

Pensieve header: A unified verification notebook for the \$sl_2\$-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

Go;

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ $\mathcal{E}$ _] := Style[ $\mathcal{E}$ , Background  $\rightarrow$  Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is \hbar , and all “coupling constants” are proportional to it.

TD

```
$p = 2; $k = 1;
If[$k == 0,  $\epsilon = 0$ ,  $\epsilon /:$   $\epsilon^{k-}$  /;  $k > $k := 0$ ];
SetAttributes[{SS, SST}, HoldAll];
TRule = { $T_i \rightarrow e^{\hbar t_i}$ ,  $T \rightarrow e^{\hbar t}$ }; qrule =  $q \rightarrow e^{\gamma e \hbar}$ ;
SS[ $\mathcal{E}$ _] := Block[{ $\hbar$ ,  $\epsilon$ }, (* Shielded Series *)
  Collect[Normal@Series[ $\mathcal{E}$ , { $\hbar$ , 0, $p}],  $\hbar$ , Together] ];
SST[ $\mathcal{E}$ _] :=
  Block[{ $\hbar$ ,  $\epsilon$ }, Collect[Normal@Series[ $\mathcal{E}$  /. TRule, { $\hbar$ , 0, $p}],  $\hbar$ , Together] ];
Simp[ $\mathcal{E}$ _, op_] := Collect[ $\mathcal{E}$ , _CU | _QU, op];
Simp[ $\mathcal{E}$ _] := Simp[ $\mathcal{E}$ , Collect[Normal@Series[#, { $\hbar$ , 0, $p}],  $\hbar$ , Expand] &];
SimpT[ $\mathcal{E}$ _] :=
  Collect[ $\mathcal{E}$ , _CU | _QU, Collect[Normal@Series[#, TRule, { $\hbar$ , 0, $p}],  $\hbar$ , Expand] &];
```

Differential polynomials (DP):

Utils

```
DP $_{\alpha \rightarrow D_x, \beta \rightarrow D_y}$ [P_] [ $\lambda$ _] :=
  Total[CoefficientRules[P, { $\alpha$ ,  $\beta$ }] /. ({ $m$ _,  $n$ _}  $\rightarrow$   $c$ _)  $\Rightarrow$   $c$  D[ $\lambda$ , { $x$ ,  $m$ }, { $y$ ,  $n$ }]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[ $x$ _] :=  $x$ ;
NCM[ $x$ _,  $y$ _,  $z$ _] := ( $x$  **  $y$ ) **  $z$ ;
0 ** _ = _ ** 0 = 0;
( $x$ _ Plus) **  $y$ _ := (# **  $y$ ) & /@  $x$ ;  $x$ _ ** ( $y$ _ Plus) := ( $x$  ** #) & /@  $y$ ;
B[ $x$ _,  $x$ _] = 0; B[ $x$ _,  $y$ _] :=  $x$  **  $y$  -  $y$  **  $x$ ;
```

QLImplementation

```

DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
  gs = Generators /. {opts}, cs = Centrals /. {opts}},
  (#U = U@#) & /@gs;
  gp = Alternatives @@ gs; gp = gp | gp; (* gens *)
  sr = Flatten@Table[{g → ++k, gi_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[ε_] := Collect[ε, _U, Expand] /.  $\hbar^d$  /; d > $p → 0;
  Ui_ [ε_] := ε /. {t : cp ⇒ ti, u_U ⇒ Replace[u, x_ ⇒ xi, 1]};
  Ui_ [NCM[]] = pow[ε_, 0] = U@{} = 1U = U[];
  B[U@(x_)i, U@(y_)i] := B[U@xi, U@yi] = Ui@B[U@x, U@y];
  B[U@(x_)i, U@(y_)j] /; i != j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1U) := CE[c x]; (c_. 1U) ** x_ := CE[c x];
  (* (a_. ** x_U) ** (b_. ** y_U) := CE[M[a b, x ** y]]; *)
  (a_. U[xx___, x_] ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b, U[xx, x, y, yy]],
    U@xx ** CE@M[a b, U@y ** U@x + B[U@x, U@y]] ** U@yy];
  U@{c_. * (L : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[L, {n}] ** U@{r}];
  U@{c_. * L : gp, r___} := CE[c U[L] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
  U@{L_, r___} := U@{Expand[L], r};
  U[ε_NonCommutativeMultiply] := U /@ ε;
  0U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, L_List ⇒ Lnull, {1}];
    vs = Join@@(First /@ sp);
    us = Join@@(sp /. L_s_ ⇒ (L /. x_i_ ⇒ xs));
    CE[Total[
      CoefficientRules[poly, vs] /. (p_ → c_) ⇒ c U@(us^p)
    ]] /. x_nnull ⇒ x];
  pow[ε_, n_] := pow[ε, n - 1] ** ε;
  S_U[ε_, ss__Rule] := CE@Total[
    CoefficientRules[ε, First /@ {ss}] /.
      (p_ → c_) ⇒ c NCM@@MapThread[pow, {Last /@ {ss}, p}]];
  m_j→k_ [c_. * u_U] := CE[ ((c /. (t : cp)_j → tk) DeleteCases[u, _j|k]) **
    U@@Cases[u, w_j ⇒ wk] ** U@@Cases[u, _k] ];
  Si_ [c_. * u_U] := CE[ ((c /. Si[U, Centrals]) DeleteCases[u, _i]) **
    Ui_ [NCM@@Reverse@Cases[u, x_i ⇒ S@U@x]] ] ]

```

DeclareMorphism

QLImplementation

```

DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) :=> (m[U[g]] = img), {1}];
  m[1_U] = 1_V;
  m[U[g_i]] := V_i[m[U@g]];
  m[U[vs_]] := NCM@@(m/@U/@{vs});
  m[ε_] := Simp[ε /. oncs /. u_U :=> m[u]];
)

```

Meta-Operations

QLImplementation

```

m_j→j_ = Identity;
m_j→k_[ε_Plus] := Simp[m_j→k_/@ε];
m_is____,i_,j→k_[ε_] := m_j→k_@m_is,i→j@ε;
S_i_[ε_Plus] := Simp[S_i_/@ε];

```

Implementing $CU = \mathcal{U}(sl_2^{\vee \epsilon})$

CU

```

DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ε a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU);
S_i_[CU, Centrals] = {t_i → -t_i};

```

Verifying associativity on triples of generators:

```

With[{bas = CU/@{y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

```

Verifying associativity on a “random” triple:

```

With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp // HL
}] // Timing
{1.65625,
 {(28 t^2 γ^4 + 116 t γ^5 ε) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
    {z1, bas}, {z2, bas} ] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t1 t3 CU[y1, a1, x2] + t1 t1 CU[y1, a2, x2] // m1->3
CU[a2, x2, y3] t3^2 + CU[x2, y3, a3] t3^2
```

Verifying meta-associativity:

```
Module[{z, u},
  Table[u = CU[z[[1]]1, z[[2]]2, z[[3]]3]; z -> HL[m1,3->3@m2,3->3@u == m2,3->3@m1,2->2@u],
    {z, Tuples[{y, a, x}, 3]}]]
{{y, y, y} -> True, {y, y, a} -> True, {y, y, x} -> True, {y, a, y} -> True,
 {y, a, a} -> True, {y, a, x} -> True, {y, x, y} -> True, {y, x, a} -> True,
 {y, x, x} -> True, {a, y, y} -> True, {a, y, a} -> True, {a, y, x} -> True, {a, a, y} -> True,
 {a, a, a} -> True, {a, a, x} -> True, {a, x, y} -> True, {a, x, a} -> True, {a, x, x} -> True,
 {x, y, y} -> True, {x, y, a} -> True, {x, y, x} -> True, {x, a, y} -> True, {x, a, a} -> True,
 {x, a, x} -> True, {x, x, y} -> True, {x, x, a} -> True, {x, x, x} -> True}
```

Implementing QU = $\mathcal{U}_q(\mathfrak{sl}_2^{\vee \epsilon})$

Aside

```
Series[(1 - T e^{-2 e a h}) / h, {a, 0, 3}]
```

Aside

$$\frac{1 - T}{h} + 2 e T a - 2 (e^2 h T) a^2 + \frac{4}{3} e^3 h^2 T a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators -> {y, a, x}, Centrals -> {t, T}];
B[aQU, yQU] = -\gamma yQU; B[xQU, aQU] = -\gamma QU@x;
B[xQU, yQU] = SS[q - 1 /. qrule /. e -> \epsilon] QU@{y, x} + OQU[{a}, SS[(1 - T e^{-2 \epsilon a h}) / \hbar]];
(S@yQU = OQU[{a, y}, SS[-T^{-1} e^{\hbar \epsilon a} y]]; S@aQU = -aQU; S@xQU = OQU[{a, x}, SS[-e^{\hbar \epsilon a} x]]);
S_i[QU, Centrals] = {t_i -> -t_i, T_i -> T_i^{-1}};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas} ] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
 {QU[y], QU[x]} →  $\frac{(-1 + T) QU[]}{\hbar} - 2 T \in QU[a] - \gamma \in \hbar QU[y, x]$ },
 {{QU[a], QU[y]} → -γ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} →  $\frac{(1 - T) QU[]}{\hbar} + 2 T \in QU[a] + \gamma \in \hbar QU[y, x]$ ,
 {QU[x], QU[a]} → -γ QU[x], {QU[x], QU[x]} → 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
 {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ \$p=5, \$k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
 (rhs = (z1 ** z2) ** z3 // Simp) // Short,
 HL[z1 ** (z2 ** z3) - rhs // Simp]
}] // Timing
{4.73438, { $\left(\frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \in - 280 \ll 3 \gg + 198 T^2 \gamma^5 \in}{\hbar}\right) QU[y, y, y, x, x] +$ 
 $\ll 18 \gg + (1 + 8 \gamma \in \hbar) QU[\ll 1 \gg, 0]}$ }
```

Verifying that S is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
 {z1, bas}, {z2, bas} ] ]
{{{QU[y1], QU[y1]} → 0, {QU[y1], QU[a1]} → 0, {QU[y1], QU[x1]} → 0},
 {{QU[a1], QU[y1]} → 0, {QU[a1], QU[a1]} → 0, {QU[a1], QU[x1]} → 0},
 {{QU[x1], QU[y1]} → 0, {QU[x1], QU[a1]} → 0, {QU[x1], QU[x1]} → 0}}
```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs @ \$p=5, \$k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
 Short[lhs = z1 ** (z2 ** z3)],
 Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
 Expand[Limit[rhs /. TRule[QU → CU], ħ → 0] - lhs] // HL
}] // Timing
{11.6563, {28 t^2 γ^4 CU[y, y, y, x, x] +
 116 t γ^5 ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
 2  $\left(\frac{\gamma^4}{\hbar^2} - \frac{2 T \gamma^4}{\hbar^2} + \frac{T^2 \gamma^4}{\hbar^2} + \frac{\gamma^5 \in}{\hbar} - \frac{2 T \gamma^5 \in}{\hbar} + \frac{T^2 \gamma^5 \in}{\hbar}\right) QU[y, y, y, x, x] +$ 
 $\ll 209 \gg + (1 + 8 \gamma \in \hbar) QU[y, \ll 11 \gg, x], 0}}$ }
```

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -xCU, a → -aCU, x → -yCU}, {t → -t, T → T-1]];
DeclareMorphism[Qθ, QU → QU, {y → 0QU[{a, x}, SS[-T-1/2 eħ ea x]],
  a → -aQU, x → 0QU[{a, y}, SS[-T-1/2 eħ ea y]]}, {t → -t, T → T-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
{QU[y] → - $\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon \hbar QU[a, x]}{\sqrt{T}}$  → QU[y], QU[a] → -QU[a] → QU[a],
  QU[x] →  $\left(-\frac{1}{\sqrt{T}} + \frac{\gamma \epsilon \hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon \hbar QU[y, a]}{\sqrt{T}}$  → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$AD\$\mathbf{f} = \gamma \left(\left(\text{Cosh} \left[\hbar \left(a e + \frac{\gamma e}{2} - \frac{t}{2} \right) \right] - \text{Cosh} \left[\hbar \sqrt{\left(\frac{t - \gamma e}{2} \right)^2 + e \omega} \right] \right) / \right. \\ \left. \left(\hbar e^{\hbar ((a+\gamma) e - t/2)} \text{Sinh} \left[\frac{\gamma e \hbar}{2} \right] (a^2 e + a \gamma e - a t - \omega) \right) \right);$$

Scaling behaviour of AD\$f:

```
HL@Simplify[AD$f == ((AD$f /.  $\gamma \rightarrow 1$ ) /. { $e \rightarrow \gamma e$ ,  $a \rightarrow \gamma^{-1} a$ ,  $\omega \rightarrow \gamma^{-1} \omega$ })]
```

True

```
HL@FullSimplify[
  AD$f == ((AD$f /.  $\gamma \rightarrow 1$ ) /. { $\hbar \rightarrow \gamma^2 \hbar$ ,  $e \rightarrow e / \gamma$ ,  $a \rightarrow a / \gamma$ ,  $t \rightarrow \gamma^{-2} t$ ,  $\omega \rightarrow \gamma^{-3} \omega$ })]
```

True

ADeq

```
AD$ $\omega$  =  $\gamma$  CU[y, x] +  $\epsilon$  CU[a, a] - (t -  $\gamma \epsilon$ ) CU[a];
```

ADeq

```
DeclareMorphism[AD, QU  $\rightarrow$  CU,
  {a  $\rightarrow$  aCU, x  $\rightarrow$  CU@x, y  $\rightarrow$  SCU[SS[AD$f] /. e  $\rightarrow$   $\epsilon$ , a  $\rightarrow$  aCU,  $\omega \rightarrow$  AD$ $\omega$ ] ** yCU}]
```

Verifying that the asymmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2}  $\rightarrow$  HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]}  $\rightarrow$  0, {QU[y], QU[a]}  $\rightarrow$  0, {QU[y], QU[x]}  $\rightarrow$  0},
 {{QU[a], QU[y]}  $\rightarrow$  0, {QU[a], QU[a]}  $\rightarrow$  0, {QU[a], QU[x]}  $\rightarrow$  0},
 {{QU[x], QU[y]}  $\rightarrow$  0, {QU[x], QU[a]}  $\rightarrow$  0, {QU[x], QU[x]}  $\rightarrow$  0}}
```

The Symmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$SD$g = \sqrt{\left(\left(2\gamma \left(\cosh\left[\frac{\hbar}{2} \sqrt{t^2 + \gamma^2 e^2 + 4e\omega}\right] - \cosh\left[\frac{t - e\gamma - 2ea}{2/\hbar}\right] \right) \right) / \left(\sinh\left[\frac{\gamma e \hbar}{2}\right] (t(2a + \gamma) - 2a(a + \gamma)e + 2\omega)\hbar \right) \right)}$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

$$\{SD\$P = \frac{\text{Cosh}[\hbar \left(\frac{e-t}{2} + e a \right)] - \text{Cosh}[\hbar \sqrt{\frac{t^2+e^2}{4} + e \varpi}]}{\hbar \text{Sinh}[\frac{-e\hbar}{2}] (\varpi - e a^2 + (t-e) a + t/2)},$$

`Simplify[SD\$P == (SD\$P /. {a -> -a-1, t -> -t})] // HL,`
`PowerExpand@Simplify[(SD\$P /. {h -> \gamma^2 h, e -> e/\gamma, a -> a/\gamma, t -> \gamma^{-2} t, \varpi -> \gamma^{-3} \varpi}) ==`
`SD\$g (SD\$g /. {a -> -a-\gamma, t -> -t})] // HL,`
`SD\$Q = Simplify[SD\$P /. {a -> c-1/2}],`
`Simplify[SD\$Q == (SD\$Q /. {c -> -c, t -> -t})] // HL,`
`FullSimplify[SD\$g == FullSimplify[`
`\sqrt{SD\$Q} /. c -> a+1/2 /. {h -> \gamma^2 h, e -> e/\gamma, a -> a/\gamma, t -> \gamma^{-2} t, \varpi -> \gamma^{-3} \varpi}]] // HL`
`}`

$$\left\{ - \left(\left(\left(\text{Cosh} \left[\left(a e + \frac{e-t}{2} \right) \hbar \right] - \text{Cosh} \left[\sqrt{\frac{1}{4} (e^2 + t^2) + e \varpi} \hbar \right] \text{Csch} \left[\frac{e \hbar}{2} \right] \right) \right) / \right.$$

$$\left. \left(\left(-a^2 e + \frac{t}{2} + a (-e + t) + \varpi \right) \hbar \right) \right\}, \text{True, True},$$

$$\left(4 \left(-\text{Cosh} \left[\frac{1}{2} \sqrt{e^2 + t^2 + 4 e \varpi} \hbar \right] + \text{Cosh} \left[c e \hbar - \frac{t \hbar}{2} \right] \right) \text{Csch} \left[\frac{e \hbar}{2} \right] \right) / \left((-1 + 4 c^2) e - 4 (c t + \varpi) \hbar \right),$$

True, True

SDeq

```
SD$f = Simplify[ e^{\hbar(t/2 - e a)} (SD$g /. {a -> -a, t -> -t});
```

SDeq

```
SD$w = \gamma CU[y, x] + e CU[a, a] - (t - \gamma e) CU[a] - t \gamma 1_{CU}/2;
```

SDeq

```
DeclareMorphism[SD, QU -> CU, {a -> a_{CU},
  x -> S_{CU}[SS[SD$f] /. e -> e, a -> a_{CU}, \varpi -> SD$w] ** x_{CU},
  y -> S_{CU}[SS[SD$g] /. e -> e, a -> a_{CU}, \varpi -> SD$w] ** y_{CU}}]
```

Verifying the θ -symmetry:

```
Table[HL@SimpT[C\theta[SD[z]] == SD[Q\theta[z]]], {z, QU/@{y, a, x}}]
{True, True, True}
```

Verifying that the symmetric dequantizer is a homomorphism:

```
With[{bas = QU/@{y, a, x}},
  Table[{z1, z2} -> HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}]]
{{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> 0, {QU[y], QU[x]} -> 0},
 {{QU[a], QU[y]} -> 0, {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> 0},
 {{QU[x], QU[y]} -> 0, {QU[x], QU[a]} -> 0, {QU[x], QU[x]} -> 0}}
```


The representation ρ

rho

```

rho@yCU = rho@yQU =  $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$ ; rho@aCU = rho@aQU =  $\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ ;
rho@xCU =  $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ ; rho@xQU = SS@ $\begin{pmatrix} 0 & (1 - e^{-\gamma \epsilon \hbar}) / (\epsilon \hbar) \\ 0 & 0 \end{pmatrix}$ ;
rho[e^-] := MatrixExp[rho[epsilon]];
rho[epsilon_] :=
  (epsilon /. {t -> gamma epsilon, T -> e^{hbar gamma epsilon}} /. (U : CU | QU)[u___] => Fold[Dot,  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , rho /@ U /@ {u}])

```

Verifying that ρ represents CU and QU:

```

Table[rho[z1 ** z2] == rho[z1].rho[z2] // SS // HL,
  {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}

```

\mathbb{C} and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```

CU@C[CUspecs___, Q_, P_] := OCU[specs, SS[e^Q P]];
QU@C[QUspecs___, Q_, P_] := OQU[specs, SS[e^Q P]];

```

```
HL[rho[e^xi CUex].rho[e^alpha CUea] == rho[e^alpha CUea].rho[e^{-gamma xi} CUex]]
```

True

Logos

```

Lambda_U[{xi_, alpha_}, {x, a}] := C_U[{a, x}, alpha a + e^{-gamma xi} xi x, 1];
Lambda_U[{alpha_, eta_}, {a, y}] := C_U[{y, a}, alpha a + e^{-gamma eta} eta y, 1];

```

```

{Lambda_{#}[{xi, alpha}, {x, a}], lhs = #@Lambda_{#}[{x, a}, hbar (xi x + alpha a), 1],
  HL[lhs == #@Lambda_{#}[hbar {xi, alpha}, {x, a}]]} & /@ {CU, QU}
{{C_CU[{a, x}, a alpha + e^{-alpha gamma} x xi, 1],
  CU[] + alpha hbar CU[a] + (xi hbar - alpha gamma xi hbar^2) CU[x] + 1/2 alpha^2 hbar^2 CU[a, a] + alpha xi hbar^2 CU[a, x] + 1/2 xi^2 hbar^2 CU[x, x],
  True}, {C_QU[{a, x}, a alpha + e^{-alpha gamma} x xi, 1], QU[] + alpha hbar QU[a] + (xi hbar - alpha gamma xi hbar^2) QU[x] +
  1/2 alpha^2 hbar^2 QU[a, a] + alpha xi hbar^2 QU[a, x] + 1/2 xi^2 hbar^2 QU[x, x], True}}

```

```
{Lambda[{{alpha, eta}, {a, y}], lhs = #@C#[{a, y}, hbar (eta y + alpha a), 1],
  HL[lhs = #@Lambda[hbar {alpha, eta}, {a, y}]] & /@ {CU, QU}
}{C[CU[{y, a}, a alpha + e^{-alpha y} eta, 1],
  CU[] + alpha hbar CU[a] + (eta hbar - alpha gamma eta hbar^2) CU[y] + 1/2 alpha^2 hbar^2 CU[a, a] + alpha eta hbar^2 CU[y, a] + 1/2 eta^2 hbar^2 CU[y, y],
  True}, {C[QU[{y, a}, a alpha + e^{-alpha y} eta, 1], QU[] + alpha hbar QU[a] + (eta hbar - alpha gamma eta hbar^2) QU[y] +
  1/2 alpha^2 hbar^2 QU[a, a] + alpha eta hbar^2 QU[y, a] + 1/2 eta^2 hbar^2 QU[y, y], True}]}
```

Goal. In either U , compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta=0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
    G = Echo@Simp[Table[\xi^k/k!, {k, 0, $k+1}].NestList[Simp[B[x_U, #]] &, y_U, $k+1]];
    fs = Echo@Flatten@Table[f_{1,i,j,k}[\eta], {1, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = Echo[fs.(bs = fs /. f_{L_,i_,j_,k_}[\eta] => e^L U[{y^i, a^j, x^k}])];
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1_U /. eta -> 0, F ** G - y_U ** F - partial_eta F}}, {b, bs}]];
    sol = Echo@First[F /. DSolve[es, fs, eta]];
    Echo[sol /. {e -> 1, U -> Times}];
    Collect[sol /. {e -> 1, U -> Times}, e, Simplify]
  ]]
```

```
“ -t \xi CU[] + 2 e \xi CU[a] - \gamma e \xi^2 CU[x] + CU[y]
“ {f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta],
  f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]}
“ CU[] f_{0,0,0,0}[\eta] + e CU[] f_{1,0,0,0}[\eta] + e CU[x] f_{1,0,0,1}[\eta] + e CU[a] f_{1,0,1,0}[\eta] + e CU[a, x] f_{1,0,1,1}[\eta] +
  e CU[y] f_{1,1,0,0}[\eta] + e CU[y, x] f_{1,1,0,1}[\eta] + e CU[y, a] f_{1,1,1,0}[\eta] + e CU[y, a, x] f_{1,1,1,1}[\eta]
» e^{-t \eta \xi} CU[] + 1/2 e^{-t \eta \xi} t \gamma e \eta^2 \xi^2 CU[] + 2 e^{-t \eta \xi} e \eta \xi CU[a] - e^{-t \eta \xi} \gamma e \eta \xi^2 CU[x] - e^{-t \eta \xi} \gamma e \eta^2 \xi CU[y]
» 1 + 2 a e \eta \xi - y \gamma e \eta^2 \xi - x \gamma e \eta \xi^2 + 1/2 t \gamma e \eta^2 \xi^2
1 + 1/2 e \eta \xi (4 a + \gamma (-2 y \eta - 2 x \xi + t \eta \xi))
```

Logos

```

 $\Delta_U[\{\xi_1, \eta_1\}, \{x, y\}] :=$ 
 $\Delta_U[\{\xi_1, \eta_1\}, \{x, y\}] = \text{Module}[\{\xi, \eta, G, F, fs, f, bs, eq, b, eqs\},$ 
 $G = \text{Simp}[\text{Table}[\xi^k/k!, \{k, 0, \$k+1\}].\text{NestList}[\text{Simp}[B[x_U, \#]] \&, y_U, \$k+1];$ 
 $fs = \text{Flatten}@\text{Table}[f_{1,i,j,k}[\eta], \{1, 0, \$k\}, \{i, 0, 1\}, \{j, 0, 1\}, \{k, 0, 1\}];$ 
 $F = fs.(bs = fs /. f_{l-,i-,j-,k-}[\eta] \Rightarrow e^l U@ \{y^i, a^j, x^k\} /. e \rightarrow \epsilon);$ 
 $eqs = \text{Flatten}[\text{Table}[\text{Coefficient}[eq, b] == 0,$ 
 $\{eq, \{F - 1_U /. \eta \rightarrow 0, F ** G - y_U ** F - \partial_\eta F\}, \{b, bs\}]]];$ 
 $F = F /. \text{DSolve}[eqs, fs, \eta][[1];$ 
 $\mathbb{C}_U[\{y, a, x\},$ 
 $\xi x + \eta y + (U /. \{CU \rightarrow -t \eta \xi, QU \rightarrow \eta \xi (1 - T) / \hbar\}),$ 
 $F /. \{e \rightarrow 1, U \rightarrow \text{Times}\}$ 
 $] /. \{\xi \rightarrow \xi_1, \eta \rightarrow \eta_1\};$ 

```

```

 $\Delta_{CU}[\{\xi, \eta\}, \{x, y\}], \text{lhs} = \text{CU}@\mathbb{C}_U[\{x, y\}, \hbar (\xi x + \eta y), 1],$ 
 $\text{HL}[\text{lhs} == \text{CU}@\Delta_{CU}[\hbar \{\xi, \eta\}, \{x, y\}]]$ 

```

$$\{\mathbb{C}_U[\{y, a, x\}, y \eta + x \xi - t \eta \xi, 1 + 2 a \epsilon \eta \xi - y \gamma \epsilon \eta^2 \xi - x \gamma \epsilon \eta \xi^2 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2],$$

$$(1 - t \eta \xi \hbar^2) \text{CU}[] + 2 \epsilon \eta \xi \hbar^2 \text{CU}[a] + \xi \hbar \text{CU}[x] + \eta \hbar \text{CU}[y] +$$

$$\frac{1}{2} \xi^2 \hbar^2 \text{CU}[x, x] + \eta \xi \hbar^2 \text{CU}[y, x] + \frac{1}{2} \eta^2 \hbar^2 \text{CU}[y, y], \text{True}\}$$

```

 $\Delta_{QU}[\{\xi, \eta\}, \{x, y\}], \text{lhs} = \text{QU}@\mathbb{C}_U[\{x, y\}, \hbar (\xi x + \eta y), 1],$ 
 $\text{HL}@\text{SimpT}[\text{lhs} == \text{QU}@\Delta_{QU}[\hbar \{\xi, \eta\}, \{x, y\}]]$ 

```

$$\{\mathbb{C}_U[\{y, a, x\}, y \eta + x \xi + \frac{(1 - T) \eta \xi}{\hbar}, 1 + 2 a T \epsilon \eta \xi - \frac{1}{2} (-1 + 3 T) y \gamma \epsilon \eta^2 \xi -$$

$$\frac{1}{2} (-1 + 3 T) x \gamma \epsilon \eta \xi^2 + \frac{(-1 + T) (-1 + 3 T) \gamma \epsilon \eta^2 \xi^2}{4 \hbar} + x y \gamma \epsilon \eta \xi \hbar],$$

$$(1 + \eta \xi \hbar - T \eta \xi \hbar) \text{QU}[] + 2 T \epsilon \eta \xi \hbar^2 \text{QU}[a] + \xi \hbar \text{QU}[x] + \eta \hbar \text{QU}[y] +$$

$$\frac{1}{2} \xi^2 \hbar^2 \text{QU}[x, x] + \eta \xi \hbar^2 \text{QU}[y, x] + \frac{1}{2} \eta^2 \hbar^2 \text{QU}[y, y], \text{True}\}$$

```

 $\{tt = \text{Last}[\Delta_{CU}[\{\xi, \eta\}, \{x, y\}]], \text{Normal}@\text{Series}[\text{Log}[tt], \{\epsilon, 0, \$k\}]\}$ 

```

$$\{1 + 2 a \epsilon \eta \xi - y \gamma \epsilon \eta^2 \xi - x \gamma \epsilon \eta \xi^2 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2, \epsilon \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right)\}$$

```

 $\{tt = \text{Last}[\Delta_{QU}[\{\xi, \eta\}, \{x, y\}]], \text{Normal}@\text{Series}[\text{Log}[tt], \{\epsilon, 0, \$k\}]\}$ 

```

$$\{1 + 2 a T \epsilon \eta \xi - \frac{1}{2} (-1 + 3 T) y \gamma \epsilon \eta^2 \xi - \frac{1}{2} (-1 + 3 T) x \gamma \epsilon \eta \xi^2 +$$

$$\frac{(-1 + T) (-1 + 3 T) \gamma \epsilon \eta^2 \xi^2}{4 \hbar} + x y \gamma \epsilon \eta \xi \hbar, \frac{1}{4 \hbar} \epsilon (\gamma \eta^2 \xi^2 - 4 T \gamma \eta^2 \xi^2 + 3 T^2 \gamma \eta^2 \xi^2 +$$

$$8 a T \eta \xi \hbar + 2 y \gamma \eta^2 \xi \hbar - 6 T y \gamma \eta^2 \xi \hbar + 2 x \gamma \eta \xi^2 \hbar - 6 T x \gamma \eta \xi^2 \hbar + 4 x y \gamma \eta \xi \hbar^2)\}$$

Logos

```

Simp[ $\mathbb{C}_U$ [specs___, Q_, P_] :=
 $\mathbb{C}_U$ [specs, ExpandNumerator@Together[Q], Collect[P,  $\epsilon$ , ExpandNumerator@*Together]];
 $\Delta_U$ [{ $\nu_1$ _,  $\omega_1$ _,  $\delta$ _}, {u_, w_}] := Simp@Module[{ $\nu$ ,  $\omega$ , yax, q, p, Q, d},
{yax, q, p} = List@@ $\Delta_U$ [{ $\nu$ ,  $\omega$ }, {u, w}]];
 $\mathbb{C}_U$ [yax, Q = ( $\nu u + \omega w + \delta u w + d \nu w$ ) / (1 - d  $\delta$ ),
Expand[(1 - d  $\delta$ )-1 e-Q DP $\nu \rightarrow D_u, \omega \rightarrow D_w$ [p][eQ]] /. {d  $\rightarrow$   $\partial_{\nu, \omega} q$ } /. { $\nu \rightarrow \nu_1, \omega \rightarrow \omega_1$ }]

```

```

{ $\Delta_{CU}$ [{ $\xi$ ,  $\eta$ ,  $\delta$ }, {x, y}], lhs = CU@ $\mathbb{C}_{CU}$ [{x, y},  $\hbar$  ( $\xi x + \eta y + \delta x y$ ), 1],
HL[lhs == CU@ $\Delta_{CU}$ [ $\hbar$  { $\xi$ ,  $\eta$ ,  $\delta$ }, {x, y}]]]

```

```

{ $\mathbb{C}_{CU}$ [{y, a, x},  $\frac{xy \delta + y \eta + x \xi - t \eta \xi}{1 + t \delta}, \frac{1}{1 + t \delta}$  +
 $\frac{1}{2(1 + t \delta)^5} \epsilon$  (4 a  $\delta$  + 12 a t  $\delta^2$  + 4 a x y  $\delta^2$  + 2 t  $\gamma \delta^2$  - 8 x y  $\gamma \delta^2$  + 12 a t2  $\delta^3$  + 8 a t x y  $\delta^3$  + 4 t2  $\gamma \delta^3$  -
12 t x y  $\gamma \delta^3$  - 4 x2 y2  $\gamma \delta^3$  + 4 a t3  $\delta^4$  + 4 a t2 x y  $\delta^4$  + 2 t3  $\gamma \delta^4$  - 4 t2 x y  $\gamma \delta^4$  - 3 t x2 y2  $\gamma \delta^4$  +
4 a y  $\delta \eta$  - 4 y  $\gamma \delta \eta$  + 8 a t y  $\delta^2 \eta$  - 4 t y  $\gamma \delta^2 \eta$  - 6 x y2  $\gamma \delta^2 \eta$  + 4 a t2 y  $\delta^3 \eta$  - 4 t x y2  $\gamma \delta^3 \eta$  -
2 y2  $\gamma \delta \eta^2$  - t y2  $\gamma \delta^2 \eta^2$  + 4 a x  $\delta \xi$  - 4 x  $\gamma \delta \xi$  + 8 a t x  $\delta^2 \xi$  - 4 t x  $\gamma \delta^2 \xi$  - 6 x2 y  $\gamma \delta^2 \xi$  +
4 a t2 x  $\delta^3 \xi$  - 4 t x2 y  $\gamma \delta^3 \xi$  + 4 a  $\eta \xi$  + 8 a t  $\delta \eta \xi$  + 4 t  $\gamma \delta \eta \xi$  - 8 x y  $\gamma \delta \eta \xi$  + 4 a t2  $\delta^2 \eta \xi$  +
4 t2  $\gamma \delta^2 \eta \xi$  - 4 t x y  $\gamma \delta^2 \eta \xi$  - 2 y  $\gamma \eta^2 \xi$  - 2 x2  $\gamma \delta \xi^2$  - t x2  $\gamma \delta^2 \xi^2$  - 2 x  $\gamma \eta \xi^2$  + t  $\gamma \eta^2 \xi^2$ ),
(1 - t  $\delta \hbar$  + t2  $\delta^2 \hbar^2$  + t  $\gamma \delta^2 \in \hbar^2$  - t  $\eta \xi \hbar^2$ ) CU[] + (2  $\delta \in \hbar$  - 4 t  $\delta^2 \in \hbar^2$  + 2  $\in \eta \xi \hbar^2$ )
CU[a] +
( $\xi \hbar$  - 2 t  $\delta \xi \hbar^2$  - 2  $\gamma \delta \in \xi \hbar^2$ ) CU[x] +
( $\eta \hbar$  - 2 t  $\delta \eta \hbar^2$  - 2  $\gamma \delta \in \eta \hbar^2$ ) CU[y] +
4  $\delta \in \xi \hbar^2$  CU[a, x] +
 $\frac{1}{2} \xi^2 \hbar^2$  CU[x, x] +
4  $\delta \in \eta \hbar^2$  CU[y, a] +
( $\delta \hbar$  - 2 t  $\delta^2 \hbar^2$  - 4  $\gamma \delta^2 \in \hbar^2$  +  $\eta \xi \hbar^2$ ) CU[y, x] +
 $\frac{1}{2} \eta^2 \hbar^2$  CU[y, y] +
4  $\delta^2 \in \hbar^2$  CU[y, a, x] +
 $\delta \xi \hbar^2$  CU[y, x, x] +
 $\delta \eta \hbar^2$  CU[y, y, x] +
 $\frac{1}{2} \delta^2 \hbar^2$  CU[y, y, x, x], True}

```

{ $\Delta_{\text{QU}}[\{\xi, \eta, \delta\}, \{x, y\}], \text{lhs} = \text{QU}@\mathbb{C}_{\text{QU}}[\{x, y\}, \hbar(\xi x + \eta y + \delta x y), 1],$

$\text{HL}@\text{SimpT}[\text{lhs} = \text{QU}@\Delta_{\text{QU}}[\hbar\{\xi, \eta, \delta\}, \{x, y\}]]\}$

{ $\mathbb{C}_{\text{QU}}[\{y, a, x\}, \frac{\eta\xi - T\eta\xi + xy\delta\hbar + y\eta\hbar + x\xi\hbar}{-\delta + T\delta + \hbar},$

$\frac{\hbar}{-\delta + T\delta + \hbar} + \frac{1}{4(-\delta + T\delta + \hbar)^5} \in (-8aT\delta^4\hbar^2 + 24aT^2\delta^4\hbar^2 - 24aT^3\delta^4\hbar^2 + 8aT^4\delta^4\hbar^2 + 2\gamma\delta^4\hbar^2 -$

$12T\gamma\delta^4\hbar^2 + 24T^2\gamma\delta^4\hbar^2 - 20T^3\gamma\delta^4\hbar^2 + 6T^4\gamma\delta^4\hbar^2 + 24aT\delta^3\hbar^3 - 48aT^2\delta^3\hbar^3 +$
 $24aT^3\delta^3\hbar^3 - 4\gamma\delta^3\hbar^3 + 20T\gamma\delta^3\hbar^3 - 28T^2\gamma\delta^3\hbar^3 + 12T^3\gamma\delta^3\hbar^3 + 8aTx\gamma\delta^4\hbar^3 -$
 $16aT^2xy\delta^4\hbar^3 + 8aT^3xy\delta^4\hbar^3 - 8Tx\gamma\gamma\delta^4\hbar^3 + 16T^2xy\gamma\delta^4\hbar^3 - 8T^3xy\gamma\delta^4\hbar^3 +$
 $8aTy\delta^3\eta\hbar^3 - 16aT^2y\delta^3\eta\hbar^3 + 8aT^3y\delta^3\eta\hbar^3 + 8aTx\delta^3\xi\hbar^3 - 16aT^2x\delta^3\xi\hbar^3 +$
 $8aT^3x\delta^3\xi\hbar^3 + 8aT\delta^2\eta\xi\hbar^3 - 16aT^2\delta^2\eta\xi\hbar^3 + 8aT^3\delta^2\eta\xi\hbar^3 - 4\gamma\delta^2\eta\xi\hbar^3 +$
 $20T\gamma\delta^2\eta\xi\hbar^3 - 28T^2\gamma\delta^2\eta\xi\hbar^3 + 12T^3\gamma\delta^2\eta\xi\hbar^3 - 24aT\delta^2\hbar^4 + 24aT^2\delta^2\hbar^4 +$
 $2\gamma\delta^2\hbar^4 - 8T\gamma\delta^2\hbar^4 + 6T^2\gamma\delta^2\hbar^4 - 16aTx\gamma\delta^3\hbar^4 + 16aT^2xy\delta^3\hbar^4 + 24Tx\gamma\gamma\delta^3\hbar^4 -$
 $24T^2xy\gamma\delta^3\hbar^4 + x^2y^2\gamma\delta^4\hbar^4 + 4Tx^2y^2\gamma\delta^4\hbar^4 - 5T^2x^2y^2\gamma\delta^4\hbar^4 - 16aTy\delta^2\eta\hbar^4 +$
 $16aT^2y\delta^2\eta\hbar^4 - 4y\gamma\delta^2\eta\hbar^4 + 16Ty\gamma\delta^2\eta\hbar^4 - 12T^2y\gamma\delta^2\eta\hbar^4 + 8Tx\gamma^2\delta^3\eta\hbar^4 -$
 $8T^2x\gamma^2\delta^3\eta\hbar^4 - y^2\gamma\delta^2\eta^2\hbar^4 + 4Ty^2\gamma\delta^2\eta^2\hbar^4 - 3T^2y^2\gamma\delta^2\eta^2\hbar^4 - 16aTx\delta^2\xi\hbar^4 +$
 $16aT^2x\delta^2\xi\hbar^4 - 4x\gamma\delta^2\xi\hbar^4 + 16Tx\gamma\delta^2\xi\hbar^4 - 12T^2x\gamma\delta^2\xi\hbar^4 + 8Tx^2y\gamma\delta^3\xi\hbar^4 -$
 $8T^2x^2y\gamma\delta^3\xi\hbar^4 - 16aT\delta\eta\xi\hbar^4 + 16aT^2\delta\eta\xi\hbar^4 + 4\gamma\delta\eta\xi\hbar^4 - 16T\gamma\delta\eta\xi\hbar^4 +$
 $12T^2\gamma\delta\eta\xi\hbar^4 + 8Tx\gamma\gamma\delta^2\eta\xi\hbar^4 - 8T^2xy\gamma\delta^2\eta\xi\hbar^4 - x^2\gamma\delta^2\xi^2\hbar^4 + 4Tx^2\gamma\delta^2\xi^2\hbar^4 -$
 $3T^2x^2\gamma\delta^2\xi^2\hbar^4 + \gamma\eta^2\xi^2\hbar^4 - 4T\gamma\eta^2\xi^2\hbar^4 + 3T^2\gamma\eta^2\xi^2\hbar^4 + 8aT\delta\hbar^5 + 8aTx\gamma\delta^2\hbar^5 -$
 $4xy\gamma\delta^2\hbar^5 - 12Tx\gamma\gamma\delta^2\hbar^5 - 4x^2y^2\gamma\delta^3\hbar^5 - 4Tx^2y^2\gamma\delta^3\hbar^5 + 8aTy\delta\eta\hbar^5 + 4y\gamma\delta\eta\hbar^5 -$
 $12Ty\gamma\delta\eta\hbar^5 - 2xy^2\gamma\delta^2\eta\hbar^5 - 10Tx\gamma^2\delta^2\eta\hbar^5 + 2y^2\gamma\delta\eta^2\hbar^5 - 6Ty^2\gamma\delta\eta^2\hbar^5 +$
 $8aTx\delta\xi\hbar^5 + 4x\gamma\delta\xi\hbar^5 - 12Tx\gamma\delta\xi\hbar^5 - 2x^2y\gamma\delta^2\xi\hbar^5 - 10Tx^2y\gamma\delta^2\xi\hbar^5 + 8aT\eta\xi\hbar^5 -$
 $16Tx\gamma\gamma\delta\eta\xi\hbar^5 + 2y\gamma\eta^2\xi\hbar^5 - 6Ty\gamma\eta^2\xi\hbar^5 + 2x^2\gamma\delta\xi^2\hbar^5 - 6Tx^2\gamma\delta\xi^2\hbar^5 + 2x\gamma\eta\xi^2\hbar^5 -$
 $6Tx\gamma\eta\xi^2\hbar^5 + 4xy\gamma\delta\hbar^6 + 4x^2y^2\gamma\delta^2\hbar^6 + 4xy^2\gamma\delta\eta\hbar^6 + 4x^2y\gamma\delta\xi\hbar^6 + 4xy\gamma\eta\xi\hbar^6)],$

$(1 + \delta - T\delta + \delta^2 - 2T\delta^2 + T^2\delta^2 + \frac{1}{2}\gamma\delta^2 \in \hbar - 2T\gamma\delta^2 \in \hbar + \frac{3}{2}T^2\gamma\delta^2 \in \hbar + \eta\xi\hbar - T\eta\xi\hbar)$

$\text{QU}[] +$

$(2T\delta \in \hbar + 4T\delta^2 \in \hbar - 4T^2\delta^2 \in \hbar + 2T \in \eta\xi\hbar^2)$

$\text{QU}[a] +$

$(\xi\hbar + 2\delta\xi\hbar - 2T\delta\xi\hbar + \gamma\delta \in \xi\hbar^2 - 3T\gamma\delta \in \xi\hbar^2)$

$\text{QU}[x] +$

$(\eta\hbar + 2\delta\eta\hbar - 2T\delta\eta\hbar + \gamma\delta \in \eta\hbar^2 - 3T\gamma\delta \in \eta\hbar^2)$

$\text{QU}[y] +$

$4T\delta \in \xi\hbar^2 \text{QU}[a, x] + \frac{1}{2}\xi^2\hbar^2 \text{QU}[x, x] +$

$4T\delta \in$

$\eta\hbar^2 \text{QU}[y, a] +$

$(\delta\hbar + 2\delta^2\hbar - 2T\delta^2\hbar + \gamma\delta \in \hbar^2 + 4\gamma\delta^2 \in \hbar^2 - 8T\gamma\delta^2 \in \hbar^2 + \eta\xi\hbar^2)$

$\text{QU}[y, x] +$

$\frac{1}{2}\eta^2\hbar^2 \text{QU}[y, y] + 4T\delta^2 \in \hbar^2 \text{QU}[y, a, x] +$

$\delta\xi\hbar^2 \text{QU}[y, x, x] +$

$\delta\eta\hbar^2 \text{QU}[y, y, x] +$

$\frac{1}{2}\delta^2\hbar^2 \text{QU}[y, y, x, x], \text{True}\}$

{tt = Last[ACU[{ξ, η, δ}, {x, y}], Normal@Series[Log[tt], {ε, 0, \$k}]]}

$$\left\{ \frac{1}{1+t\delta} + \frac{1}{2(1+t\delta)^5} \left(4a\delta + 12at\delta^2 + 4axy\delta^2 + 2t\gamma\delta^2 - 8xy\gamma\delta^2 + 12at^2\delta^3 + 8atxy\delta^3 + 4t^2\gamma\delta^3 - 12txy\gamma\delta^3 - 4x^2y^2\gamma\delta^3 + 4at^3\delta^4 + 4at^2xy\delta^4 + 2t^3\gamma\delta^4 - 4t^2xy\gamma\delta^4 - 3tx^2y^2\gamma\delta^4 + 4ay\delta\eta - 4y\gamma\delta\eta + 8aty\delta^2\eta - 4ty\gamma\delta^2\eta - 6xy^2\gamma\delta^2\eta + 4at^2y\delta^3\eta - 4txy^2\gamma\delta^3\eta - 2y^2\gamma\delta\eta^2 - ty^2\gamma\delta^2\eta^2 + 4ax\delta\xi - 4x\gamma\delta\xi + 8atx\delta^2\xi - 4tx\gamma\delta^2\xi - 6x^2y\gamma\delta^2\xi + 4at^2x\delta^3\xi - 4tx^2y\gamma\delta^3\xi + 4a\eta\xi + 8at\delta\eta\xi + 4t\gamma\delta\eta\xi - 8xy\gamma\delta\eta\xi + 4at^2\delta^2\eta\xi + 4t^2\gamma\delta^2\eta\xi - 4txy\gamma\delta^2\eta\xi - 2y\gamma\eta^2\xi - 2x^2\gamma\delta\xi^2 - tx^2\gamma\delta^2\xi^2 - 2x\gamma\eta\xi^2 + t\gamma\eta^2\xi^2 \right), \frac{1}{2(1+t\delta)^4} \left(4a\delta + 12at\delta^2 + 4axy\delta^2 + 2t\gamma\delta^2 - 8xy\gamma\delta^2 + 12at^2\delta^3 + 8atxy\delta^3 + 4t^2\gamma\delta^3 - 12txy\gamma\delta^3 - 4x^2y^2\gamma\delta^3 + 4at^3\delta^4 + 4at^2xy\delta^4 + 2t^3\gamma\delta^4 - 4t^2xy\gamma\delta^4 - 3tx^2y^2\gamma\delta^4 + 4ay\delta\eta - 4y\gamma\delta\eta + 8aty\delta^2\eta - 4ty\gamma\delta^2\eta - 6xy^2\gamma\delta^2\eta + 4at^2y\delta^3\eta - 4txy^2\gamma\delta^3\eta - 2y^2\gamma\delta\eta^2 - ty^2\gamma\delta^2\eta^2 + 4ax\delta\xi - 4x\gamma\delta\xi + 8atx\delta^2\xi - 4tx\gamma\delta^2\xi - 6x^2y\gamma\delta^2\xi + 4at^2x\delta^3\xi - 4tx^2y\gamma\delta^3\xi + 4a\eta\xi + 8at\delta\eta\xi + 4t\gamma\delta\eta\xi - 8xy\gamma\delta\eta\xi + 4at^2\delta^2\eta\xi + 4t^2\gamma\delta^2\eta\xi - 4txy\gamma\delta^2\eta\xi - 2y\gamma\eta^2\xi - 2x^2\gamma\delta\xi^2 - tx^2\gamma\delta^2\xi^2 - 2x\gamma\eta\xi^2 + t\gamma\eta^2\xi^2 \right) + \text{Log}\left[\frac{1}{1+t\delta}\right] \right\}$$

{tt = Last[AQu[{ξ, η, δ}, {x, y}], Normal@Series[Log[tt], {ε, 0, \$k}]]

$$\left\{ \frac{\hbar}{-\delta + T\delta + \hbar} + \frac{1}{4(-\delta + T\delta + \hbar)^5} \in \left(-8aT\delta^4\hbar^2 + 24aT^2\delta^4\hbar^2 - 24aT^3\delta^4\hbar^2 + 8aT^4\delta^4\hbar^2 + 2\gamma\delta^4\hbar^2 - 12T\gamma\delta^4\hbar^2 + 24T^2\gamma\delta^4\hbar^2 - 20T^3\gamma\delta^4\hbar^2 + 6T^4\gamma\delta^4\hbar^2 + 24aT\delta^3\hbar^3 - 48aT^2\delta^3\hbar^3 + 24aT^3\delta^3\hbar^3 - 4\gamma\delta^3\hbar^3 + 20T\gamma\delta^3\hbar^3 - 28T^2\gamma\delta^3\hbar^3 + 12T^3\gamma\delta^3\hbar^3 + 8aTx\gamma\delta^4\hbar^3 - 16aT^2xy\delta^4\hbar^3 + 8aT^3xy\delta^4\hbar^3 - 8Tx\gamma\delta^4\hbar^3 + 16T^2xy\gamma\delta^4\hbar^3 - 8T^3xy\gamma\delta^4\hbar^3 + 8aTy\delta^3\eta\hbar^3 - 16aT^2y\delta^3\eta\hbar^3 + 8aT^3y\delta^3\eta\hbar^3 + 8aTx\delta^3\xi\hbar^3 - 16aT^2x\delta^3\xi\hbar^3 + 8aT^3x\delta^3\xi\hbar^3 + 8aT\delta^2\eta\xi\hbar^3 - 16aT^2\delta^2\eta\xi\hbar^3 + 8aT^3\delta^2\eta\xi\hbar^3 - 4\gamma\delta^2\eta\xi\hbar^3 + 20T\gamma\delta^2\eta\xi\hbar^3 - 28T^2\gamma\delta^2\eta\xi\hbar^3 + 12T^3\gamma\delta^2\eta\xi\hbar^3 - 24aT\delta^2\hbar^4 + 24aT^2\delta^2\hbar^4 + 2\gamma\delta^2\hbar^4 - 8T\gamma\delta^2\hbar^4 + 6T^2\gamma\delta^2\hbar^4 - 16aTx\gamma\delta^3\hbar^4 + 16aT^2xy\delta^3\hbar^4 + 24Tx\gamma\delta^3\hbar^4 - 24T^2xy\gamma\delta^3\hbar^4 + x^2y^2\gamma\delta^4\hbar^4 + 4Tx^2y^2\gamma\delta^4\hbar^4 - 5T^2x^2y^2\gamma\delta^4\hbar^4 - 16aTy\delta^2\eta\hbar^4 + 16aT^2y\delta^2\eta\hbar^4 - 4y\gamma\delta^2\eta\hbar^4 + 16Ty\gamma\delta^2\eta\hbar^4 - 12T^2y\gamma\delta^2\eta\hbar^4 + 8Tx\gamma^2\delta^3\eta\hbar^4 - 8T^2xy^2\delta^3\eta\hbar^4 - y^2\gamma\delta^2\eta^2\hbar^4 + 4Ty^2\gamma\delta^2\eta^2\hbar^4 - 3T^2y^2\gamma\delta^2\eta^2\hbar^4 - 16aTx\delta^2\xi\hbar^4 + 16aT^2x\delta^2\xi\hbar^4 - 4x\gamma\delta^2\xi\hbar^4 + 16Tx\gamma\delta^2\xi\hbar^4 - 12T^2x\gamma\delta^2\xi\hbar^4 + 8Tx^2y\delta^3\xi\hbar^4 - 8T^2x^2y\delta^3\xi\hbar^4 - 16aT\delta\eta\xi\hbar^4 + 16aT^2\delta\eta\xi\hbar^4 + 4\gamma\delta\eta\xi\hbar^4 - 16T\gamma\delta\eta\xi\hbar^4 + 12T^2\gamma\delta\eta\xi\hbar^4 + 8Tx\gamma\delta^2\eta\xi\hbar^4 - 8T^2xy\gamma\delta^2\eta\xi\hbar^4 - x^2\gamma\delta^2\xi^2\hbar^4 + 4Tx^2\gamma\delta^2\xi^2\hbar^4 - 3T^2x^2\gamma\delta^2\xi^2\hbar^4 + \gamma\eta^2\xi^2\hbar^4 - 4T\gamma\eta^2\xi^2\hbar^4 + 3T^2\gamma\eta^2\xi^2\hbar^4 + 8aT\delta\hbar^5 + 8aTx\gamma\delta^2\hbar^5 - 4xy\gamma\delta^2\hbar^5 - 12Tx\gamma\delta^2\hbar^5 - 4x^2y^2\gamma\delta^3\hbar^5 - 4Tx^2y^2\gamma\delta^3\hbar^5 + 8aTy\delta\eta\hbar^5 + 4y\gamma\delta\eta\hbar^5 - 12Ty\gamma\delta\eta\hbar^5 - 2xy^2\gamma\delta^2\eta\hbar^5 - 10Txy^2\gamma\delta^2\eta\hbar^5 + 2y^2\gamma\delta\eta^2\hbar^5 - 6Ty^2\gamma\delta\eta^2\hbar^5 + 8aTx\delta\xi\hbar^5 + 4x\gamma\delta\xi\hbar^5 - 12Tx\gamma\delta\xi\hbar^5 - 2x^2y\gamma\delta^2\xi\hbar^5 - 10Tx^2y\gamma\delta^2\xi\hbar^5 + 8aT\eta\xi\hbar^5 - 16Tx\gamma\delta\eta\xi\hbar^5 + 2y\gamma\eta^2\xi\hbar^5 - 6Ty\gamma\eta^2\xi\hbar^5 + 2x^2\gamma\delta\xi^2\hbar^5 - 6Tx^2\gamma\delta\xi^2\hbar^5 + 2x\gamma\eta\xi^2\hbar^5 - 6Tx\gamma\eta\xi^2\hbar^5 + 4xy\gamma\delta\hbar^6 + 4x^2y^2\gamma\delta^2\hbar^6 + 4xy^2\gamma\delta\eta\hbar^6 + 4x^2y\gamma\delta\xi\hbar^6 + 4xy\gamma\eta\xi\hbar^6 \right), \frac{1}{4(-\delta + T\delta + \hbar)^4} \in \left(-8aT\delta^4\hbar + 24aT^2\delta^4\hbar - 24aT^3\delta^4\hbar + 8aT^4\delta^4\hbar + 2\gamma\delta^4\hbar - 12T\gamma\delta^4\hbar + 24T^2\gamma\delta^4\hbar - 20T^3\gamma\delta^4\hbar + 6T^4\gamma\delta^4\hbar + 24aT\delta^3\hbar^2 - 48aT^2\delta^3\hbar^2 + 24aT^3\delta^3\hbar^2 - 4\gamma\delta^3\hbar^2 + 20T\gamma\delta^3\hbar^2 - 28T^2\gamma\delta^3\hbar^2 + 12T^3\gamma\delta^3\hbar^2 + 8aTx\gamma\delta^4\hbar^2 - 16aT^2xy\delta^4\hbar^2 + 8aT^3xy\delta^4\hbar^2 - 8Tx\gamma\delta^4\hbar^2 + 16T^2xy\gamma\delta^4\hbar^2 - 8T^3xy\gamma\delta^4\hbar^2 + 8aTy\delta^3\eta\hbar^2 - 16aT^2y\delta^3\eta\hbar^2 + 8aT^3y\delta^3\eta\hbar^2 + 8aTx\delta^3\xi\hbar^2 - 16aT^2x\delta^3\xi\hbar^2 + 8aT^3x\delta^3\xi\hbar^2 + 8aT\delta^2\eta\xi\hbar^2 - 16aT^2\delta^2\eta\xi\hbar^2 - 4\gamma\delta^2\eta\xi\hbar^2 + 20T\gamma\delta^2\eta\xi\hbar^2 - 28T^2\gamma\delta^2\eta\xi\hbar^2 + 12T^3\gamma\delta^2\eta\xi\hbar^2 - 24aT\delta^2\hbar^3 + 24aT^2\delta^2\hbar^3 + 2\gamma\delta^2\hbar^3 - 8T\gamma\delta^2\hbar^3 + 6T^2\gamma\delta^2\hbar^3 - 16aTx\gamma\delta^3\hbar^3 + 16aT^2xy\delta^3\hbar^3 + 24Tx\gamma\gamma\delta^3\hbar^3 - 24T^2xy\gamma\delta^3\hbar^3 + x^2y^2\gamma\delta^4\hbar^3 + 4Tx^2y^2\gamma\delta^4\hbar^3 - 5T^2x^2y^2\gamma\delta^4\hbar^3 - 16aTy\delta^2\eta\hbar^3 + 16aT^2y\delta^2\eta\hbar^3 - 4y\gamma\delta^2\eta\hbar^3 + 16Ty\gamma\delta^2\eta\hbar^3 - 12T^2y\gamma\delta^2\eta\hbar^3 + 8Tx\gamma^2\delta^3\eta\hbar^3 - 8T^2xy^2\delta^3\eta\hbar^3 - y^2\gamma\delta^2\eta^2\hbar^3 + 4Ty^2\gamma\delta^2\eta^2\hbar^3 - 3T^2y^2\gamma\delta^2\eta^2\hbar^3 - 16aTx\delta^2\xi\hbar^3 + 16aT^2x\delta^2\xi\hbar^3 - 4x\gamma\delta^2\xi\hbar^3 + 16Tx\gamma\delta^2\xi\hbar^3 - 12T^2x\gamma\delta^2\xi\hbar^3 + 8Tx^2y\delta^3\xi\hbar^3 - 8T^2x^2y\delta^3\xi\hbar^3 - 16aT\delta\eta\xi\hbar^3 + 16aT^2\delta\eta\xi\hbar^3 + 4\gamma\delta\eta\xi\hbar^3 - 16T\gamma\delta\eta\xi\hbar^3 + 12T^2\gamma\delta\eta\xi\hbar^3 + 8Tx\gamma\delta^2\eta\xi\hbar^3 - 8T^2xy\gamma\delta^2\eta\xi\hbar^3 - x^2\gamma\delta^2\xi^2\hbar^3 + 4Tx^2\gamma\delta^2\xi^2\hbar^3 - 3T^2x^2\gamma\delta^2\xi^2\hbar^3 + \gamma\eta^2\xi^2\hbar^3 - 4T\gamma\eta^2\xi^2\hbar^3 + 3T^2\gamma\eta^2\xi^2\hbar^3 + 8aT\delta\hbar^4 + 8aTx\gamma\delta^2\hbar^4 - 4xy\gamma\delta^2\hbar^4 - 12Tx\gamma\delta^2\hbar^4 - 4x^2y^2\gamma\delta^3\hbar^4 - 4Tx^2y^2\gamma\delta^3\hbar^4 + 8aTy\delta\eta\hbar^4 + 4y\gamma\delta\eta\hbar^4 - 12Ty\gamma\delta\eta\hbar^4 - 2xy^2\gamma\delta^2\eta\hbar^4 - 10Txy^2\gamma\delta^2\eta\hbar^4 + 2y^2\gamma\delta\eta^2\hbar^4 - 6Ty^2\gamma\delta\eta^2\hbar^4 + 8aTx\delta\xi\hbar^4 + 4x\gamma\delta\xi\hbar^4 - 12Tx\gamma\delta\xi\hbar^4 - 2x^2y\gamma\delta^2\xi\hbar^4 - 10Tx^2y\gamma\delta^2\xi\hbar^4 + 8aT\eta\xi\hbar^4 - 16Tx\gamma\delta\eta\xi\hbar^4 + 2y\gamma\eta^2\xi\hbar^4 - 6Ty\gamma\eta^2\xi\hbar^4 + 2x^2\gamma\delta\xi^2\hbar^4 - 6Tx^2\gamma\delta\xi^2\hbar^4 + 2x\gamma\eta\xi^2\hbar^4 - 6Tx\gamma\eta\xi^2\hbar^4 + 4xy\gamma\delta\hbar^5 + 4x^2y^2\gamma\delta^2\hbar^5 + 4xy^2\gamma\delta\eta\hbar^5 + 4x^2y\gamma\delta\xi\hbar^5 + 4xy\gamma\eta\xi\hbar^5 \right) + \text{Log}\left[\frac{\hbar}{-\delta + T\delta + \hbar}\right] \}$$

Reorderings with Rord

Rord

```

Rordui, wj → k [CU [L---, {L---, ui, wj, r---}S, R---, Q---, P---]] :=
Simp@Module[{u, ω, δ, Δ1, yax, q, p, δ1 = ∂ui, wj Q},
  {yax, q, p} = List@@If[δ1 == 0, ΔU[{u, ω}, {u, w}], ΔU[{u, ω, δ}, {u, w}]] /.
  {y → yk, a → ak, x → xk, t → tS, T → TS};
CU[L, {L, Sequence@@yax, r}S, R, q + (Q /. ui | wj → 0), e-q DPui → Du, wj → Dω [P] [p eq]] /.
  {u → ∂ui Q /. wj → 0, ω → ∂wj Q /. ui → 0, δ → δ1}]

```

With[{c0 = C_{CU} [{y₁, x₁}₁, {x₂, a₂, y₂}₂, ħ t₁ a₂ + ħ t₁⁻¹ (e^{t₁} - 1) y₁ x₂, 1 + e x₁ y₂]}],
 {Short[rhs = c0 // Rord_{x₂, a₂ → 3}, 3], HL[CU[c0] == CU[rhs]]}]

$$\left\{ \mathbb{C}_{\text{CU}} \left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma \hbar t_1} (e^{\gamma \hbar t_1} \hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1} \hbar x_3 y_1)}{t_1}, 1 + e x_1 y_2 \right], \text{True} \right\}$$

With[{c0 = C_{CU} [{y₁, a₁, x₁}₁, {x₂, a₂, y₂}₂,
 ħ (l₁₁ t₁ a₁ + l₁₂ t₁ a₂ + l₂₁ t₂ a₁ + l₂₂ t₂ a₂ + γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
 1 + e (l₁ a₁ + l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)}],
 {Short[rhs = c0 // Rord_{x₂, a₂ → 3}, 3], HL[CU[c0] == CU[rhs]]}]

$$\left\{ \mathbb{C}_{\text{CU}} \left[\{y_1, a_1, x_1\}_1, \{\langle\langle 1 \rangle\rangle\}_2, \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle, 1 + e^{-\gamma \hbar (l_{12} t_1 + l_{22} t_2)} \in (e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 + p_{21} x_3 y_1 + e^{\langle\langle 1 \rangle\rangle} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \right], \text{True} \right\}$$

With[{q0 = C_{QU} [{y₁, a₁, x₁}₁, {x₂, a₂, y₂}₂,
 ħ (l₁₁ t₁ a₁ + l₁₂ t₁ a₂ + l₂₁ t₂ a₁ + l₂₂ t₂ a₂ + γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
 1 + e (l₁ a₁ + l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)}],
 {Short[rhs = q0 // Rord_{x₂, a₂ → 3}, 3], HL[QU[q0] == QU[rhs]]}]

$$\left\{ \mathbb{C}_{\text{QU}} \left[\{y_1, a_1, x_1\}_1, \{\langle\langle 1 \rangle\rangle\}_2, \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle, 1 + e^{-\gamma \hbar (l_{12} t_1 + l_{22} t_2)} \in (e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 + p_{21} x_3 y_1 + e^{\langle\langle 1 \rangle\rangle} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \right], \text{True} \right\}$$

With[{q0 = C_{QU} [{y₁, a₁, x₁}₁, {x₂, a₂, y₂}₂,
 ħ (l₁₁ t₁ a₁ + l₁₂ t₁ a₂ + l₂₁ t₂ a₁ + l₂₂ t₂ a₂ + γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
 1 + e (l₁ a₁ + l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)}],
 {Short[rhs = q0 // Rord_{a₂, y₂ → 3}, 3], HL[QU[q0] == QU[rhs]]}]

$$\left\{ \mathbb{C}_{\text{QU}} \left[\{y_1, a_1, x_1\}_1, \{\langle\langle 1 \rangle\rangle\}_2, \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle, 1 + e^{-\gamma \hbar (l_{12} t_1 + l_{22} t_2)} \in (e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma \hbar (l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 + e^{\langle\langle 1 \rangle\rangle} p_{21} x_2 y_1 + p_{12} x_1 y_3 + p_{22} x_2 y_3 - \gamma \hbar l_2 x_1 y_3 \gamma_{12} - \gamma \hbar l_2 x_2 y_3 \gamma_{22}) \right], \text{True} \right\}$$


```
With[{q0 = QU[{x1, y1}1, {x2, a2, y2}2,
  hbar (l12 t1 a2 + l22 t2 a2 + gamma11 x1 y1 + gamma12 x1 y2 + gamma21 x2 y1 + gamma22 x2 y2),
  1 + e (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2) ]},
{Short[rhs = q0 // Rord[x1, y1 -> 3, 5], HL@SimpT[QU[q0] == QU[rhs]]]}
{QU[{y3, a3, x3}1, {x2, a2, y2}2,
  hbar a2 l12 t1 + <<16>> + hbar T1 x2 y2 gamma11 gamma22,
  1 - gamma11 + T1 gamma11},
  1 / (1 - gamma11 + T1 gamma11) + (e (4 hbar a2 l2 + 4 p11 - 4 p11 T1 + 4 hbar p22 x2 y2 + <<339>> + gamma hbar^4 x2^2 y2^2 gamma12^2 gamma21^2 -
  4 gamma hbar^4 T1 x2^2 y2^2 gamma12^2 gamma21^2 + 3 gamma hbar^4 T1^2 x2^2 y2^2 gamma12^2 gamma21^2)) / (4 hbar (1 - gamma11 + T1 gamma11)^5)], True}
```

R in QU.

Faddeev-Quesne's formula:

Faddeev

```
e_{q-,k-}[x-] := e^{\sum_{j=1}^k \frac{(1-q)^j x^j}{j(1-q^j)}}; e_{q-,k}[x] := e_{q-,k}[x]
```

Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)}\right\}$$

Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]

{1, 1, 1, 1, 1, 1}

R

```
QU[R_{i-,j-}] := QU[{y1, a1}_i, {a2, x2}_j, SS[e^{hbar b1 a2} e_{q-/.qrule/.e->e}[hbar y1 x2] /. b1 -> gamma^{-1} (e a1 - t_i)]];
QU[R_{i-,j-}^{-1}] := S_j@QU[R_{i-,j-}];
```

QU[R_{3,4}] // Short

$$QU[] + \frac{e \hbar QU[a_3, a_4]}{\gamma} + \hbar QU[y_3, x_4] + \frac{\langle\langle 1 \rangle\rangle}{\gamma} + \langle\langle 1 \rangle\rangle - \frac{\langle\langle 1 \rangle\rangle}{\gamma} - \frac{e \langle\langle 3 \rangle\rangle}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4] t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ \$p=4, \$k=2):

QU[R_{1,2} ** R_{1,2}^{-1}] // Simp // HL // Timing

{0.03125, QU[]}

Verifying R3 (~156 secs @ \$p=4, \$k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]] // Timing
{0.1875, {QU[] +  $\frac{\epsilon \hbar \text{QU}[a_1, a_2]}{\gamma} + \ll 81 \gg + \text{QU}[y_1, a_3, x_3] \left( -\frac{\hbar^2 t_2}{\gamma} + \frac{\hbar^2 t_2 T_2}{\gamma} \right), \mathbf{0}} \}}$ 
```

R in \mathbb{C}_{QU} .

RinOE

```
 $\mathbb{C}_{\text{QU}}[R_{i,j}] := \mathbb{C}_{\text{QU}}[\{y_i, a_i, x_i\}_i, \{y_j, a_j, x_j\}_j, -\hbar \gamma^{-1} t_i a_j + \hbar y_i x_j, \text{Normal@Series}[$   

 $e^{\hbar \gamma^{-1} t_i a_j - \hbar y_i x_j} (e^{\hbar b_i a_j} e_q[\hbar y_i x_j] /. b_i \rightarrow \gamma^{-1} (e a_i - t_i)) /. \text{qrule}, \{e, \mathbf{0}, \$k\}] /. e \rightarrow \epsilon]$ 
```

```
 $\mathbb{C}_{\text{QU}}[R_{1,2}]$ 
```

```
 $\mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \frac{\epsilon \hbar a_1 a_2}{\gamma}]$ 
```

E

$E[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a 's, Q is a combination of $x_i y_j$, and P is a perturbation polynomial. It should be interpreted via $\text{CO}[E[\dots], \{x_1, a_1, y_1\}_j, \dots]$ (with some default for direct interpretation), or likewise via $\text{QO}[E[\dots], \{x_1, a_1, y_1\}_j, \dots]$. In themselves, CO and QO should have an interpretation in CU/QU by casting.

Alternative Algorithms

```
 $\lambda_{\text{alt}}[\text{CU}] := \text{If}[\$k == \mathbf{0}, \mathbf{1}, \text{Module}[\{\text{eq}, d, b, c, \text{so}\},$   

 $\text{eq} = \rho @ e^{\xi x_{\text{cu}}} . \rho @ e^{\eta y_{\text{cu}}} == \rho @ e^{d y_{\text{cu}}} . \rho @ e^{c (t^1_{\text{cu}} - 2 e a_{\text{cu}})} . \rho @ e^{b x_{\text{cu}}};$   

 $\{\text{so}\} = \text{Solve}[\text{Thread}[\text{Flatten} / @ \text{eq}], \{d, b, c\}] /. \text{C} @ \mathbf{1} \rightarrow \mathbf{0};$   

 $\text{Normal@Series}[e^{-\eta y - \xi x + \eta \xi t + c t + d y - 2 e c a + b x} /. \text{so}, \{\epsilon, \mathbf{0}, \$k\}]]];$ 
```

```
{ $\lambda_{\text{alt}}[\text{CU}], \text{HL@Simplify}[\lambda_{\text{alt}}[\text{CU}] == \text{Last}[\Delta_{\text{CU}}[\{\xi, \eta\}, \{x, y\}]]]}$ 
```

```
{ $1 + \left( 2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2 \right), \text{True}}$ 
```