\delta^2\hbar^3 + 2\gamma\delta^2\hbar^3 - 8T\gamma\delta^2\hbar^3 + 6T^2\gamma\delta^2\hbar^3 - 16aTx\gamma\delta^3\hbar^3 + 16aT^2xy\delta^3\hbar^3 + 24Tx\gamma\delta^3\hbar^3 - 24T^2xy\gamma\delta^3\hbar^3 + x^2y^2\gamma\delta^4\hbar^3 + 4Tx^2y^2\gamma\delta^4\hbar^3 - 5T^2x^2y^2\gamma\delta^4\hbar^3 - 16aTy\delta^2\eta\hbar^3 + 16aT^2y\delta^2\eta\hbar^3 - 4y\gamma\delta^2\eta\hbar^3 + 16Ty\gamma\delta^2\eta\hbar^3 - 12T^2y\gamma\delta^2\eta\hbar^3 + 8Tx\gamma^2\delta^3\eta\hbar^3 - 8T^2xy^2\delta^3\eta\hbar^3 - y^2\gamma\delta^2\eta^2\hbar^3 + 4Ty^2\gamma\delta^2\eta^2\hbar^3 - 3T^2y^2\gamma\delta^2\eta^2\hbar^3 - 16aTx\delta^2\xi\hbar^3 + 16aT^2x\delta^2\xi\hbar^3 - 4x\gamma\delta^2\xi\hbar^3 + 16Tx\gamma\delta^2\xi\hbar^3 - 12T^2x\gamma\delta^2\xi\hbar^3 + 8Tx^2y\delta^3\xi\hbar^3 - 8T^2x^2y\delta^3\xi\hbar^3 - 16aT\delta\eta\xi\hbar^3 + 16aT^2\delta\eta\xi\hbar^3 + 4\gamma\delta\eta\xi\hbar^3 - 16T\gamma\delta\eta\xi\hbar^3 + 12T^2\gamma\delta\eta\xi\hbar^3 + 8Tx\gamma\delta^2\eta\xi\hbar^3 - 8T^2xy\gamma\delta^2\eta\xi\hbar^3 - x^2\gamma\delta^2\xi^2\hbar^3 + 4Tx^2\gamma\delta^2\xi^2\hbar^3 - 3T^2x^2\gamma\delta^2\xi^2\hbar^3 + \gamma\eta^2\xi^2\hbar^3 - 4T\gamma\eta^2\xi^2\hbar^3 + 3T^2\gamma\eta^2\xi^2\hbar^3 + 8aT\delta\hbar^4 + 8aTx\gamma\delta^2\hbar^4 - 4xy\gamma\delta^2\hbar^4 - 12Tx\gamma\delta^2\hbar^4 - 4x^2y^2\gamma\delta^3\hbar^4 - 4Tx^2y^2\gamma\delta^3\hbar^4 + 8aTy\delta\eta\hbar^4 + 4y\gamma\delta\eta\hbar^4 - 12Ty\gamma\delta\eta\hbar^4 - 2xy^2\gamma\delta^2\eta\hbar^4 - 10Tx\gamma^2\delta^2\eta\hbar^4 + 2y^2\gamma\delta\eta^2\hbar^4 - 6Ty^2\gamma\delta\eta^2\hbar^4 + 8aTx\delta\xi\hbar^4 + 4x\gamma\delta\xi\hbar^4 - 12Tx\gamma\delta\xi\hbar^4 - 2x^2y\gamma\delta^2\xi\hbar^4 - 10Tx^2y\gamma\delta^2\xi\hbar^4 + 8aT\eta\xi\hbar^4 - 16Tx\gamma\delta\eta\xi\hbar^4 + 2y\gamma\eta^2\xi\hbar^4 - 6Ty\gamma\eta^2\xi\hbar^4 + 2x^2\gamma\delta\xi^2\hbar^4 - 6Tx^2\gamma\delta\xi^2\hbar^4 + 2x\gamma\eta\xi^2\hbar^4 - 6Tx\gamma\eta\xi^2\hbar^4 + 4xy\gamma\delta\hbar^5 + 4x^2y^2\gamma\delta^2\hbar^5 + 4xy^2\gamma\delta\eta\hbar^5 + 4x^2y\gamma\delta\xi\hbar^5 + 4xy\gamma\eta\xi\hbar^5 \right) + \text{Log}\left[\frac{\hbar}{-\delta + T\delta + \hbar}\right] \}$$

Reorderings with Rord

Rord

```

Rordui, wj → k [CU[L---, {L---, ui, wj, r---}S, R---, Q---, P---]] :=
Simp@Module[{u, w, δ, Δ1, yax, q, p, n, δ1 = ∂ui, wj Q},
  {yax, q, p} = List@@If[δ1 == 0, ΔU[{u, w}, {u, w}], ΔU[{u, w, δ}, {u, w}]] /.
  {y → yn, a → an, x → xn, t → tS, T → TS};
CU[L, {L, Sequence@@yax, r}S, R, q + (Q /. ui | wj → 0), e-q SP{ui → v, wj → ω}[P p eq]] /.
  {n → k, v → ∂ui Q /. wj → 0, ω → ∂wj Q /. ui → 0, δ → δ1}];

```

```

With[{c0 = CCU[{y1, x1}1, {x2, a2, y2}2, ħ t1 a2 + ħ t1-1 (et1 - 1) y1 x2, 1 + ε x1 y2]},
  {Short[rhs = c0 // Rordx2, a2 → 3, 3], HL[CU[c0] == CU[rhs]]}]
{CCU[{y1, x1}1, {a3, x3, y2}2,  $\frac{e^{-\gamma \hbar t_1} (e^{\gamma \hbar t_1} \hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1} \hbar x_3 y_1)}{t_1}$ , 1 + ε x1 y2], True}

```

```

With[{c0 = CCU[{y1, a1, x1}1, {x2, a2, y2}2,
  ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
  1 + ε (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = c0 // Rordx2, a2 → 3, 3], HL[CU[c0] == CU[rhs]]}]
{CCU[{y1, a1, x1}1, {<<1>>2, <<1>> <<1>>,
  1 + e-γ ħ (l12 t1 + l22 t2) ∈ (eγ ħ (l12 t1 + l22 t2) a1 l1 + eγ ħ (l12 t1 + l22 t2) a3 l2 + eγ ħ (l12 t1 + l22 t2) p11 x1 y1 +
  p21 x3 y1 + e<<1>> p12 x1 y2 + p22 x3 y2 - γ ħ l2 x3 y1 γ21 - γ ħ l2 x3 y2 γ22)], True}

```

```

With[{q0 = CQU[{y1, a1, x1}1, {x2, a2, y2}2,
  ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
  1 + ε (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q0 // Rordx2, a2 → 3, 3], HL[QU[q0] == QU[rhs]]}]
{CQU[{y1, a1, x1}1, {<<1>>2, <<1>> <<1>>,
  1 + e-γ ħ (l12 t1 + l22 t2) ∈ (eγ ħ (l12 t1 + l22 t2) a1 l1 + eγ ħ (l12 t1 + l22 t2) a3 l2 + eγ ħ (l12 t1 + l22 t2) p11 x1 y1 +
  p21 x3 y1 + e<<1>> p12 x1 y2 + p22 x3 y2 - γ ħ l2 x3 y1 γ21 - γ ħ l2 x3 y2 γ22)], True}

```

```

With[{q0 = CQU[{y1, a1, x1}1, {x2, a2, y2}2,
  ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
  1 + ε (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q0 // Rorda2, y2 → 3, 3], HL[QU[q0] == QU[rhs]]}]
{CQU[{y1, a1, x1}1, {<<1>>2, <<1>> <<1>>,
  1 + e-γ ħ (l12 t1 + l22 t2) ∈ (eγ ħ (l12 t1 + l22 t2) a1 l1 + eγ ħ (l12 t1 + l22 t2) a3 l2 + eγ ħ (l12 t1 + l22 t2) p11 x1 y1 +
  e<<1>> p21 x2 y1 + p12 x1 y3 + p22 x2 y3 - γ ħ l2 x1 y3 γ12 - γ ħ l2 x2 y3 γ22)], True}

```

```
Timing@With[{q0 = CQU[{x1, y1}1, {x2, a2, y2}2,
  h (l12 t1 a2 + l22 t2 a2 + r11 x1 y1 + r12 x1 y2 + r21 x2 y1 + r22 x2 y2),
  1 + e (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2) ]},
{Short[rhs = q0 // Rordx1,y1->3, 5], HL@SimpT[QU[q0] == QU[rhs]]}]
{137.516, {CQU[{y3, a3, x3}1, {x2, a2, y2}2,
  h a2 l12 t1 + <<17>>,
  1 - r11 + T1 r11,
  1 / (1 - r11 + T1 r11) + (e (4 h a2 l2 + <<346>> + 3 r h^4 T1^2 x2^2 y2^2 r12^2 r21^2)) / (4 h (1 - r11 + T1 r11)^5) +
  1 / (4 (1 - r11 + T1 r11)^7) e^2 (8 a3 p11 T1 + 4 r h p11 x3 y3 + 4 r p11 r11 + <<2722>> +
  12 r h^3 p11 T1^2 x2^3 y2^3 r12^3 r21^3 - 10 r h^3 p11 T1^3 x2^3 y2^3 r12^3 r21^3 + 3 r h^3 p11 T1^4 x2^3 y2^3 r12^3 r21^3) ]}, True}}
```

```
Timing@With[{q0 = CQU[{x1, y1}1, {x2, a2, y2}2,
  h (l12 t1 a2 + l22 t2 a2 + r11 x1 y1 + r12 x1 y2 + r21 x2 y1 + r22 x2 y2),
  1 + e (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2) ]},
{Short[rhs = q0 // Rordx1,y1->1, 5], HL@SimpT[QU[q0] == QU[rhs]]}]
{125.703, {CQU[{y1, a1, x1}1, {x2, a2, y2}2,
  h a2 l12 t1 + <<17>>,
  1 - r11 + T1 r11,
  1 / (1 - r11 + T1 r11) + (e (4 h a2 l2 + <<346>> + 3 r h^4 T1^2 x2^2 y2^2 r12^2 r21^2)) / (4 h (1 - r11 + T1 r11)^5) +
  1 / (4 (1 - r11 + T1 r11)^7) e^2 (8 a1 p11 T1 + 4 r h p11 x1 y1 + 4 r p11 r11 + <<2722>> +
  12 r h^3 p11 T1^2 x2^3 y2^3 r12^3 r21^3 - 10 r h^3 p11 T1^3 x2^3 y2^3 r12^3 r21^3 + 3 r h^3 p11 T1^4 x2^3 y2^3 r12^3 r21^3) ]}, True}}
```

R in QU.

Faddeev-Quesne's formula:

Faddeev

```
e_{q-,k-}[x-] := e ^ (sum_{j=1}^k ((1-q)^j x^j) / (j (1-q^j))); e_{q-,k}[x]
```

```
Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)}\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
{1, 1, 1, 1, 1, 1}
```

R

```
QU[R_{i-,j-}] := OQU[{y1, a1}i, {a2, x2}j, SS[e^{h b1 a2} e_{qh}[h y1 x2] /. b1 -> r^{-1} (e a1 - t1)]];
QU[R_{i-,j-}^{-1}] := S_j@QU[R_{i-,j-}];
```

QU[R_{3,4}] // Short

$$QU[] + \frac{\epsilon \hbar QU[a_3, a_4]}{\gamma} + \hbar QU[y_3, x_4] + \frac{\langle\langle 1 \rangle\rangle}{\gamma} + \langle\langle 1 \rangle\rangle - \frac{\langle\langle 1 \rangle\rangle}{\gamma} - \frac{\epsilon \langle\langle 3 \rangle\rangle}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4] t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ \$p=4, \$k=2):

QU[R_{1,2} ** R_{1,2}⁻¹] // Simp // HL // Timing

{0.046875, QU[]}

Verifying R3 (~156 secs @ \$p=4, \$k=2):

{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]], HL@SimpT[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]} // Timing

{0.140625, {QU[] + \frac{\epsilon \hbar QU[a_1, a_2]}{\gamma} + \langle\langle 74 \rangle\rangle + 2 \epsilon \hbar^2 QU[y_1, a_2, x_3] T_2 + QU[y_1, x_3] (\hbar - \hbar T_2), \mathbf{0}}}

R in \mathbb{C}_{QU} .

RinOE

```
 $\mathbb{C}_{QU}[R_{i,j}] := \mathbb{C}_{QU}[\{y_i, a_i, x_i\}_i, \{y_j, a_j, x_j\}_j, -\hbar \gamma^{-1} t_i a_j + \hbar y_i x_j, \text{Normal@Series}[e^{\hbar \gamma^{-1} t_i a_j - \hbar y_i x_j} (e^{\hbar b_i a_j} e_{q_b}[\hbar y_i x_j] /. b_i \rightarrow \gamma^{-1} (\epsilon a_i - t_i)), \{\epsilon, \theta, \$k\}]]]$ 
```

$\mathbb{C}_{QU}[R_{1,2}]$

$$\mathbb{C}_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \frac{\epsilon \hbar a_1 a_2}{\gamma}]$$

E

$E[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a 's, Q is a combination of $x_i y_j$, and P is a perturbation polynomial. It should be interpreted via $\text{CO}[E[\dots], \{x_1, a_1, y_1\}_j, \dots]$ (with some default for direct interpretation), or likewise via $\text{QO}[E[\dots], \{x_1, a_1, y_1\}_j, \dots]$. In themselves, CO and QO should have an interpretation in CU/QU by casting.

Alternative Algorithms

AllLogos

```
 $\lambda_{alt}[CU] := \text{If}[\$k == \theta, 1, \text{Module}[\{\text{eq}, d, b, c, \text{so}\}, \text{eq} = \rho @ e^{\xi x_{cu}} . \rho @ e^{\eta y_{cu}} == \rho @ e^{d y_{cu}} . \rho @ e^{c (t_{1cu} - 2 \epsilon a_{cu})} . \rho @ e^{b x_{cu}}; \{\text{so}\} = \text{Solve}[\text{Thread}[\text{Flatten} / @ \text{eq}], \{d, b, c\}] /. C @ 1 \rightarrow \theta; \text{Normal@Series}[e^{-\eta y - \xi x + \eta \xi t + c t + d y - 2 \epsilon c a + b x} /. \text{so}, \{\epsilon, \theta, \$k\}]]];$ 
```

{ $\lambda_{alt}[CU]$, HL@Simplify[$\lambda_{alt}[CU] == \text{Last}[\Delta_{CU}[\{\xi, \eta\}, \{x, y\}]]]}$ }

$$\{1 + \epsilon \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right), \text{True}\}$$