Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The "degree carrier / filtration parameter" is ℏ, and all "coupling constants" are proportional to it.

TD

```
$p = 2; $k = 1; $E := {$k, $p};
$trim := {ℏ^(p_.) /; p > $p → 0, ϵ^(k_.) /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → ⅇ^(ℏ t_i), T → ⅇ^(ℏ t)}; q_ℏ = ⅇ^(γ ϵ ℏ);
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. TRule], {ℏ, 0, $p}],
   ℏ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. TRule, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

Self-Pair (SP):

```
D[x^5, {x, 3}]
```

$60\,x^2$

```
∂_{x,3} x^5
```

$60\,x^2$

```
SP_{}[P_] := P; SP_{ξ_→x_, ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^(d_.) :→ ∂_{x,d} f
```

```
SP_{ξ→x}[(ξ^2 + ξ + 3) (x^5 ⅇ^x + 7 x) + 99 a]
```

$7 + 99\,a + 21\,x + 20\,e^x x^3 + 15\,e^x x^4 + 5\,e^x x^5$

```
SP_{ξ→x,η→y}[(ξ^2 + ξ + 3 + 2 ξ η) (x^5 ⅇ^x + 7 x) + 99 a + ⅇ^(δ x y) ξ η]
```

$7 + 99\,a + 21\,x + 20\,e^{x}\,x^{3} + 15\,e^{x}\,x^{4} + 5\,e^{x}\,x^{5} + e^{x\,y\,\delta}\,\delta + e^{x\,y\,\delta}\,x\,y\,\delta^{2}$

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
   (#_U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
   CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
   U_i_[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ Replace[u, x_ ⧴ x_i, 1]};
   U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
   (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
     CE@M[a b /. $trim, U[xx, x, y, yy]],
     U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List ⧴ l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
       ]] /. x_null ⧴ x];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
      CoefficientRules[ℰ, First /@ {ss}] /.
       (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   m_j_→k_[c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
      U @@ Cases[u, w_j ⧴ w_k] ** U @@ Cases[u, _k]];
   S_i_[c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
      U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]] ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
   {(g_ → img_) ⧴ (m[U[g]] = img), (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

Meta-Operations

QLImplementation

```
m_{j_→j_} = Identity;
m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing $CU = \mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.67188,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
CU[a₂, x₂, y₃] t₃² + CU[x₂, y₃, a₃] t₃²
```

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z[[1]]₁, z[[2]]₂, z[[3]]₃]; z → HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}]]
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T e^(-2 ϵ a ℏ)) / ℏ, {a, 0, 3}]
```

Aside

$$\frac{1-T}{\hbar} + 2\,T\,\epsilon\,a - 2\left(T\,\epsilon^2\,\hbar\right) a^2 + \frac{4}{3}\,T\,\epsilon^3\,\hbar^2\,a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ℏ - 1] QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T e^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 e^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := 𝕆_QU[{a, x}, SS[-e^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
  {QU[y], QU[x]} → ((-1 + T) QU[])/ℏ - 2 T ∈ QU[a] - γ ∈ ℏ QU[y, x]},
 {{QU[a], QU[y]} → -γ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} → ((1 - T) QU[])/ℏ + 2 T ∈ QU[a] + γ ∈ ℏ QU[y, x],
  {QU[x], QU[a]} → -γ QU[x], {QU[x], QU[x]} → 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

```
{3.51563, {((28 γ^4 - 56 T γ^4 + 28 T^2 γ^4)/ℏ^2 + (82 γ^5 ∈ - 280 <<3>> + 198 T^2 γ^5 ∈)/ℏ) QU[y, y, y, x, x] +
   <<18>> + (1 + 8 γ ∈ ℏ) QU[<<1>>], 0}}
```

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y₁, a₁, x₁}},
 Table[{z1, z2} → HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{{QU[y₁], QU[y₁]} → 0, {QU[y₁], QU[a₁]} → 0, {QU[y₁], QU[x₁]} → 0},
 {{QU[a₁], QU[y₁]} → 0, {QU[a₁], QU[a₁]} → 0, {QU[a₁], QU[x₁]} → 0},
 {{QU[x₁], QU[y₁]} → 0, {QU[x₁], QU[a₁]} → 0, {QU[x₁], QU[x₁]} → 0}}
```

Verifying that $\mathrm{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

```
{9.9375, {28 t^2 γ^4 CU[y, y, y, x, x] +
   116 t γ^5 ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
  2 (γ^4/ℏ^2 - (2 T γ^4)/ℏ^2 + (T^2 γ^4)/ℏ^2 + (γ^5 ∈)/ℏ - (2 T γ^5 ∈)/ℏ + (T^2 γ^5 ∈)/ℏ) QU[y, y, y, x, x] +
   <<209>> + (1 + 8 γ ∈ ℏ) QU[y, <<11>>, x], 0}}
```

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y :→ 𝕆_QU[{a, x}, SS[-T^(-1/2) ⅇ^(ℏ ϵ a) x]],
  a → -a_QU, x :→ 𝕆_QU[{a, y}, SS[-T^(-1/2) ⅇ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$$\left\{QU[y]\to -\frac{QU[x]}{\sqrt{T}}-\frac{\epsilon\hbar\, QU[a,x]}{\sqrt{T}}\to QU[y],\ QU[a]\to -QU[a]\to QU[a],\right.$$
$$\left. QU[x]\to\left(-\frac{1}{\sqrt{T}}+\frac{\gamma\epsilon\hbar}{\sqrt{T}}\right)QU[y]-\frac{\epsilon\hbar\, QU[y,a]}{\sqrt{T}}\to QU[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\mathbb{AD}\$f=\gamma\left(\left(\text{Cosh}\left[\hbar\left(a\epsilon+\frac{\gamma\epsilon}{2}-\frac{t}{2}\right)\right]-\text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\epsilon}{2}\right)^2+\epsilon\omega}\right]\right)\Big/\left(\hbar\, e^{\hbar((a+\gamma)\epsilon-t/2)}\,\text{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(a^2\epsilon+a\gamma\epsilon-a t-\omega\right)\right)\right);$$

Scaling behaviour of AD$f:

```
HL@Simplify[AD$f == ((AD$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  AD$f == ((AD$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

ADeq

```
AD$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[AD, QU → CU,
  {a → a_CU, x → CU@x, y :→ 𝕊_CU[SS[AD$f], a → a_CU, ω → AD$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$SD\$g = \sqrt{\left(\left(2\,\gamma\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\,\epsilon^2+4\,\epsilon\,\varpi}\right]-\mathrm{Cosh}\left[\frac{t-\epsilon\,\gamma-2\,\epsilon\,a}{2/\hbar}\right]\right)\right)\Big/\left(\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(t\,(2\,a+\gamma)-2\,a\,(a+\gamma)\,\epsilon+2\,\varpi\right)\hbar\right)\right)};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ħ ((ϵ-t)/2 + ϵ a)] - Cosh[ħ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) / (ħ Sinh[-ϵ ħ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
     SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
     Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}\,(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}\,(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$
$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}\,(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\,\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True}\right\}$$

SDeq

```
SD$f = Simplify[e^(ħ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x :→ S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y :→ S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = (0 (1 - ⅇ^(-γ ϵ ℏ)) / (ϵ ℏ); 0 0);
ρ[ⅇ^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] := (ℰ /. TRule /. t → γ ϵ /. (U : CU | QU) [u___] ↦ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ϵ^(k_·) /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[ⅇ^(ξ U@x)].ρ[ⅇ^(α U@a)] == ρ[ⅇ^(α U@a)].ρ[ⅇ^(ⅇ^(-γ α) ξ U@x)]], {U, {CU, QU}}]
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[ⅇ^Q P]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[ⅇ^Q P]];
```

Logos

```
Λ_U_[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + ⅇ^(-γ α) ξ x, 1];
Λ_U_[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + ⅇ^(-γ α) η y, 1];
```

```
{Λ_#[{ξ, α}, {x, a}], lhs = #@Œ_#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ_#[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
{{Œ_CU[{a, x}, a α + ⅇ^(-α γ) x ξ, 1],
  CU[] + α ℏ CU[a] + (ξ ℏ - α γ ξ ℏ^2) CU[x] + 1/2 α^2 ℏ^2 CU[a, a] + α ξ ℏ^2 CU[a, x] + 1/2 ξ^2 ℏ^2 CU[x, x],
  True}, {Œ_QU[{a, x}, a α + ⅇ^(-α γ) x ξ, 1], QU[] + α ℏ QU[a] + (ξ ℏ - α γ ξ ℏ^2) QU[x] +
  1/2 α^2 ℏ^2 QU[a, a] + α ξ ℏ^2 QU[a, x] + 1/2 ξ^2 ℏ^2 QU[x, x], True}}
```

```
{Λ#[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{\mathbb{E}_{CU}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y\eta, 1],$
$CU[\,] + \alpha\hbar\, CU[a] + (\eta\hbar - \alpha\gamma\eta\hbar^2)\, CU[y] + \frac{1}{2}\alpha^2\hbar^2\, CU[a, a] + \alpha\eta\hbar^2\, CU[y, a] + \frac{1}{2}\eta^2\hbar^2\, CU[y, y],$
$\text{True}\}, \{\mathbb{E}_{QU}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y\eta, 1], QU[\,] + \alpha\hbar\, QU[a] + (\eta\hbar - \alpha\gamma\eta\hbar^2)\, QU[y] +$
$\frac{1}{2}\alpha^2\hbar^2\, QU[a, a] + \alpha\eta\hbar^2\, QU[y, a] + \frac{1}{2}\eta^2\hbar^2\, QU[y, y], \text{True}\}\}$

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {ⅇ- → 1, U → Times}];
   Collect[sol /. {ⅇ- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

″ $-t\xi\, CU[\,] + 2\epsilon\xi\, CU[a] - \gamma\epsilon\xi^2\, CU[x] + CU[y]$

″ $\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta],$
$f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]\}$

″ $CU[\,]\, f_{0,0,0,0}[\eta] + \epsilon\, CU[\,]\, f_{1,0,0,0}[\eta] + \epsilon\, CU[x]\, f_{1,0,0,1}[\eta] + \epsilon\, CU[a]\, f_{1,0,1,0}[\eta] + \epsilon\, CU[a, x]\, f_{1,0,1,1}[\eta] +$
$\epsilon\, CU[y]\, f_{1,1,0,0}[\eta] + \epsilon\, CU[y, x]\, f_{1,1,0,1}[\eta] + \epsilon\, CU[y, a]\, f_{1,1,1,0}[\eta] + \epsilon\, CU[y, a, x]\, f_{1,1,1,1}[\eta]$

» $e^{-t\eta\xi}\, CU[\,] + \frac{1}{2} e^{-t\eta\xi}\, t\gamma\epsilon\eta^2\xi^2\, CU[\,] + 2 e^{-t\eta\xi}\epsilon\eta\xi\, CU[a] - e^{-t\eta\xi}\gamma\epsilon\eta\xi^2\, CU[x] - e^{-t\eta\xi}\gamma\epsilon\eta^2\xi\, CU[y]$

» $1 + 2a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2} t\gamma\epsilon\eta^2\xi^2$

$1 + \frac{1}{2}\epsilon\eta\xi\,(4a + \gamma(-2y\eta - 2x\xi + t\eta\xi))$

Logos

```
Λ_U_[{ξ1_, η1_}, {x, y}] := Λ_U[{ξ1, η1}, {x, y}] = Module[{ξ, η, G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
    fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
    F = F /. DSolve[es, fs, η][[1]];
    Œ_U[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
      F /. {ⅇ- → 1, U → Times}
     ] /. {ξ → ξ1, η → η1}];
```

```
{ΛCU[{ξ, η}, {x, y}], lhs = CU@𝔈CU[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@ΛCU[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{\mathbb{E}_{CU}\left[\{y,a,x\},\ y\eta+x\xi-t\eta\xi,\ 1+2a\epsilon\eta\xi-y\gamma\epsilon\eta^2\xi-x\gamma\epsilon\eta\xi^2+\frac{1}{2}t\gamma\epsilon\eta^2\xi^2\right],\right.$$
$$\left(1-t\eta\xi\hbar^2\right)CU[]+2\epsilon\eta\xi\hbar^2CU[a]+\xi\hbar CU[x]+\eta\hbar CU[y]+$$
$$\left.\frac{1}{2}\xi^2\hbar^2CU[x,x]+\eta\xi\hbar^2CU[y,x]+\frac{1}{2}\eta^2\hbar^2CU[y,y],\ \text{True}\right\}$$

```
{ΛQU[{ξ, η}, {x, y}], lhs = QU@𝔈QU[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@ΛQU[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{\mathbb{E}_{QU}\left[\{y,a,x\},\ y\eta+x\xi+\frac{(1-T)\eta\xi}{\hbar},\ 1+2aT\epsilon\eta\xi-\frac{1}{2}(-1+3T)y\gamma\epsilon\eta^2\xi-\right.\right.$$
$$\left.\frac{1}{2}(-1+3T)x\gamma\epsilon\eta\xi^2+\frac{(-1+T)(-1+3T)\gamma\epsilon\eta^2\xi^2}{4\hbar}+xy\gamma\epsilon\eta\xi\hbar\right],$$
$$(1+\eta\xi\hbar-T\eta\xi\hbar)QU[]+2T\epsilon\eta\xi\hbar^2QU[a]+\xi\hbar QU[x]+\eta\hbar QU[y]+$$
$$\left.\frac{1}{2}\xi^2\hbar^2QU[x,x]+\eta\xi\hbar^2QU[y,x]+\frac{1}{2}\eta^2\hbar^2QU[y,y],\ \text{True}\right\}$$

```
{tt = Last[ΛCU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\left\{1+2a\epsilon\eta\xi-y\gamma\epsilon\eta^2\xi-x\gamma\epsilon\eta\xi^2+\frac{1}{2}t\gamma\epsilon\eta^2\xi^2,\ \epsilon\left(2a\eta\xi-y\gamma\eta^2\xi-x\gamma\eta\xi^2+\frac{1}{2}t\gamma\eta^2\xi^2\right)\right\}$$

```
{tt = Last[ΛQU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\left\{1+2aT\epsilon\eta\xi-\frac{1}{2}(-1+3T)y\gamma\epsilon\eta^2\xi-\frac{1}{2}(-1+3T)x\gamma\epsilon\eta\xi^2+\right.$$
$$\frac{(-1+T)(-1+3T)\gamma\epsilon\eta^2\xi^2}{4\hbar}+xy\gamma\epsilon\eta\xi\hbar,\ \frac{1}{4\hbar}\epsilon\left(\gamma\eta^2\xi^2-4T\gamma\eta^2\xi^2+3T^2\gamma\eta^2\xi^2+\right.$$
$$\left.\left.8aT\eta\xi\hbar+2y\gamma\eta^2\xi\hbar-6Ty\gamma\eta^2\xi\hbar+2x\gamma\eta\xi^2\hbar-6Tx\gamma\eta\xi^2\hbar+4xy\gamma\eta\xi\hbar^2\right)\right\}$$

Logos

```
Simp[𝔈U_[specs___, Q_, P_]] :=
  𝔈U[specs, ExpandNumerator@Together[Q], Collect[P, ϵ, ExpandNumerator@*Together]];
ΛU_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
   {yax, q, p} = List @@ ΛU[{υ, ω}, {u, w}];
   𝔈U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
     Expand[(1 - d δ)^-1 𝕖^-Q DP_{υ→D_u, ω→D_w}[p][𝕖^Q]] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}]
```

```
{ΛCU[{ξ, η, δ}, {x, y}], lhs = CU@𝔈CU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL[lhs == CU@ΛCU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{\mathbb{E}_{CU}\Big[\{y, a, x\}, \frac{x\,y\,\delta + y\,\eta + x\,\xi - t\,\eta\,\xi}{1 + t\,\delta}, \frac{1}{1 + t\,\delta} + \frac{1}{2\,(1 + t\,\delta)^5}\,\epsilon\,\big(4\,a\,\delta + 12\,a\,t\,\delta^2 + 4\,a\,x\,y\,\delta^2 + 2\,t\,\gamma\,\delta^2 - 8\,x\,y\,\gamma\,\delta^2 + 12\,a\,t^2\,\delta^3 + 8\,a\,t\,x\,y\,\delta^3 + 4\,t^2\,\gamma\,\delta^3 - 12\,t\,x\,y\,\gamma\,\delta^3 - 4\,x^2\,y^2\,\gamma\,\delta^3 + 4\,a\,t^3\,\delta^4 + 4\,a\,t^2\,x\,y\,\delta^4 + 2\,t^3\,\gamma\,\delta^4 - 4\,t^2\,x\,y\,\gamma\,\delta^4 - 3\,t\,x^2\,y^2\,\gamma\,\delta^4 + 4\,a\,y\,\delta\,\eta - 4\,y\,\gamma\,\delta\,\eta + 8\,a\,t\,y\,\delta^2\,\eta - 4\,t\,y\,\gamma\,\delta^2\,\eta - 6\,x\,y^2\,\gamma\,\delta^2\,\eta + 4\,a\,t^2\,y\,\delta^3\,\eta - 4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta - 2\,y^2\,\gamma\,\delta\,\eta^2 - t\,y^2\,\gamma\,\delta^2\,\eta^2 + 4\,a\,x\,\delta\,\xi - 4\,x\,\gamma\,\delta\,\xi + 8\,a\,t\,x\,\delta^2\,\xi - 4\,t\,x\,\gamma\,\delta^2\,\xi - 6\,x^2\,y\,\gamma\,\delta^2\,\xi + 4\,a\,t^2\,x\,\delta^3\,\xi - 4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 4\,a\,\eta\,\xi + 8\,a\,t\,\delta\,\eta\,\xi + 4\,t\,\gamma\,\delta\,\eta\,\xi - 8\,x\,y\,\gamma\,\delta\,\eta\,\xi + 4\,a\,t^2\,\delta^2\,\eta\,\xi + 4\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 2\,y\,\gamma\,\eta^2\,\xi - 2\,x^2\,\gamma\,\delta\,\xi^2 - t\,x^2\,\gamma\,\delta^2\,\xi^2 - 2\,x\,\gamma\,\eta\,\xi^2 + t\,\gamma\,\eta^2\,\xi^2\big)\Big],$$

$$\big(1 - t\,\delta\,\hbar + t^2\,\delta^2\,\hbar^2 + t\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - t\,\eta\,\xi\,\hbar^2\big)\,CU[\,] + \big(2\,\delta\,\epsilon\,\hbar - 4\,t\,\delta^2\,\epsilon\,\hbar^2 + 2\,\epsilon\,\eta\,\xi\,\hbar^2\big)\,CU[a] + \big(\xi\,\hbar - 2\,t\,\delta\,\xi\,\hbar^2 - 2\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2\big)\,CU[x] + \big(\eta\,\hbar - 2\,t\,\delta\,\eta\,\hbar^2 - 2\,\gamma\,\delta\,\epsilon\,\eta\,\hbar^2\big)\,CU[y] + 4\,\delta\,\epsilon\,\xi\,\hbar^2\,CU[a, x] + \frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] + 4\,\delta\,\epsilon\,\eta\,\hbar^2\,CU[y, a] + \big(\delta\,\hbar - 2\,t\,\delta^2\,\hbar^2 - 4\,\gamma\,\delta^2\,\epsilon\,\hbar^2 + \eta\,\xi\,\hbar^2\big)\,CU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y] + 4\,\delta^2\,\epsilon\,\hbar^2\,CU[y, a, x] + \delta\,\xi\,\hbar^2\,CU[y, x, x] + \delta\,\eta\,\hbar^2\,CU[y, y, x] + \frac{1}{2}\,\delta^2\,\hbar^2\,CU[y, y, x, x], \text{True}\Big\}$$

```
{ΛQU[{ξ, η, δ}, {x, y}], lhs = QU@ŒQU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@ΛQU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$
\begin{aligned}
&\Big\{\text{Œ}_{QU}\Big[\{y, a, x\}, \frac{\eta\,\xi - T\,\eta\,\xi + x\,y\,\delta\,\hbar + y\,\eta\,\hbar + x\,\xi\,\hbar}{-\delta + T\,\delta + \hbar},\\
&\quad \frac{\hbar}{-\delta + T\,\delta + \hbar} + \frac{1}{4\,(-\delta + T\,\delta + \hbar)^5}\,\epsilon\,\Big(-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + 2\,\gamma\,\delta^4\,\hbar^2 -\\
&\qquad 12\,T\,\gamma\,\delta^4\,\hbar^2 + 24\,T^2\,\gamma\,\delta^4\,\hbar^2 - 20\,T^3\,\gamma\,\delta^4\,\hbar^2 + 6\,T^4\,\gamma\,\delta^4\,\hbar^2 + 24\,a\,T\,\delta^3\,\hbar^3 - 48\,a\,T^2\,\delta^3\,\hbar^3 +\\
&\qquad 24\,a\,T^3\,\delta^3\,\hbar^3 - 4\,\gamma\,\delta^3\,\hbar^3 + 20\,T\,\gamma\,\delta^3\,\hbar^3 - 28\,T^2\,\gamma\,\delta^3\,\hbar^3 + 12\,T^3\,\gamma\,\delta^3\,\hbar^3 + 8\,a\,T\,x\,y\,\delta^4\,\hbar^3 -\\
&\qquad 16\,a\,T^2\,x\,y\,\delta^4\,\hbar^3 + 8\,a\,T^3\,x\,y\,\delta^4\,\hbar^3 - 8\,T\,x\,y\,\gamma\,\delta^4\,\hbar^3 + 16\,T^2\,x\,y\,\gamma\,\delta^4\,\hbar^3 - 8\,T^3\,x\,y\,\gamma\,\delta^4\,\hbar^3 +\\
&\qquad 8\,a\,T\,y\,\delta^3\,\eta\,\hbar^3 - 16\,a\,T^2\,y\,\delta^3\,\eta\,\hbar^3 + 8\,a\,T^3\,y\,\delta^3\,\eta\,\hbar^3 + 8\,a\,T\,x\,\delta^3\,\xi\,\hbar^3 - 16\,a\,T^2\,x\,\delta^3\,\xi\,\hbar^3 +\\
&\qquad 8\,a\,T^3\,x\,\delta^3\,\xi\,\hbar^3 + 8\,a\,T\,\delta^2\,\eta\,\xi\,\hbar^3 - 16\,a\,T^2\,\delta^2\,\eta\,\xi\,\hbar^3 + 8\,a\,T^3\,\delta^2\,\eta\,\xi\,\hbar^3 - 4\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 +\\
&\qquad 20\,T\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 - 28\,T^2\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 + 12\,T^3\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 - 24\,a\,T\,\delta^2\,\hbar^4 + 24\,a\,T^2\,\delta^2\,\hbar^4 +\\
&\qquad 2\,\gamma\,\delta^2\,\hbar^4 - 8\,T\,\gamma\,\delta^2\,\hbar^4 + 6\,T^2\,\gamma\,\delta^2\,\hbar^4 - 16\,a\,T\,x\,y\,\delta^3\,\hbar^4 + 16\,a\,T^2\,x\,y\,\delta^3\,\hbar^4 + 24\,T\,x\,y\,\gamma\,\delta^3\,\hbar^4 -\\
&\qquad 24\,T^2\,x\,y\,\gamma\,\delta^3\,\hbar^4 + x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 + 4\,T\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 - 5\,T^2\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 - 16\,a\,T\,y\,\delta^2\,\eta\,\hbar^4 +\\
&\qquad 16\,a\,T^2\,y\,\delta^2\,\eta\,\hbar^4 - 4\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 + 16\,T\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 - 12\,T^2\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 + 8\,T\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 -\\
&\qquad 8\,T^2\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 - y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 + 4\,T\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 - 3\,T^2\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 - 16\,a\,T\,x\,\delta^2\,\xi\,\hbar^4 +\\
&\qquad 16\,a\,T^2\,x\,\delta^2\,\xi\,\hbar^4 - 4\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 + 16\,T\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 - 12\,T^2\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 + 8\,T\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 -\\
&\qquad 8\,T^2\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 - 16\,a\,T\,\delta\,\eta\,\xi\,\hbar^4 + 16\,a\,T^2\,\delta\,\eta\,\xi\,\hbar^4 + 4\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 - 16\,T\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 +\\
&\qquad 12\,T^2\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 + 8\,T\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - 8\,T^2\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + 4\,T\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 -\\
&\qquad 3\,T^2\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + \gamma\,\eta^2\,\xi^2\,\hbar^4 - 4\,T\,\gamma\,\eta^2\,\xi^2\,\hbar^4 + 3\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar^4 + 8\,a\,T\,\delta\,\hbar^5 + 8\,a\,T\,x\,y\,\delta^2\,\hbar^5 -\\
&\qquad 4\,x\,y\,\gamma\,\delta^2\,\hbar^5 - 12\,T\,x\,y\,\gamma\,\delta^2\,\hbar^5 - 4\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^5 - 4\,T\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^5 + 8\,a\,T\,y\,\delta\,\eta\,\hbar^5 + 4\,y\,\gamma\,\delta\,\eta\,\hbar^5 -\\
&\qquad 12\,T\,y\,\gamma\,\delta\,\eta\,\hbar^5 - 2\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^5 - 10\,T\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^5 + 2\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^5 - 6\,T\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^5 +\\
&\qquad 8\,a\,T\,x\,\delta\,\xi\,\hbar^5 + 4\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 12\,T\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 2\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^5 - 10\,T\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^5 + 8\,a\,T\,\eta\,\xi\,\hbar^5 -\\
&\qquad 16\,T\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^5 + 2\,y\,\gamma\,\eta^2\,\xi\,\hbar^5 - 6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^5 + 2\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^5 - 6\,T\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^5 + 2\,x\,\gamma\,\eta\,\xi^2\,\hbar^5 -\\
&\qquad 6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^5 + 4\,x\,y\,\gamma\,\delta\,\hbar^6 + 4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6\Big)\Big],\\
&\Big(1 + \delta - T\,\delta + \delta^2 - 2\,T\,\delta^2 + T^2\,\delta^2 + \frac{1}{2}\,\gamma\,\delta^2\,\epsilon\,\hbar - 2\,T\,\gamma\,\delta^2\,\epsilon\,\hbar + \frac{3}{2}\,T^2\,\gamma\,\delta^2\,\epsilon\,\hbar + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\Big)\\
&\quad QU[\,] +\\
&\big(2\,T\,\delta\,\epsilon\,\hbar + 4\,T\,\delta^2\,\epsilon\,\hbar - 4\,T^2\,\delta^2\,\epsilon\,\hbar + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\big)\\
&\quad QU[a] +\\
&\big(\xi\,\hbar + 2\,\delta\,\xi\,\hbar - 2\,T\,\delta\,\xi\,\hbar + \gamma\,\delta\,\epsilon\,\xi\,\hbar^2 - 3\,T\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2\big)\\
&\quad QU[x] +\\
&\big(\eta\,\hbar + 2\,\delta\,\eta\,\hbar - 2\,T\,\delta\,\eta\,\hbar + \gamma\,\delta\,\epsilon\,\eta\,\hbar^2 - 3\,T\,\gamma\,\delta\,\epsilon\,\eta\,\hbar^2\big)\\
&\quad QU[y] +\\
&4\,T\,\delta\,\epsilon\,\xi\,\hbar^2\,QU[a, x] + \frac{1}{2}\,\xi^2\,\hbar^2\,QU[x, x] +\\
&4\,T\,\delta\,\epsilon\\
&\quad \eta\,\hbar^2\,QU[y, a] +\\
&\big(\delta\,\hbar + 2\,\delta^2\,\hbar - 2\,T\,\delta^2\,\hbar + \gamma\,\delta\,\epsilon\,\hbar^2 + 4\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - 8\,T\,\gamma\,\delta^2\,\epsilon\,\hbar^2 + \eta\,\xi\,\hbar^2\big)\\
&\quad QU[y, x] +\\
&\frac{1}{2}\,\eta^2\,\hbar^2\,QU[y, y] + 4\,T\,\delta^2\,\epsilon\,\hbar^2\,QU[y, a, x] +\\
&\delta\,\xi\,\hbar^2\,QU[y, x, x] +\\
&\delta\,\eta\,\hbar^2\,QU[y, y, x] +\\
&\frac{1}{2}\,\delta^2\,\hbar^2\,QU[y, y, x, x],\ \text{True}\Big\}
\end{aligned}
$$

```
{tt = Last[Λ_CU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\Bigg\{\frac{1}{1+t\,\delta}+\frac{1}{2\,(1+t\,\delta)^5}\,\epsilon\,\big(4\,a\,\delta+12\,a\,t\,\delta^2+4\,a\,x\,y\,\delta^2+2\,t\,\gamma\,\delta^2-8\,x\,y\,\gamma\,\delta^2+12\,a\,t^2\,\delta^3+8\,a\,t\,x\,y\,\delta^3+4\,t^2\,\gamma\,\delta^3-12\,t\,x\,y\,\gamma\,\delta^3-4\,x^2\,y^2\,\gamma\,\delta^3+4\,a\,t^3\,\delta^4+4\,a\,t^2\,x\,y\,\delta^4+2\,t^3\,\gamma\,\delta^4-4\,t^2\,x\,y\,\gamma\,\delta^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4+4\,a\,y\,\delta\,\eta-4\,y\,\gamma\,\delta\,\eta+8\,a\,t\,y\,\delta^2\,\eta-4\,t\,y\,\gamma\,\delta^2\,\eta-6\,x\,y^2\,\gamma\,\delta^2\,\eta+4\,a\,t^2\,y\,\delta^3\,\eta-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta-2\,y^2\,\gamma\,\delta\,\eta^2-t\,y^2\,\gamma\,\delta^2\,\eta^2+4\,a\,x\,\delta\,\xi-4\,x\,\gamma\,\delta\,\xi+8\,a\,t\,x\,\delta^2\,\xi-4\,t\,x\,\gamma\,\delta^2\,\xi-6\,x^2\,y\,\gamma\,\delta^2\,\xi+4\,a\,t^2\,x\,\delta^3\,\xi-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi+4\,a\,\eta\,\xi+8\,a\,t\,\delta\,\eta\,\xi+4\,t\,\gamma\,\delta\,\eta\,\xi-8\,x\,y\,\gamma\,\delta\,\eta\,\xi+4\,a\,t^2\,\delta^2\,\eta\,\xi+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x^2\,\gamma\,\delta\,\xi^2-t\,x^2\,\gamma\,\delta^2\,\xi^2-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\big),$$

$$\frac{1}{2\,(1+t\,\delta)^4}\,\epsilon\,\big(4\,a\,\delta+12\,a\,t\,\delta^2+4\,a\,x\,y\,\delta^2+2\,t\,\gamma\,\delta^2-8\,x\,y\,\gamma\,\delta^2+12\,a\,t^2\,\delta^3+8\,a\,t\,x\,y\,\delta^3+4\,t^2\,\gamma\,\delta^3-12\,t\,x\,y\,\gamma\,\delta^3-4\,x^2\,y^2\,\gamma\,\delta^3+4\,a\,t^3\,\delta^4+4\,a\,t^2\,x\,y\,\delta^4+2\,t^3\,\gamma\,\delta^4-4\,t^2\,x\,y\,\gamma\,\delta^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4+4\,a\,y\,\delta\,\eta-4\,y\,\gamma\,\delta\,\eta+8\,a\,t\,y\,\delta^2\,\eta-4\,t\,y\,\gamma\,\delta^2\,\eta-6\,x\,y^2\,\gamma\,\delta^2\,\eta+4\,a\,t^2\,y\,\delta^3\,\eta-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta-2\,y^2\,\gamma\,\delta\,\eta^2-t\,y^2\,\gamma\,\delta^2\,\eta^2+4\,a\,x\,\delta\,\xi-4\,x\,\gamma\,\delta\,\xi+8\,a\,t\,x\,\delta^2\,\xi-4\,t\,x\,\gamma\,\delta^2\,\xi-6\,x^2\,y\,\gamma\,\delta^2\,\xi+4\,a\,t^2\,x\,\delta^3\,\xi-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi+4\,a\,\eta\,\xi+8\,a\,t\,\delta\,\eta\,\xi+4\,t\,\gamma\,\delta\,\eta\,\xi-8\,x\,y\,\gamma\,\delta\,\eta\,\xi+4\,a\,t^2\,\delta^2\,\eta\,\xi+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x^2\,\gamma\,\delta\,\xi^2-t\,x^2\,\gamma\,\delta^2\,\xi^2-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\big)+\mathrm{Log}\Big[\frac{1}{1+t\,\delta}\Big]\Bigg\}$$

```
{tt = Last[Λ_QU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$
\begin{aligned}
&\Big\{\frac{\hbar}{-\delta+T\delta+\hbar}+\\
&\frac{1}{4\,(-\delta+T\delta+\hbar)^5}\,\epsilon\,\Big(-8aT\delta^4\hbar^2+24aT^2\delta^4\hbar^2-24aT^3\delta^4\hbar^2+8aT^4\delta^4\hbar^2+2\gamma\delta^4\hbar^2-12T\gamma\delta^4\hbar^2+\\
&\quad 24T^2\gamma\delta^4\hbar^2-20T^3\gamma\delta^4\hbar^2+6T^4\gamma\delta^4\hbar^2+24aT\delta^3\hbar^3-48aT^2\delta^3\hbar^3+24aT^3\delta^3\hbar^3-\\
&\quad 4\gamma\delta^3\hbar^3+20T\gamma\delta^3\hbar^3-28T^2\gamma\delta^3\hbar^3+12T^3\gamma\delta^3\hbar^3+8aTxy\delta^4\hbar^3-16aT^2xy\delta^4\hbar^3+\\
&\quad 8aT^3xy\delta^4\hbar^3-8Txy\gamma\delta^4\hbar^3+16T^2xy\gamma\delta^4\hbar^3-8T^3xy\gamma\delta^4\hbar^3+8aTy\delta^3\eta\hbar^3-\\
&\quad 16aT^2y\delta^3\eta\hbar^3+8aT^3y\delta^3\eta\hbar^3+8aTx\delta^3\xi\hbar^3-16aT^2x\delta^3\xi\hbar^3+8aT^3x\delta^3\xi\hbar^3+\\
&\quad 8aT\delta^2\eta\xi\hbar^3-16aT^2\delta^2\eta\xi\hbar^3+8aT^3\delta^2\eta\xi\hbar^3-4\gamma\delta^2\eta\xi\hbar^3+20T\gamma\delta^2\eta\xi\hbar^3-\\
&\quad 28T^2\gamma\delta^2\eta\xi\hbar^3+12T^3\gamma\delta^2\eta\xi\hbar^3-24aT\delta^2\hbar^4+24aT^2\delta^2\hbar^4+2\gamma\delta^2\hbar^4-8T\gamma\delta^2\hbar^4+\\
&\quad 6T^2\gamma\delta^2\hbar^4-16aTxy\delta^3\hbar^4+16aT^2xy\delta^3\hbar^4+24Txy\gamma\delta^3\hbar^4-24T^2xy\gamma\delta^3\hbar^4+\\
&\quad x^2y^2\gamma\delta^4\hbar^4+4Tx^2y^2\gamma\delta^4\hbar^4-5T^2x^2y^2\gamma\delta^4\hbar^4-16aTy\delta^2\eta\hbar^4+16aT^2y\delta^2\eta\hbar^4-\\
&\quad 4y\gamma\delta^2\eta\hbar^4+16Ty\gamma\delta^2\eta\hbar^4-12T^2y\gamma\delta^2\eta\hbar^4+8Txy^2\gamma\delta^3\eta\hbar^4-8T^2xy^2\gamma\delta^3\eta\hbar^4-\\
&\quad y^2\gamma\delta^2\eta^2\hbar^4+4Ty^2\gamma\delta^2\eta^2\hbar^4-3T^2y^2\gamma\delta^2\eta^2\hbar^4-16aTx\delta^2\xi\hbar^4+16aT^2x\delta^2\xi\hbar^4-\\
&\quad 4x\gamma\delta^2\xi\hbar^4+16Tx\gamma\delta^2\xi\hbar^4-12T^2x\gamma\delta^2\xi\hbar^4+8Tx^2y\gamma\delta^3\xi\hbar^4-8T^2x^2y\gamma\delta^3\xi\hbar^4-\\
&\quad 16aT\delta\eta\xi\hbar^4+16aT^2\delta\eta\xi\hbar^4+4\gamma\delta\eta\xi\hbar^4-16T\gamma\delta\eta\xi\hbar^4+12T^2\gamma\delta\eta\xi\hbar^4+\\
&\quad 8Txy\gamma\delta^2\eta\xi\hbar^4-8T^2xy\gamma\delta^2\eta\xi\hbar^4-x^2\gamma\delta^2\xi^2\hbar^4+4Tx^2\gamma\delta^2\xi^2\hbar^4-3T^2x^2\gamma\delta^2\xi^2\hbar^4+\\
&\quad \gamma\eta^2\xi^2\hbar^4-4T\gamma\eta^2\xi^2\hbar^4+3T^2\gamma\eta^2\xi^2\hbar^4+8aT\delta\hbar^5+8aTxy\delta^2\hbar^5-4xy\gamma\delta^2\hbar^5-\\
&\quad 12Txy\gamma\delta^2\hbar^5-4x^2y^2\gamma\delta^3\hbar^5-4Tx^2y^2\gamma\delta^3\hbar^5+8aTy\delta\eta\hbar^5+4y\gamma\delta\eta\hbar^5-\\
&\quad 12Ty\gamma\delta\eta\hbar^5-2xy^2\gamma\delta^2\eta\hbar^5-10Txy^2\gamma\delta^2\eta\hbar^5+2y^2\gamma\delta\eta^2\hbar^5-6Ty^2\gamma\delta\eta^2\hbar^5+\\
&\quad 8aTx\delta\xi\hbar^5+4x\gamma\delta\xi\hbar^5-12Tx\gamma\delta\xi\hbar^5-2x^2y\gamma\delta^2\xi\hbar^5-10Tx^2y\gamma\delta^2\xi\hbar^5+8aT\eta\xi\hbar^5-\\
&\quad 16Txy\gamma\delta\eta\xi\hbar^5+2y\gamma\eta^2\xi\hbar^5-6Ty\gamma\eta^2\xi\hbar^5+2x^2\gamma\delta\xi^2\hbar^5-6Tx^2\gamma\delta\xi^2\hbar^5+2x\gamma\eta\xi^2\hbar^5-\\
&\quad 6Tx\gamma\eta\xi^2\hbar^5+4xy\gamma\delta\hbar^6+4x^2y^2\gamma\delta^2\hbar^6+4xy^2\gamma\delta\eta\hbar^6+4x^2y\gamma\delta\xi\hbar^6+4xy\gamma\eta\xi\hbar^6\Big),\\
&\frac{1}{4\,(-\delta+T\delta+\hbar)^4}\,\epsilon\,\Big(-8aT\delta^4\hbar+24aT^2\delta^4\hbar-24aT^3\delta^4\hbar+8aT^4\delta^4\hbar+2\gamma\delta^4\hbar-12T\gamma\delta^4\hbar+\\
&\quad 24T^2\gamma\delta^4\hbar-20T^3\gamma\delta^4\hbar+6T^4\gamma\delta^4\hbar+24aT\delta^3\hbar^2-48aT^2\delta^3\hbar^2+24aT^3\delta^3\hbar^2-4\gamma\delta^3\hbar^2+\\
&\quad 20T\gamma\delta^3\hbar^2-28T^2\gamma\delta^3\hbar^2+12T^3\gamma\delta^3\hbar^2+8aTxy\delta^4\hbar^2-16aT^2xy\delta^4\hbar^2+8aT^3xy\delta^4\hbar^2-\\
&\quad 8Txy\gamma\delta^4\hbar^2+16T^2xy\gamma\delta^4\hbar^2-8T^3xy\gamma\delta^4\hbar^2+8aTy\delta^3\eta\hbar^2-16aT^2y\delta^3\eta\hbar^2+\\
&\quad 8aT^3y\delta^3\eta\hbar^2+8aTx\delta^3\xi\hbar^2-16aT^2x\delta^3\xi\hbar^2+8aT^3x\delta^3\xi\hbar^2+8aT\delta^2\eta\xi\hbar^2-\\
&\quad 16aT^2\delta^2\eta\xi\hbar^2+8aT^3\delta^2\eta\xi\hbar^2-4\gamma\delta^2\eta\xi\hbar^2+20T\gamma\delta^2\eta\xi\hbar^2-28T^2\gamma\delta^2\eta\xi\hbar^2+\\
&\quad 12T^3\gamma\delta^2\eta\xi\hbar^2-24aT\delta^2\hbar^3+24aT^2\delta^2\hbar^3+2\gamma\delta^2\hbar^3-8T\gamma\delta^2\hbar^3+6T^2\gamma\delta^2\hbar^3-\\
&\quad 16aTxy\delta^3\hbar^3+16aT^2xy\delta^3\hbar^3+24Txy\gamma\delta^3\hbar^3-24T^2xy\gamma\delta^3\hbar^3+x^2y^2\gamma\delta^4\hbar^3+\\
&\quad 4Tx^2y^2\gamma\delta^4\hbar^3-5T^2x^2y^2\gamma\delta^4\hbar^3-16aTy\delta^2\eta\hbar^3+16aT^2y\delta^2\eta\hbar^3-4y\gamma\delta^2\eta\hbar^3+\\
&\quad 16Ty\gamma\delta^2\eta\hbar^3-12T^2y\gamma\delta^2\eta\hbar^3+8Txy^2\gamma\delta^3\eta\hbar^3-8T^2xy^2\gamma\delta^3\eta\hbar^3-y^2\gamma\delta^2\eta^2\hbar^3+\\
&\quad 4Ty^2\gamma\delta^2\eta^2\hbar^3-3T^2y^2\gamma\delta^2\eta^2\hbar^3-16aTx\delta^2\xi\hbar^3+16aT^2x\delta^2\xi\hbar^3-4x\gamma\delta^2\xi\hbar^3+\\
&\quad 16Tx\gamma\delta^2\xi\hbar^3-12T^2x\gamma\delta^2\xi\hbar^3+8Tx^2y\gamma\delta^3\xi\hbar^3-8T^2x^2y\gamma\delta^3\xi\hbar^3-16aT\delta\eta\xi\hbar^3+\\
&\quad 16aT^2\delta\eta\xi\hbar^3+4\gamma\delta\eta\xi\hbar^3-16T\gamma\delta\eta\xi\hbar^3+12T^2\gamma\delta\eta\xi\hbar^3+8Txy\gamma\delta^2\eta\xi\hbar^3-\\
&\quad 8T^2xy\gamma\delta^2\eta\xi\hbar^3-x^2\gamma\delta^2\xi^2\hbar^3+4Tx^2\gamma\delta^2\xi^2\hbar^3-3T^2x^2\gamma\delta^2\xi^2\hbar^3+\gamma\eta^2\xi^2\hbar^3-\\
&\quad 4T\gamma\eta^2\xi^2\hbar^3+3T^2\gamma\eta^2\xi^2\hbar^3+8aT\delta\hbar^4+8aTxy\delta^2\hbar^4-4xy\gamma\delta^2\hbar^4-12Txy\gamma\delta^2\hbar^4-\\
&\quad 4x^2y^2\gamma\delta^3\hbar^4-4Tx^2y^2\gamma\delta^3\hbar^4+8aTy\delta\eta\hbar^4+4y\gamma\delta\eta\hbar^4-12Ty\gamma\delta\eta\hbar^4-2xy^2\gamma\delta^2\eta\hbar^4-\\
&\quad 10Txy^2\gamma\delta^2\eta\hbar^4+2y^2\gamma\delta\eta^2\hbar^4-6Ty^2\gamma\delta\eta^2\hbar^4+8aTx\delta\xi\hbar^4+4x\gamma\delta\xi\hbar^4-12Tx\gamma\delta\xi\hbar^4-\\
&\quad 2x^2y\gamma\delta^2\xi\hbar^4-10Tx^2y\gamma\delta^2\xi\hbar^4+8aT\eta\xi\hbar^4-16Txy\gamma\delta\eta\xi\hbar^4+2y\gamma\eta^2\xi\hbar^4-\\
&\quad 6Ty\gamma\eta^2\xi\hbar^4+2x^2\gamma\delta\xi^2\hbar^4-6Tx^2\gamma\delta\xi^2\hbar^4+2x\gamma\eta\xi^2\hbar^4-6Tx\gamma\eta\xi^2\hbar^4+4xy\gamma\delta\hbar^5+\\
&\quad 4x^2y^2\gamma\delta^2\hbar^5+4xy^2\gamma\delta\eta\hbar^5+4x^2y\gamma\delta\xi\hbar^5+4xy\gamma\eta\xi\hbar^5\Big)+\mathrm{Log}\Big[\frac{\hbar}{-\delta+T\delta+\hbar}\Big]\Big\}
\end{aligned}
$$

Reorderings with Rord

Rord

```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
 Simp@Module[{υ, ω, δ, Λ1, yax, q, p, δ1 = ∂_{u_i,w_j} Q},
   {yax, q, p} = List @@ If[δ1 === 0, Λ_U[{υ, ω}, {u, w}], Λ_U[{υ, ω, δ}, {u, w}]] /.
     {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s};
   𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), 𝕖^-q DP_{u_i→D_υ, w_j→D_ω}[P][p 𝕖^q]] /.
    {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}]
```

```
With[{c𝔼 = 𝔼_CU[{y_1, x_1}_1, {x_2, a_2, y_2}_2, ℏ t_1 a_2 + ℏ t_1^-1 (𝕖^t_1 - 1) y_1 x_2, 1 + ϵ x_1 y_2]},
 {Short[rhs = c𝔼 // Rord_{x_2,a_2→3}, 3], HL[CU[c𝔼] == CU[rhs]]}]
```

$$\left\{\mathbb{E}_{CU}\left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1}\hbar x_3 y_1\right)}{t_1}, 1 + \epsilon x_1 y_2\right], \text{True}\right\}$$

```
With[{c𝔼 = 𝔼_CU[{y_1, a_1, x_1}_1, {x_2, a_2, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = c𝔼 // Rord_{x_2,a_2→3}, 3], HL[CU[c𝔼] == CU[rhs]]}]
```

$$\{\mathbb{E}_{CU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg,$$
$$1 + e^{-\gamma\hbar(l_{12} t_1 + l_{22} t_2)} \epsilon \left(e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 + p_{21} x_3 y_1 + e^{\ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)], \text{True}\}$$

```
With[{q𝔼 = 𝔼_QU[{y_1, a_1, x_1}_1, {x_2, a_2, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = q𝔼 // Rord_{x_2,a_2→3}, 3], HL[QU[q𝔼] == QU[rhs]]}]
```

$$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg,$$
$$1 + e^{-\gamma\hbar(l_{12} t_1 + l_{22} t_2)} \epsilon \left(e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 + p_{21} x_3 y_1 + e^{\ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)], \text{True}\}$$

```
With[{q𝔼 = 𝔼_QU[{y_1, a_1, x_1}_1, {x_2, a_2, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = q𝔼 // Rord_{a_2,y_2→3}, 3], HL[QU[q𝔼] == QU[rhs]]}]
```

$$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg,$$
$$1 + e^{-\gamma\hbar(l_{12} t_1 + l_{22} t_2)} \epsilon \left(e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 + e^{\ll 1 \gg} p_{21} x_2 y_1 + p_{12} x_1 y_3 + p_{22} x_2 y_3 - \gamma\hbar l_2 x_1 y_3 \gamma_{12} - \gamma\hbar l_2 x_2 y_3 \gamma_{22}\right)], \text{True}\}$$

```
With[{q𝕆 = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
    ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q𝕆 // Rord_{x1,y1→3}, 5], HL@SimpT[QU[q𝕆] == QU[rhs]]}]
```

$$\left\{\mathbb{E}_{QU}\left[\{y_3, a_3, x_3\}_1, \{x_2, a_2, y_2\}_2, \frac{\hbar\, a_2\, l_{12}\, t_1 + \ll 17 \gg}{1 - \gamma_{11} + T_1 \gamma_{11}}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + \frac{\epsilon \left(4 \hbar\, a_2\, l_2 + \ll 346 \gg + 3 \gamma \hbar^4 T_1^2 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2\right)}{4 \hbar \left(1 - \gamma_{11} + T_1 \gamma_{11}\right)^5} + \frac{1}{4 \left(1 - \gamma_{11} + T_1 \gamma_{11}\right)^7} \epsilon^2 \left(8 a_3 p_{11} T_1 + 4 \gamma \hbar\, p_{11} x_3 y_3 + 4 \gamma\, p_{11} \gamma_{11} + \ll 2722 \gg + 12 \gamma \hbar^3 p_{11} T_1^2 x_2^3 y_2^3 \gamma_{12}^3 \gamma_{21}^3 - 10 \gamma \hbar^3 p_{11} T_1^3 x_2^3 y_2^3 \gamma_{12}^3 \gamma_{21}^3 + 3 \gamma \hbar^3 p_{11} T_1^4 x_2^3 y_2^3 \gamma_{12}^3 \gamma_{21}^3\right)\right], \text{True}\right\}$$

R in QU.

Faddeev-Quesne's formula:

Faddeev

```
𝕖_{q_,k_}[x_] := 𝕖^(Sum_{j=1}^{k} ((1 - q)^j x^j)/(j (1 - q^j))); 𝕖_{q_}[x_] := 𝕖_{q,$k}[x]
```

```
Table[Together@SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)\left(1+q+q^2\right)}, \frac{1}{(1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)}, 1 \Big/ \left((1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)\left(1+q+q^2+q^3+q^4\right)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

$\{1, 1, 1, 1, 1, 1\}$

R

```
QU[R_{i_,j_}] := 𝕆QU[{y1, a1}i, {a2, x2}j, SS[𝕖^(ℏ b1 a2) 𝕖_{qℏ}[ℏ y1 x2] /. b1 → γ^-1 (ϵ a1 - t_i)]];
QU[R^-1_{i_,j_}] := S_j@QU[R_{i,j}];
```

```
QU[R3,4] // Short
```

$$QU[\,] + \frac{\epsilon \hbar\, QU[a_3, a_4]}{\gamma} + \hbar\, QU[y_3, x_4] + \frac{\ll 1 \gg}{\gamma} + \ll 1 \gg - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon \ll 3 \gg}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4]\, t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R1,2 ** R^-1_{1,2}] // Simp // HL // Timing
```

$\{0.046875, QU[\,]\}$

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$\{0.140625, \{QU[\,] + \frac{\epsilon\hbar\, QU[a_1, a_2]}{\gamma} + \ll 74 \gg + 2\epsilon\hbar^2 QU[y_1, a_2, x_3]\, T_2 + QU[y_1, x_3]\,(\hbar - \hbar T_2), 0\}\}$

R in Œ$_{QU}$.

RinOE

```
ŒQU[Ri_,j_] := ŒQU[{yi, ai, xi}i, {yj, aj, xj}j, -ħ γ^-1 ti aj + ħ yi xj,
   Normal@Series[e^(ħ γ^-1 ti aj - ħ yi xj) (e^(ħ bi aj) e_qħ[ħ yi xj] /. bi → γ^-1 (ϵ ai - ti)), {ϵ, 0, $k}]]
```

```
ŒQU[R1,2]
```

$\text{Œ}_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\, a_2\, t_1}{\gamma} + \hbar\, x_2\, y_1, 1 + \frac{\epsilon\hbar\, a_1\, a_2}{\gamma}]$

𝔼

𝔼[*L*, *Q*, *P*] means $e^{\hbar(L+Q)} P$, where *L* is linear in the *a*'s, *Q* is a combination of $x_i\, y_j$, and *P* is a perturbation polynomial. It should be interpreted via C𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...] (with some default for direct interpretation), or likewise via Q𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...]. In themselves, C𝕆 and Q𝕆 should have an interpretation in CU/QU by casting.

Alternative Algorithms

```
λalt[CU] := If[$k == 0, 1, Module[{eq, d, b, c, so},
     eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
     {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
     Normal@Series[e^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, $k}]]];
```

```
{λalt[CU], HL@Simplify[λalt[CU] == Last[ΛCU[{ξ, η}, {x, y}]]]}
```

$\{1 + \epsilon\left(2\, a\, \eta\, \xi - y\, \gamma\, \eta^2\, \xi - x\, \gamma\, \eta\, \xi^2 + \frac{1}{2}\, t\, \gamma\, \eta^2\, \xi^2\right), \text{True}\}$