Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$p = 2; $k = 1; (* $k can't be ∞ at least because of Faddeev-Quesne. *)
If[$k == 0, ϵ = 0, ϵ /: ϵ^k_ /; k > $k := 0]; (* $k=0 fails in Series[..{ϵ,...}] *)
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → ⅇ^(ℏ t_i), T → ⅇ^(ℏ t)};
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
   Collect[Normal@Series[ℰ, {ℏ, 0, $p}], ℏ, Together] ];
SST[ℰ_] :=
  Block[{ℏ, ϵ}, Collect[Normal@Series[ℰ /. TRule, {ℏ, 0, $p}], ℏ, Together] ];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, Collect[Normal@Series[#, {ℏ, 0, $p}], ℏ, Expand] &];
SimpT[ℰ_] :=
  Collect[ℰ, _CU | _QU, Collect[Normal@Series[# /. TRule, {ℏ, 0, $p}], ℏ, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, CE, pow,
   gs = Generators /. {opts}, cs = Centrals /. {opts} },
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Thread[gs → Range@Length@gs]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  CE[ℰ_] := Collect[ℰ, _U, (Expand[#] /. ħ^d_ /; d > $p ⧴ 0) &];
  U_i_[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ Replace[u, x_ ⧴ x_i, 1]};
  U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** 1_U := x; 1_U ** x_ := x;
  (a_. * x_U) ** (b_. * y_U) := If[a b === 0, 0, CE[a b (x ** y)]];
  U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
    U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List ⧴ l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
      ]] /. x_null ⧴ x];
  pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  S_i_[c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i] **
      U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]]; ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) ⧴ (m[U[g]] = img), {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]]; )
```

Meta-Operations

QLImplementation

```
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{0.859375,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T e^(-2 e a h)) / h, {a, 0, 3}]
```

Aside

$$\frac{1-T}{h} + 2\,e\,T\,a - 2\,(e^2\,h\,T)\,a^2 + \frac{4}{3}\,e^3\,h^2\,T\,a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q = SS[e^(γ ϵ ℏ)];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] = (q - 1) QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T e^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU = 𝕆_QU[{a, y}, SS[-T^-1 e^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU = 𝕆_QU[{a, x}, SS[-e^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\Big\{\Big\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\,QU[y],$$
$$\{QU[y], QU[x]\} \to \frac{(-1+T)\,QU[\,]}{\hbar} - 2\,T\,\epsilon\,QU[a] - \gamma\,\epsilon\,\hbar\,QU[y, x]\Big\},$$
$$\{\{QU[a], QU[y]\} \to -\gamma\,QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\,QU[x]\},$$
$$\Big\{\{QU[x], QU[y]\} \to \frac{(1-T)\,QU[\,]}{\hbar} + 2\,T\,\epsilon\,QU[a] + \gamma\,\epsilon\,\hbar\,QU[y, x],$$
$$\{QU[x], QU[a]\} \to -\gamma\,QU[x],\ \{QU[x], QU[x]\} \to 0\Big\}\Big\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

$$\Big\{8.60938,\ \Big\{\Big(\frac{28\,\gamma^4 - 56\,T\,\gamma^4 + 28\,T^2\,\gamma^4}{\hbar^2} + \frac{82\,\gamma^5\,\epsilon - 280\,\ll 3\gg + 198\,T^2\,\gamma^5\,\epsilon}{\hbar}\Big)\,QU[y, y, y, x, x] +$$
$$\ll 18\gg + (1 + 8\,\gamma\,\epsilon\,\hbar)\,QU[\ll 1\gg],\ 0\Big\}\Big\}$$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{{QU[y1], QU[y1]} → 0, {QU[y1], QU[a1]} → 0, {QU[y1], QU[x1]} → 0},
 {{QU[a1], QU[y1]} → 0, {QU[a1], QU[a1]} → 0, {QU[a1], QU[x1]} → 0},
 {{QU[x1], QU[y1]} → 0, {QU[x1], QU[a1]} → 0, {QU[x1], QU[x1]} → 0}}
```

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

```
{13.0156, {48 t γ^5 ϵ CU[y, y, y, x, x] + <<77>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
  2 (4 γ^5 ϵ/ℏ - 8 T γ^5 ϵ/ℏ + 4 T^2 γ^5 ϵ/ℏ) QU[y, y, y, x, x] +
   <<217>> + 8 γ ϵ ℏ QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆_QU[{a, x}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) x]],
  a → -a_QU, x → 𝕆_QU[{a, y}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

```
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$$\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],$$
$$QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\mathbb{AD}\$f = \gamma\left(\left(\mathrm{Cosh}\left[\hbar\left(a\,e + \frac{\gamma\,e}{2} - \frac{t}{2}\right)\right] - \mathrm{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,e}{2}\right)^2 + e\,\omega}\,\right]\right) \Big/ \left(\hbar\,e^{\hbar\,((a+\gamma)\,e - t/2)}\,\mathrm{Sinh}\left[\frac{\gamma\,e\,\hbar}{2}\right]\left(a^2\,e + a\,\gamma\,e - a\,t - \omega\right)\right)\right);$$

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {e → γ e, a → γ^-1 a, ω → γ^-1 ω})]
```

```
True
```

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, e → e / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

```
True
```

ADeq

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[A𝔻, QU → CU,
 {a → a_CU, x → CU@x, y → 𝕊_CU[SS[A𝔻$f] /. e → ϵ, a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

```
SD$g = Sqrt[((2 γ (Cosh[ħ/2 Sqrt[t^2 + γ^2 e^2 + 4 e ϖ]] - Cosh[(t - e γ - 2 e a)/(2/ħ)]))/
    (Sinh[γ e ħ/2] (t (2 a + γ) - 2 a (a + γ) e + 2 ϖ) ħ))];
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ħ ((e-t)/2 + e a)] - Cosh[ħ Sqrt[(t^2+e^2)/4 + e ϖ]])/
   (ħ Sinh[(-e ħ)/2] (ϖ - e a^2 + (t - e) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\text{Cosh}\left[\left(a e+\frac{e-t}{2}\right)\hbar\right]-\text{Cosh}\left[\sqrt{\frac{1}{4}\left(e^2+t^2\right)+e\varpi}\;\hbar\right]\right)\text{Csch}\left[\frac{e\hbar}{2}\right]\right)\Big/\left(\left(-a^2 e+\frac{t}{2}+a\left(-e+t\right)+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$

$$\left(4\left(-\text{Cosh}\left[\frac{1}{2}\sqrt{e^2+t^2+4e\varpi}\;\hbar\right]+\text{Cosh}\left[c e\hbar-\frac{t\hbar}{2}\right]\right)\text{Csch}\left[\frac{e\hbar}{2}\right]\right)\Big/\left(\left(\left(-1+4c^2\right)e-4\left(c t+\varpi\right)\right)\hbar\right),$$

$$\left.\text{True},\ \text{True}\right\}$$

SDeq

```
SD$f = Simplify[e^(ħ (t/2 - e a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
   x → S_CU[SS[SD$f] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** x_CU,
   y → S_CU[SS[SD$g] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

R in QU.

Faddeev-Quesne's formula:

Quesne

$$\mathbb{e}_{q\_,k\_}[x\_] := \mathbb{e}^\wedge\left(\sum_{j=1}^{k}\frac{(1-q)^j x^j}{j\,(1-q^j)}\right);\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
Table[Together@SeriesCoefficient[𝕖ρ,5[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1,\ 1,\ \frac{1}{1+\rho},\ \frac{1}{(1+\rho)\,(1+\rho+\rho^2)},\ \frac{1}{(1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)},\ 1\Big/\left((1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)\,(1+\rho+\rho^2+\rho^3+\rho^4)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, ρ] SeriesCoefficient[𝕖ρ,5[x], {x, 0, n}]], {n, 0, 5}]
```

```
{1, 1, 1, 1, 1, 1}
```

R

$$\mathrm{QU}[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[\{y_1, a_1\}_i,\ \{a_2, x_2\}_j,\ \mathrm{SS}\left[\mathbb{e}^{\hbar b_1 a_2}\,\mathbb{e}_q[\hbar\, y_1 x_2]\ /.\ b_1 \to \gamma^{-1}\,(\epsilon\, a_1 - t_i)\right]\right];$$
$$\mathrm{QU}\left[R^{-1}_{i\_,j\_}\right] := S_j\mathrm{@QU}[R_{i,j}];$$

```
QU[R3,4] // Short
```

$$\mathrm{QU}[\,] + \frac{\epsilon\,\hbar\,\mathrm{QU}[a_3, a_4]}{\gamma} + \hbar\,\mathrm{QU}[y_3, x_4] + \frac{\ll 1\gg}{\gamma} + \ll 1\gg - \frac{\ll 1\gg}{\gamma} - \frac{\epsilon\,\ll 3\gg}{\gamma^2} - \frac{\hbar^2\,\mathrm{QU}[y_3, a_4, x_4]\,t_3}{\gamma} + \frac{\hbar^2\,\mathrm{QU}[a_4, a_4]\,t_3^2}{2\,\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R1,2 ** R^-1 1,2] // Simp // HL // Timing
```

```
{0.09375, QU[]}
```

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$$\left\{0.421875,\ \left\{\mathrm{QU}[\,] + \frac{\epsilon\,\hbar\,\mathrm{QU}[a_1, a_2]}{\gamma} + \ll 85\gg + \mathrm{QU}[y_1, y_1, x_3, x_3]\left(\frac{\hbar^2}{2} - \hbar^2\,T_2 + \frac{1}{2}\,\hbar^2\,T_2^2\right),\ 0\right\}\right\}$$

The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = SS@(0 (1 - e^(-γ ϵ ħ)) / (ϵ ħ); 0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] :=
  (ℰ /. {t → γ ϵ, T → e^(ħ γ ϵ)} /. (U : CU | QU)[u___] :→ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // SS // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^Q P]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^Q P]];
```

```
HL[ρ[e^(ξ CU@x)].ρ[e^(α CU@a)] == ρ[e^(α CU@a)].ρ[e^(e^(-γ α) ξ CU@x)]]
```

```
True
```

Logos

```
Λ_U_[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-γ α) ξ x, 1];
Λ_U_[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-γ α) η y, 1];
```

```
{Λ_#[{ξ, α}, {x, a}], lhs = #@Œ_#[{x, a}, ħ (ξ x + α a), 1],
   HL[lhs == #@Λ_#[ħ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

```
{{Œ_CU[{a, x}, a α + e^(-α γ) x ξ, 1],
  CU[] + α ħ CU[a] + (ξ ħ - α γ ξ ħ^2) CU[x] + 1/2 α^2 ħ^2 CU[a, a] + α ξ ħ^2 CU[a, x] + 1/2 ξ^2 ħ^2 CU[x, x],
  True}, {Œ_QU[{a, x}, a α + e^(-α γ) x ξ, 1], QU[] + α ħ QU[a] + (ξ ħ - α γ ξ ħ^2) QU[x] +
  1/2 α^2 ħ^2 QU[a, a] + α ξ ħ^2 QU[a, x] + 1/2 ξ^2 ħ^2 QU[x, x], True}}
```

```
{Λ#[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{\mathbb{E}_{CU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1],$
$CU[\,] + \alpha\,\hbar\,CU[a] + (\eta\,\hbar - \alpha\,\gamma\,\eta\,\hbar^2)\,CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\,\eta\,\hbar^2\,CU[y, a] + \frac{1}{2}\eta^2\hbar^2\,CU[y, y],$
$\text{True}\}, \{\mathbb{E}_{QU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1], QU[\,] + \alpha\,\hbar\,QU[a] + (\eta\,\hbar - \alpha\,\gamma\,\eta\,\hbar^2)\,QU[y] +$
$\frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\,\eta\,\hbar^2\,QU[y, a] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y], \text{True}\}\}$

Goal. In either U, compute $F = e^{-\eta y}\,e^{\xi x}\,e^{\eta y}\,e^{-\xi x}$. First compute $G = e^{\xi x}\,y\,e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y}\,e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
With[{U = CU},
 Module[{G, F, fs, bs, e, b, es, sol},
  G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
  fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
  F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
  es = Flatten[
    Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
  sol = Echo@First[F /. DSolve[es, fs, η]];
  Echo[sol /. {e- → 1, U → Times}];
  Collect[sol /. {e- → 1, U → Times}, ϵ, Simplify]
 ]]
```

″ $-t\,\xi\,CU[\,] + 2\,\epsilon\,\xi\,CU[a] - \gamma\,\epsilon\,\xi^2\,CU[x] + CU[y]$

″ $\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta],$
$f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]\}$

″ $CU[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,CU[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,CU[x]\,f_{1,0,0,1}[\eta] + \epsilon\,CU[a]\,f_{1,0,1,0}[\eta] + \epsilon\,CU[a, x]\,f_{1,0,1,1}[\eta] +$
$\epsilon\,CU[y]\,f_{1,1,0,0}[\eta] + \epsilon\,CU[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,CU[y, a]\,f_{1,1,1,0}[\eta] + \epsilon\,CU[y, a, x]\,f_{1,1,1,1}[\eta]$

» $e^{-t\,\eta\,\xi}\,CU[\,] + \frac{1}{2}e^{-t\,\eta\,\xi}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,CU[\,] + 2\,e^{-t\,\eta\,\xi}\,\epsilon\,\eta\,\xi\,CU[a] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta\,\xi^2\,CU[x] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta^2\,\xi\,CU[y]$

» $1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\epsilon\,\eta^2\,\xi^2$

$1 + \frac{1}{2}\epsilon\,\eta\,\xi\,(4\,a + \gamma\,(-2\,y\,\eta - 2\,x\,\xi + t\,\eta\,\xi))$

Logos

```
ΛU_[{ξ1_, η1_}, {x, y}] := ΛU[{ξ1, η1}, {x, y}] = Module[{ξ, η, G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[xU, #]] &, yU, $k + 1]];
    fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1U /. η → 0, F ** G - yU ** F - ∂η F}}, {b, bs}]];
    F = F /. DSolve[es, fs, η][[1]];
    ŒU[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
      F /. {ϵ → 1, U → Times}
     ] /. {ξ → ξ1, η → η1}];
(*λ[U_]:=Last[ΛU[{ξ,η},{x,y}]];*)
```

```
{ΛCU[{ξ, η}, {x, y}], lhs = CU@ŒCU[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@ΛCU[ℏ {ξ, η}, {x, y}]]}
```

$\{\text{Œ}_{CU}[\{y, a, x\}, y\eta + x\xi - t\eta\xi, 1 + 2a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2}t\gamma\epsilon\eta^2\xi^2],$
$(1 - t\eta\xi\hbar^2)\,CU[\,] + 2\epsilon\eta\xi\hbar^2\,CU[a] + \xi\hbar\,CU[x] + \eta\hbar\,CU[y] +$
$\frac{1}{2}\xi^2\hbar^2\,CU[x, x] + \eta\xi\hbar^2\,CU[y, x] + \frac{1}{2}\eta^2\hbar^2\,CU[y, y], \text{True}\}$

```
{ΛQU[{ξ, η}, {x, y}], lhs = QU@ŒQU[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@ΛQU[ℏ {ξ, η}, {x, y}]]}
```

$\{\text{Œ}_{QU}[\{y, a, x\}, y\eta + x\xi + \frac{(1 - T)\eta\xi}{\hbar}, 1 + 2aT\epsilon\eta\xi - \frac{1}{2}(-1 + 3T)y\gamma\epsilon\eta^2\xi -$
$\frac{1}{2}(-1 + 3T)x\gamma\epsilon\eta\xi^2 + \frac{(-1 + T)(-1 + 3T)\gamma\epsilon\eta^2\xi^2}{4\hbar} + xy\gamma\epsilon\eta\xi\hbar],$
$(1 + \eta\xi\hbar - T\eta\xi\hbar)\,QU[\,] + 2T\epsilon\eta\xi\hbar^2\,QU[a] + \xi\hbar\,QU[x] + \eta\hbar\,QU[y] +$
$\frac{1}{2}\xi^2\hbar^2\,QU[x, x] + \eta\xi\hbar^2\,QU[y, x] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y], \text{True}\}$

```
{tt = Last[ΛCU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + 2a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2}t\gamma\epsilon\eta^2\xi^2, \epsilon\left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right)\}$

```
{tt = Last[ΛQU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + 2aT\epsilon\eta\xi - \frac{1}{2}(-1 + 3T)y\gamma\epsilon\eta^2\xi - \frac{1}{2}(-1 + 3T)x\gamma\epsilon\eta\xi^2 +$
$\frac{(-1 + T)(-1 + 3T)\gamma\epsilon\eta^2\xi^2}{4\hbar} + xy\gamma\epsilon\eta\xi\hbar, \frac{1}{4\hbar}\epsilon\,(\gamma\eta^2\xi^2 - 4T\gamma\eta^2\xi^2 + 3T^2\gamma\eta^2\xi^2 +$
$8aT\eta\xi\hbar + 2y\gamma\eta^2\xi\hbar - 6Ty\gamma\eta^2\xi\hbar + 2x\gamma\eta\xi^2\hbar - 6Tx\gamma\eta\xi^2\hbar + 4xy\gamma\eta\xi\hbar^2)\}$

Logos

```
Λ_U_[{υ_, ω_, 0}, {u_, w_}] := Λ_U[{υ, ω}, {u, w}];
Simp[Œ_U_[specs___, Q_, P_]] :=
  Œ_U[specs, ExpandNumerator@Together[Q], Collect[P, ϵ, ExpandNumerator@*Together]];
Λ_U_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
   {yax, q, p} = List @@ Λ_U[{υ, ω}, {u, w}];
   Œ_U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
     Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[p][e^Q]]] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}]
```

```
{Λ_CU[{ξ, η, δ}, {x, y}], lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL[lhs == CU@Λ_CU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{Œ_{CU}\Big[\{y, a, x\}, \frac{x y \delta + y \eta + x \xi - t \eta \xi}{1 + t \delta}, \frac{1}{1 + t \delta} + \frac{1}{2 (1 + t \delta)^5} \epsilon \big(4 a \delta + 12 a t \delta^2 + 4 a x y \delta^2 + 2 t \gamma \delta^2 - 8 x y \gamma \delta^2 + 12 a t^2 \delta^3 + 8 a t x y \delta^3 + 4 t^2 \gamma \delta^3 - 12 t x y \gamma \delta^3 - 4 x^2 y^2 \gamma \delta^3 + 4 a t^3 \delta^4 + 4 a t^2 x y \delta^4 + 2 t^3 \gamma \delta^4 - 4 t^2 x y \gamma \delta^4 - 3 t x^2 y^2 \gamma \delta^4 + 4 a y \delta \eta - 4 y \gamma \delta \eta + 8 a t y \delta^2 \eta - 4 t y \gamma \delta^2 \eta - 6 x y^2 \gamma \delta^2 \eta + 4 a t^2 y \delta^3 \eta - 4 t x y^2 \gamma \delta^3 \eta - 2 y^2 \gamma \delta \eta^2 - t y^2 \gamma \delta^2 \eta^2 + 4 a x \delta \xi - 4 x \gamma \delta \xi + 8 a t x \delta^2 \xi - 4 t x \gamma \delta^2 \xi - 6 x^2 y \gamma \delta^2 \xi + 4 a t^2 x \delta^3 \xi - 4 t x^2 y \gamma \delta^3 \xi + 4 a \eta \xi + 8 a t \delta \eta \xi + 4 t \gamma \delta \eta \xi - 8 x y \gamma \delta \eta \xi + 4 a t^2 \delta^2 \eta \xi + 4 t^2 \gamma \delta^2 \eta \xi - 4 t x y \gamma \delta^2 \eta \xi - 2 y \gamma \eta^2 \xi - 2 x^2 \gamma \delta \xi^2 - t x^2 \gamma \delta^2 \xi^2 - 2 x \gamma \eta \xi^2 + t \gamma \eta^2 \xi^2\big)\Big],$$

$$(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2)\, CU[] + (2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2)\, CU[a] +$$
$$(\xi \hbar - 2 t \delta \xi \hbar^2 - 2 \gamma \delta \epsilon \xi \hbar^2)\, CU[x] +$$
$$(\eta \hbar - 2 t \delta \eta \hbar^2 - 2 \gamma \delta \epsilon \eta \hbar^2)\, CU[y] +$$
$$4 \delta \epsilon \xi \hbar^2\, CU[a, x] +$$
$$\frac{1}{2} \xi^2 \hbar^2\, CU[x, x] +$$
$$4 \delta \epsilon \eta \hbar^2\, CU[y, a] +$$
$$(\delta \hbar - 2 t \delta^2 \hbar^2 - 4 \gamma \delta^2 \epsilon \hbar^2 + \eta \xi \hbar^2)\, CU[y, x] +$$
$$\frac{1}{2} \eta^2 \hbar^2\, CU[y, y] +$$
$$4 \delta^2 \epsilon \hbar^2\, CU[y, a, x] +$$
$$\delta \xi \hbar^2\, CU[y, x, x] +$$
$$\delta \eta \hbar^2\, CU[y, y, x] +$$
$$\frac{1}{2} \delta^2 \hbar^2\, CU[y, y, x, x], \text{True}\Big\}$$

```
{ΛQU[{ξ, η, δ}, {x, y}], lhs = QU@ŒQU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@ΛQU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{\text{Œ}_{QU}\Big[\{y,a,x\},\ \frac{\eta\,\xi - T\,\eta\,\xi + x\,y\,\delta\,\hbar + y\,\eta\,\hbar + x\,\xi\,\hbar}{-\delta + T\,\delta + \hbar},$$

$$\begin{aligned}
&\frac{\hbar}{-\delta+T\,\delta+\hbar} + \frac{1}{4\,(-\delta+T\,\delta+\hbar)^5}\in\Big(-8\,a\,T\,\delta^4\hbar^2 + 24\,a\,T^2\delta^4\hbar^2 - 24\,a\,T^3\delta^4\hbar^2 + 8\,a\,T^4\delta^4\hbar^2 + 2\,\gamma\,\delta^4\hbar^2 -\\
&12\,T\,\gamma\,\delta^4\hbar^2 + 24\,T^2\gamma\,\delta^4\hbar^2 - 20\,T^3\gamma\,\delta^4\hbar^2 + 6\,T^4\gamma\,\delta^4\hbar^2 + 24\,a\,T\,\delta^3\hbar^3 - 48\,a\,T^2\delta^3\hbar^3 +\\
&24\,a\,T^3\delta^3\hbar^3 - 4\,\gamma\,\delta^3\hbar^3 + 20\,T\,\gamma\,\delta^3\hbar^3 - 28\,T^2\gamma\,\delta^3\hbar^3 + 12\,T^3\gamma\,\delta^3\hbar^3 + 8\,a\,T\,x\,y\,\delta^4\hbar^3 -\\
&16\,a\,T^2x\,y\,\delta^4\hbar^3 + 8\,a\,T^3x\,y\,\delta^4\hbar^3 - 8\,T\,x\,y\,\gamma\,\delta^4\hbar^3 + 16\,T^2x\,y\,\gamma\,\delta^4\hbar^3 - 8\,T^3x\,y\,\gamma\,\delta^4\hbar^3 +\\
&8\,a\,T\,y\,\delta^3\eta\,\hbar^3 - 16\,a\,T^2y\,\delta^3\eta\,\hbar^3 + 8\,a\,T^3y\,\delta^3\eta\,\hbar^3 + 8\,a\,T\,x\,\delta^3\xi\,\hbar^3 - 16\,a\,T^2x\,\delta^3\xi\,\hbar^3 +\\
&8\,a\,T^3x\,\delta^3\xi\,\hbar^3 + 8\,a\,T\,\delta^2\eta\,\xi\,\hbar^3 - 16\,a\,T^2\delta^2\eta\,\xi\,\hbar^3 + 8\,a\,T^3\delta^2\eta\,\xi\,\hbar^3 - 4\,\gamma\,\delta^2\eta\,\xi\,\hbar^3 +\\
&20\,T\,\gamma\,\delta^2\eta\,\xi\,\hbar^3 - 28\,T^2\gamma\,\delta^2\eta\,\xi\,\hbar^3 + 12\,T^3\gamma\,\delta^2\eta\,\xi\,\hbar^3 - 24\,a\,T\,\delta^2\hbar^4 + 24\,a\,T^2\delta^2\hbar^4 +\\
&2\,\gamma\,\delta^2\hbar^4 - 8\,T\,\gamma\,\delta^2\hbar^4 + 6\,T^2\gamma\,\delta^2\hbar^4 - 16\,a\,T\,x\,y\,\delta^3\hbar^4 + 16\,a\,T^2x\,y\,\delta^3\hbar^4 + 24\,T\,x\,y\,\gamma\,\delta^3\hbar^4 -\\
&24\,T^2x\,y\,\gamma\,\delta^3\hbar^4 + x^2y^2\gamma\,\delta^4\hbar^4 + 4\,T\,x^2y^2\gamma\,\delta^4\hbar^4 - 5\,T^2x^2y^2\gamma\,\delta^4\hbar^4 - 16\,a\,T\,y\,\delta^2\eta\,\hbar^4 +\\
&16\,a\,T^2y\,\delta^2\eta\,\hbar^4 - 4\,y\,\gamma\,\delta^2\eta\,\hbar^4 + 16\,T\,y\,\gamma\,\delta^2\eta\,\hbar^4 - 12\,T^2y\,\gamma\,\delta^2\eta\,\hbar^4 + 8\,T\,x\,y^2\gamma\,\delta^3\eta\,\hbar^4 -\\
&8\,T^2x\,y^2\gamma\,\delta^3\eta\,\hbar^4 - y^2\gamma\,\delta^2\eta^2\hbar^4 + 4\,T\,y^2\gamma\,\delta^2\eta^2\hbar^4 - 3\,T^2y^2\gamma\,\delta^2\eta^2\hbar^4 - 16\,a\,T\,x\,\delta^2\xi\,\hbar^4 +\\
&16\,a\,T^2x\,\delta^2\xi\,\hbar^4 - 4\,x\,\gamma\,\delta^2\xi\,\hbar^4 + 16\,T\,x\,\gamma\,\delta^2\xi\,\hbar^4 - 12\,T^2x\,\gamma\,\delta^2\xi\,\hbar^4 + 8\,T\,x^2y\,\gamma\,\delta^3\xi\,\hbar^4 -\\
&8\,T^2x^2y\,\gamma\,\delta^3\xi\,\hbar^4 - 16\,a\,T\,\delta\,\eta\,\xi\,\hbar^4 + 16\,a\,T^2\delta\,\eta\,\xi\,\hbar^4 + 4\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 - 16\,T\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 +\\
&12\,T^2\gamma\,\delta\,\eta\,\xi\,\hbar^4 + 8\,T\,x\,y\,\gamma\,\delta^2\eta\,\xi\,\hbar^4 - 8\,T^2x\,y\,\gamma\,\delta^2\eta\,\xi\,\hbar^4 - x^2\gamma\,\delta^2\xi^2\hbar^4 + 4\,T\,x^2\gamma\,\delta^2\xi^2\hbar^4 -\\
&3\,T^2x^2\gamma\,\delta^2\xi^2\hbar^4 + \gamma\,\eta^2\xi^2\hbar^4 - 4\,T\,\gamma\,\eta^2\xi^2\hbar^4 + 3\,T^2\gamma\,\eta^2\xi^2\hbar^4 + 8\,a\,T\,\delta\,\hbar^5 + 8\,a\,T\,x\,y\,\delta^2\hbar^5 -\\
&4\,x\,y\,\gamma\,\delta^2\hbar^5 - 12\,T\,x\,y\,\gamma\,\delta^2\hbar^5 - 4\,x^2y^2\gamma\,\delta^3\hbar^5 - 4\,T\,x^2y^2\gamma\,\delta^3\hbar^5 + 8\,a\,T\,y\,\delta\,\eta\,\hbar^5 + 4\,y\,\gamma\,\delta\,\eta\,\hbar^5 -\\
&12\,T\,y\,\gamma\,\delta\,\eta\,\hbar^5 - 2\,x\,y^2\gamma\,\delta^2\eta\,\hbar^5 - 10\,T\,x\,y^2\gamma\,\delta^2\eta\,\hbar^5 + 2\,y^2\gamma\,\delta\,\eta^2\hbar^5 - 6\,T\,y^2\gamma\,\delta\,\eta^2\hbar^5 +\\
&8\,a\,T\,x\,\delta\,\xi\,\hbar^5 + 4\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 12\,T\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 2\,x^2y\,\gamma\,\delta^2\xi\,\hbar^5 - 10\,T\,x^2y\,\gamma\,\delta^2\xi\,\hbar^5 + 8\,a\,T\,\eta\,\xi\,\hbar^5 -\\
&16\,T\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^5 + 2\,y\,\gamma\,\eta^2\xi\,\hbar^5 - 6\,T\,y\,\gamma\,\eta^2\xi\,\hbar^5 + 2\,x^2\gamma\,\delta\,\xi^2\hbar^5 - 6\,T\,x^2\gamma\,\delta\,\xi^2\hbar^5 + 2\,x\,\gamma\,\eta\,\xi^2\hbar^5 -\\
&6\,T\,x\,\gamma\,\eta\,\xi^2\hbar^5 + 4\,x\,y\,\gamma\,\delta\,\hbar^6 + 4\,x^2y^2\gamma\,\delta^2\hbar^6 + 4\,x\,y^2\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
&\Big(1 + \delta - T\,\delta + \delta^2 - 2\,T\,\delta^2 + T^2\delta^2 + \frac{1}{2}\gamma\,\delta^2\in\hbar - 2\,T\,\gamma\,\delta^2\in\hbar + \frac{3}{2}T^2\gamma\,\delta^2\in\hbar + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\Big)\\
&\quad QU[\,] +\\
&\big(2\,T\,\delta\in\hbar + 4\,T\,\delta^2\in\hbar - 4\,T^2\delta^2\in\hbar + 2\,T\in\eta\,\xi\,\hbar^2\big)\\
&\quad QU[a] +\\
&\big(\xi\,\hbar + 2\,\delta\,\xi\,\hbar - 2\,T\,\delta\,\xi\,\hbar + \gamma\,\delta\in\xi\,\hbar^2 - 3\,T\,\gamma\,\delta\in\xi\,\hbar^2\big)\\
&\quad QU[x] +\\
&\big(\eta\,\hbar + 2\,\delta\,\eta\,\hbar - 2\,T\,\delta\,\eta\,\hbar + \gamma\,\delta\in\eta\,\hbar^2 - 3\,T\,\gamma\,\delta\in\eta\,\hbar^2\big)\\
&\quad QU[y] +\\
&4\,T\,\delta\in\xi\,\hbar^2\,QU[a,x] + \frac{1}{2}\xi^2\hbar^2\,QU[x,x] +\\
&4\,T\,\delta\in\\
&\quad \eta\,\hbar^2\,QU[y,a] +\\
&\big(\delta\,\hbar + 2\,\delta^2\hbar - 2\,T\,\delta^2\hbar + \gamma\,\delta\in\hbar^2 + 4\,\gamma\,\delta^2\in\hbar^2 - 8\,T\,\gamma\,\delta^2\in\hbar^2 + \eta\,\xi\,\hbar^2\big)\\
&\quad QU[y,x] +\\
&\frac{1}{2}\eta^2\hbar^2\,QU[y,y] + 4\,T\,\delta^2\in\hbar^2\,QU[y,a,x] +\\
&\delta\,\xi\,\hbar^2\,QU[y,x,x] +\\
&\delta\,\eta\,\hbar^2\,QU[y,y,x] +\\
&\frac{1}{2}\delta^2\hbar^2\,QU[y,y,x,x],\ \text{True}\Big\}
\end{aligned}$$

```
{tt = Last[ΛCU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\Big\{\frac{1}{1+t\,\delta}+\frac{1}{2\,(1+t\,\delta)^5}\,\epsilon\,\big(4\,a\,\delta+12\,a\,t\,\delta^2+4\,a\,x\,y\,\delta^2+2\,t\,\gamma\,\delta^2-8\,x\,y\,\gamma\,\delta^2+12\,a\,t^2\,\delta^3+8\,a\,t\,x\,y\,\delta^3+4\,t^2\,\gamma\,\delta^3-12\,t\,x\,y\,\gamma\,\delta^3-4\,x^2\,y^2\,\gamma\,\delta^3+4\,a\,t^3\,\delta^4+4\,a\,t^2\,x\,y\,\delta^4+2\,t^3\,\gamma\,\delta^4-4\,t^2\,x\,y\,\gamma\,\delta^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4+4\,a\,y\,\delta\,\eta-4\,y\,\gamma\,\delta\,\eta+8\,a\,t\,y\,\delta^2\,\eta-4\,t\,y\,\gamma\,\delta^2\,\eta-6\,x\,y^2\,\gamma\,\delta^2\,\eta+4\,a\,t^2\,y\,\delta^3\,\eta-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta-2\,y^2\,\gamma\,\delta\,\eta^2-t\,y^2\,\gamma\,\delta^2\,\eta^2+4\,a\,x\,\delta\,\xi-4\,x\,\gamma\,\delta\,\xi+8\,a\,t\,x\,\delta^2\,\xi-4\,t\,x\,\gamma\,\delta^2\,\xi-6\,x^2\,y\,\gamma\,\delta^2\,\xi+4\,a\,t^2\,x\,\delta^3\,\xi-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi+4\,a\,\eta\,\xi+8\,a\,t\,\delta\,\eta\,\xi+4\,t\,\gamma\,\delta\,\eta\,\xi-8\,x\,y\,\gamma\,\delta\,\eta\,\xi+4\,a\,t^2\,\delta^2\,\eta\,\xi+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x^2\,\gamma\,\delta\,\xi^2-t\,x^2\,\gamma\,\delta^2\,\xi^2-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\big),$$

$$\frac{1}{2\,(1+t\,\delta)^4}\,\epsilon\,\big(4\,a\,\delta+12\,a\,t\,\delta^2+4\,a\,x\,y\,\delta^2+2\,t\,\gamma\,\delta^2-8\,x\,y\,\gamma\,\delta^2+12\,a\,t^2\,\delta^3+8\,a\,t\,x\,y\,\delta^3+4\,t^2\,\gamma\,\delta^3-12\,t\,x\,y\,\gamma\,\delta^3-4\,x^2\,y^2\,\gamma\,\delta^3+4\,a\,t^3\,\delta^4+4\,a\,t^2\,x\,y\,\delta^4+2\,t^3\,\gamma\,\delta^4-4\,t^2\,x\,y\,\gamma\,\delta^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4+4\,a\,y\,\delta\,\eta-4\,y\,\gamma\,\delta\,\eta+8\,a\,t\,y\,\delta^2\,\eta-4\,t\,y\,\gamma\,\delta^2\,\eta-6\,x\,y^2\,\gamma\,\delta^2\,\eta+4\,a\,t^2\,y\,\delta^3\,\eta-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta-2\,y^2\,\gamma\,\delta\,\eta^2-t\,y^2\,\gamma\,\delta^2\,\eta^2+4\,a\,x\,\delta\,\xi-4\,x\,\gamma\,\delta\,\xi+8\,a\,t\,x\,\delta^2\,\xi-4\,t\,x\,\gamma\,\delta^2\,\xi-6\,x^2\,y\,\gamma\,\delta^2\,\xi+4\,a\,t^2\,x\,\delta^3\,\xi-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi+4\,a\,\eta\,\xi+8\,a\,t\,\delta\,\eta\,\xi+4\,t\,\gamma\,\delta\,\eta\,\xi-8\,x\,y\,\gamma\,\delta\,\eta\,\xi+4\,a\,t^2\,\delta^2\,\eta\,\xi+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x^2\,\gamma\,\delta\,\xi^2-t\,x^2\,\gamma\,\delta^2\,\xi^2-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\big)+\mathrm{Log}\Big[\frac{1}{1+t\,\delta}\Big]\Big\}$$

```
{tt = Last[Λ_QU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$
\begin{aligned}
&\Big\{\frac{\hbar}{-\delta+T\delta+\hbar}+\\
&\quad\frac{1}{4\,(-\delta+T\delta+\hbar)^5}\,\epsilon\,\Big(-8aT\delta^4\hbar^2+24aT^2\delta^4\hbar^2-24aT^3\delta^4\hbar^2+8aT^4\delta^4\hbar^2+2\gamma\delta^4\hbar^2-12T\gamma\delta^4\hbar^2+\\
&\qquad 24T^2\gamma\delta^4\hbar^2-20T^3\gamma\delta^4\hbar^2+6T^4\gamma\delta^4\hbar^2+24aT\delta^3\hbar^3-48aT^2\delta^3\hbar^3+24aT^3\delta^3\hbar^3-\\
&\qquad 4\gamma\delta^3\hbar^3+20T\gamma\delta^3\hbar^3-28T^2\gamma\delta^3\hbar^3+12T^3\gamma\delta^3\hbar^3+8aTxy\delta^4\hbar^3-16aT^2xy\delta^4\hbar^3+\\
&\qquad 8aT^3xy\delta^4\hbar^3-8Txy\gamma\delta^4\hbar^3+16T^2xy\gamma\delta^4\hbar^3-8T^3xy\gamma\delta^4\hbar^3+8aTy\delta^3\eta\hbar^3-\\
&\qquad 16aT^2y\delta^3\eta\hbar^3+8aT^3y\delta^3\eta\hbar^3+8aTx\delta^3\xi\hbar^3-16aT^2x\delta^3\xi\hbar^3+8aT^3x\delta^3\xi\hbar^3+\\
&\qquad 8aT\delta^2\eta\xi\hbar^3-16aT^2\delta^2\eta\xi\hbar^3+8aT^3\delta^2\eta\xi\hbar^3-4\gamma\delta^2\eta\xi\hbar^3+20T\gamma\delta^2\eta\xi\hbar^3-\\
&\qquad 28T^2\gamma\delta^2\eta\xi\hbar^3+12T^3\gamma\delta^2\eta\xi\hbar^3-24aT\delta^2\hbar^4+24aT^2\delta^2\hbar^4+2\gamma\delta^2\hbar^4-8T\gamma\delta^2\hbar^4+\\
&\qquad 6T^2\gamma\delta^2\hbar^4-16aTxy\delta^3\hbar^4+16aT^2xy\delta^3\hbar^4+24Txy\gamma\delta^3\hbar^4-24T^2xy\gamma\delta^3\hbar^4+\\
&\qquad x^2y^2\gamma\delta^4\hbar^4+4Tx^2y^2\gamma\delta^4\hbar^4-5T^2x^2y^2\gamma\delta^4\hbar^4-16aTy\delta^2\eta\hbar^4+16aT^2y\delta^2\eta\hbar^4-\\
&\qquad 4y\gamma\delta^2\eta\hbar^4+16Ty\gamma\delta^2\eta\hbar^4-12T^2y\gamma\delta^2\eta\hbar^4+8Txy^2\gamma\delta^3\eta\hbar^4-8T^2xy^2\gamma\delta^3\eta\hbar^4-\\
&\qquad y^2\gamma\delta^2\eta^2\hbar^4+4Ty^2\gamma\delta^2\eta^2\hbar^4-3T^2y^2\gamma\delta^2\eta^2\hbar^4-16aTx\delta^2\xi\hbar^4+16aT^2x\delta^2\xi\hbar^4-\\
&\qquad 4x\gamma\delta^2\xi\hbar^4+16Tx\gamma\delta^2\xi\hbar^4-12T^2x\gamma\delta^2\xi\hbar^4+8Tx^2y\gamma\delta^3\xi\hbar^4-8T^2x^2y\gamma\delta^3\xi\hbar^4-\\
&\qquad 16aT\delta\eta\xi\hbar^4+16aT^2\delta\eta\xi\hbar^4+4\gamma\delta\eta\xi\hbar^4-16T\gamma\delta\eta\xi\hbar^4+12T^2\gamma\delta\eta\xi\hbar^4+\\
&\qquad 8Txy\gamma\delta^2\eta\xi\hbar^4-8T^2xy\gamma\delta^2\eta\xi\hbar^4-x^2\gamma\delta^2\xi^2\hbar^4+4Tx^2\gamma\delta^2\xi^2\hbar^4-3T^2x^2\gamma\delta^2\xi^2\hbar^4+\\
&\qquad \gamma\eta^2\xi^2\hbar^4-4T\gamma\eta^2\xi^2\hbar^4+3T^2\gamma\eta^2\xi^2\hbar^4+8aT\delta\hbar^5+8aTxy\delta^2\hbar^5-4xy\gamma\delta^2\hbar^5-\\
&\qquad 12Txy\gamma\delta^2\hbar^5-4x^2y^2\gamma\delta^3\hbar^5-4Tx^2y^2\gamma\delta^3\hbar^5+8aTy\delta\eta\hbar^5+4y\gamma\delta\eta\hbar^5-\\
&\qquad 12Ty\gamma\delta\eta\hbar^5-2xy^2\gamma\delta^2\eta\hbar^5-10Txy^2\gamma\delta^2\eta\hbar^5+2y^2\gamma\delta\eta^2\hbar^5-6Ty^2\gamma\delta\eta^2\hbar^5+\\
&\qquad 8aTx\delta\xi\hbar^5+4x\gamma\delta\xi\hbar^5-12Tx\gamma\delta\xi\hbar^5-2x^2y\gamma\delta^2\xi\hbar^5-10Tx^2y\gamma\delta^2\xi\hbar^5+8aT\eta\xi\hbar^5-\\
&\qquad 16Txy\gamma\delta\eta\xi\hbar^5+2y\gamma\eta^2\xi\hbar^5-6Ty\gamma\eta^2\xi\hbar^5+2x^2\gamma\delta\xi^2\hbar^5-6Tx^2\gamma\delta\xi^2\hbar^5+2x\gamma\eta\xi^2\hbar^5-\\
&\qquad 6Tx\gamma\eta\xi^2\hbar^5+4xy\gamma\delta\hbar^6+4x^2y^2\gamma\delta^2\hbar^6+4xy^2\gamma\delta\eta\hbar^6+4x^2y\gamma\delta\xi\hbar^6+4xy\gamma\eta\xi\hbar^6\Big),\\
&\frac{1}{4\,(-\delta+T\delta+\hbar)^4}\,\epsilon\,\Big(-8aT\delta^4\hbar+24aT^2\delta^4\hbar-24aT^3\delta^4\hbar+8aT^4\delta^4\hbar+2\gamma\delta^4\hbar-12T\gamma\delta^4\hbar+\\
&\qquad 24T^2\gamma\delta^4\hbar-20T^3\gamma\delta^4\hbar+6T^4\gamma\delta^4\hbar+24aT\delta^3\hbar^2-48aT^2\delta^3\hbar^2+24aT^3\delta^3\hbar^2-4\gamma\delta^3\hbar^2+\\
&\qquad 20T\gamma\delta^3\hbar^2-28T^2\gamma\delta^3\hbar^2+12T^3\gamma\delta^3\hbar^2+8aTxy\delta^4\hbar^2-16aT^2xy\delta^4\hbar^2+8aT^3xy\delta^4\hbar^2-\\
&\qquad 8Txy\gamma\delta^4\hbar^2+16T^2xy\gamma\delta^4\hbar^2-8T^3xy\gamma\delta^4\hbar^2+8aTy\delta^3\eta\hbar^2-16aT^2y\delta^3\eta\hbar^2+\\
&\qquad 8aT^3y\delta^3\eta\hbar^2+8aTx\delta^3\xi\hbar^2-16aT^2x\delta^3\xi\hbar^2+8aT^3x\delta^3\xi\hbar^2+8aT\delta^2\eta\xi\hbar^2-\\
&\qquad 16aT^2\delta^2\eta\xi\hbar^2+8aT^3\delta^2\eta\xi\hbar^2-4\gamma\delta^2\eta\xi\hbar^2+20T\gamma\delta^2\eta\xi\hbar^2-28T^2\gamma\delta^2\eta\xi\hbar^2+\\
&\qquad 12T^3\gamma\delta^2\eta\xi\hbar^2-24aT\delta^2\hbar^3+24aT^2\delta^2\hbar^3+2\gamma\delta^2\hbar^3-8T\gamma\delta^2\hbar^3+6T^2\gamma\delta^2\hbar^3-\\
&\qquad 16aTxy\delta^3\hbar^3+16aT^2xy\delta^3\hbar^3+24Txy\gamma\delta^3\hbar^3-24T^2xy\gamma\delta^3\hbar^3+x^2y^2\gamma\delta^4\hbar^3+\\
&\qquad 4Tx^2y^2\gamma\delta^4\hbar^3-5T^2x^2y^2\gamma\delta^4\hbar^3-16aTy\delta^2\eta\hbar^3+16aT^2y\delta^2\eta\hbar^3-4y\gamma\delta^2\eta\hbar^3+\\
&\qquad 16Ty\gamma\delta^2\eta\hbar^3-12T^2y\gamma\delta^2\eta\hbar^3+8Txy^2\gamma\delta^3\eta\hbar^3-8T^2xy^2\gamma\delta^3\eta\hbar^3-y^2\gamma\delta^2\eta^2\hbar^3+\\
&\qquad 4Ty^2\gamma\delta^2\eta^2\hbar^3-3T^2y^2\gamma\delta^2\eta^2\hbar^3-16aTx\delta^2\xi\hbar^3+16aT^2x\delta^2\xi\hbar^3-4x\gamma\delta^2\xi\hbar^3+\\
&\qquad 16Tx\gamma\delta^2\xi\hbar^3-12T^2x\gamma\delta^2\xi\hbar^3+8Tx^2y\gamma\delta^3\xi\hbar^3-8T^2x^2y\gamma\delta^3\xi\hbar^3-16aT\delta\eta\xi\hbar^3+\\
&\qquad 16aT^2\delta\eta\xi\hbar^3+4\gamma\delta\eta\xi\hbar^3-16T\gamma\delta\eta\xi\hbar^3+12T^2\gamma\delta\eta\xi\hbar^3+8Txy\gamma\delta^2\eta\xi\hbar^3-\\
&\qquad 8T^2xy\gamma\delta^2\eta\xi\hbar^3-x^2\gamma\delta^2\xi^2\hbar^3+4Tx^2\gamma\delta^2\xi^2\hbar^3-3T^2x^2\gamma\delta^2\xi^2\hbar^3+\gamma\eta^2\xi^2\hbar^3-\\
&\qquad 4T\gamma\eta^2\xi^2\hbar^3+3T^2\gamma\eta^2\xi^2\hbar^3+8aT\delta\hbar^4+8aTxy\delta^2\hbar^4-4xy\gamma\delta^2\hbar^4-12Txy\gamma\delta^2\hbar^4-\\
&\qquad 4x^2y^2\gamma\delta^3\hbar^4-4Tx^2y^2\gamma\delta^3\hbar^4+8aTy\delta\eta\hbar^4+4y\gamma\delta\eta\hbar^4-12Ty\gamma\delta\eta\hbar^4-2xy^2\gamma\delta^2\eta\hbar^4-\\
&\qquad 10Txy^2\gamma\delta^2\eta\hbar^4+2y^2\gamma\delta\eta^2\hbar^4-6Ty^2\gamma\delta\eta^2\hbar^4+8aTx\delta\xi\hbar^4+4x\gamma\delta\xi\hbar^4-12Tx\gamma\delta\xi\hbar^4-\\
&\qquad 2x^2y\gamma\delta^2\xi\hbar^4-10Tx^2y\gamma\delta^2\xi\hbar^4+8aT\eta\xi\hbar^4-16Txy\gamma\delta\eta\xi\hbar^4+2y\gamma\eta^2\xi\hbar^4-\\
&\qquad 6Ty\gamma\eta^2\xi\hbar^4+2x^2\gamma\delta\xi^2\hbar^4-6Tx^2\gamma\delta\xi^2\hbar^4+2x\gamma\eta\xi^2\hbar^4-6Tx\gamma\eta\xi^2\hbar^4+4xy\gamma\delta\hbar^5+\\
&\qquad 4x^2y^2\gamma\delta^2\hbar^5+4xy^2\gamma\delta\eta\hbar^5+4x^2y\gamma\delta\xi\hbar^5+4xy\gamma\eta\xi\hbar^5\Big)+\mathrm{Log}\Big[\frac{\hbar}{-\delta+T\delta+\hbar}\Big]\Big\}
\end{aligned}
$$

Reorderings with Rord

SW

```
SW_{x_i_,a_j_}[Œ_U_[OrderlessPatternSequence[{lh___, x_i_, a_j_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = e^(-γ α) ξ x_i + α a_j},
  Œ_U[{lh, a_j, x_i, rh}_s, more, e^(-γ α) ξ x_i + (Q /. x_i → 0), e^-q DP_{x_i→D_ξ, a_j→D_α}[P][e^q]] /.
   {α → ∂_{a_j} Q, ξ → ∂_{x_i} Q}]
```

```
c⊙ = Œ_CU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (e^t1 - 1) y1 x2, 1 + ϵ x1 y2]
```

$\text{Œ}_{CU}\left[\{y_1, x_1\}_1, \{x_2, a_2, y_2\}_2, \hbar a_2 t_1 + \frac{(-1+e^{t_1})\hbar x_2 y_1}{t_1}, 1 + \in x_1 y_2\right]$

```
SW_{x2,a2}[c⊙]
```

$\text{Œ}_{CU}\left[\{a_2, x_2, y_2\}_2, \{y_1, x_1\}_1, \hbar a_2 t_1 + \frac{e^{-\gamma\hbar t_1}(-1+e^{t_1})\hbar x_2 y_1}{t_1}, 1 + \in x_1 y_2\right]$

```
With[{c⊙ = Œ_CU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (e^t1 - 1) y1 x2, 1 + ϵ x1 y2]},
 HL[CU[c⊙] == CU[c⊙ // SW_{x2,a2}]]]
```

True

```
With[{c⊙ = Œ_CU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {CU[c⊙] // Short, HL[CU[c⊙] == CU[c⊙ // SW_{x2,a2}]]}
]
```

$\{CU[a_1, a_1, a_1] \left(\frac{1}{2}\in\hbar^2 l_1 l_{11}^2 t_1^2 + \in\hbar^2 l_1 l_{11} l_{21} t_1 t_2 + \frac{1}{2}\in\hbar^2 l_1 l_{21}^2 t_2^2\right) + \ll 75 \gg + CU[\,]\ (\ll 1 \gg),$

True}

```
With[{q⊙ = Œ_QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {QU[q⊙] // Short, HL[QU[q⊙] == QU[q⊙ // SW_{x2,a2}]]}
]
```

$\{QU[a_1, a_1, a_1] \left(\frac{1}{2}\in\hbar^2 l_1 l_{11}^2 t_1^2 + \in\hbar^2 l_1 l_{11} l_{21} t_1 t_2 + \frac{1}{2}\in\hbar^2 l_1 l_{21}^2 t_2^2\right) + \ll 75 \gg + QU[\,]\ (\ll 1 \gg),$

True}

SW

```
SW_{a_j_,y_i_}[Œ_U_[OrderlessPatternSequence[{lh___, a_j_, y_i_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = e^(-γ α) η y_i + α a_j},
  Œ_U[{lh, y_i, a_j, rh}_s, more, e^(-γ α) η y_i + (Q /. y_i → 0), e^-q DP_{y_i→D_η, a_j→D_α}[P][e^q]] /.
   {α → ∂_{a_j} Q, η → ∂_{y_i} Q}]
```

```
With[{q⊕ = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {QU[q⊕] // Short, HL@Simp[QU[q⊕] - QU[q⊕ // SW_{a2,y2}]]}
]
```

$\{\mathrm{QU}[a_1, a_1, a_1]\left(\frac{1}{2}\epsilon\hbar^2 l_1 l_{11}^2 t_1^2 + \epsilon\hbar^2 l_1 l_{11} l_{21} t_1 t_2 + \frac{1}{2}\epsilon\hbar^2 l_1 l_{21}^2 t_2^2\right) + \ll 75 \gg + \mathrm{QU}[\,](\ll 1 \gg), 0\}$

SW

```
SW_{x_i_,y_j_→k_}[𝔼CU[OrderlessPatternSequence[{lh___, x_i_, y_j_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = ν (ξ x_k + η y_k + δ x_k y_k - t_k ξ η)},
  𝔼CU[{lh, y_k, a_k, x_k, rh}_s, more, q + (Q /. x_i | y_j → 0),
    ⅇ^-q DP_{x_i→D_ξ, y_j→D_η}[P][Λ[CU, t_k, T_k, y_k, a_k, x_k, ξ, η, δ] ⅇ^q]]
   /. ν → (1 + t_k δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]
```

```
With[{c⊕ = 𝔼CU[{x1, y1}1, {x2, a2, y2}2,
    ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {CU[c⊕] // Short, HL[CU[c⊕] == CU[c⊕ // SW_{x1,y1→1}]]}
]
```

$\{\mathrm{CU}[a_2, a_2, a_2]\left(\frac{1}{2}\epsilon\hbar^2 l_2 l_{12}^2 t_1^2 + \epsilon\hbar^2 l_2 l_{12} l_{22} t_1 t_2 + \frac{1}{2}\epsilon\hbar^2 l_2 l_{22}^2 t_2^2\right) + \ll 54 \gg + \mathrm{CU}[\,](\ll 1 \gg),$ True$\}$

SW

```
SW_{x_i_,y_j_→k_}[𝔼QU[OrderlessPatternSequence[{lh___, x_i_, y_j_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = ν (ξ x_k + η y_k + δ x_k y_k - ℏ^-1 (T_k - 1) ξ η)},
  𝔼QU[{lh, y_k, a_k, x_k, rh}_s, more, q + (Q /. x_i | y_j → 0),
    ⅇ^-q DP_{x_i→D_ξ, y_j→D_η}[P][Λ[QU, t_k, T_k, y_k, a_k, x_k, ξ, η, δ] ⅇ^q]]
    /. ν → (1 + ℏ^-1 (T_k - 1) δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]
```

```
With[
 {q⊕ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2, ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {QU[q⊕] // Short, HL[err = SimpT[QU[q⊕] - QU[q⊕ // SW_{x1,y1→1}]]]}
]
```

$\{\mathrm{QU}[a_2, a_2, a_2]\left(\frac{1}{2}\epsilon\hbar^2 l_2 l_{12}^2 t_1^2 + \epsilon\hbar^2 l_2 l_{12} l_{22} t_1 t_2 + \frac{1}{2}\epsilon\hbar^2 l_2 l_{22}^2 t_2^2\right) + \ll 54 \gg + \mathrm{QU}[\,](\ll 1 \gg), 0\}$

Rewrite Rules

RR: Rewrite Rule. RQ: Revised Quadratic.

RR

```
RR[{u_i_, w_j_} → {vs__, k_}, {𝓋_, ω_}, RQ_, λ_][(𝒪 : C𝕆 | Q𝕆)[
    OrderlessPatternSequence[{lh___, u_i_, w_j_, rh___}s_, more___, 𝔼[Q_, P_]]]] :=
  𝒪[{lh, Sequence @@ (#k & /@ {vs}), rh}s, more, 𝔼[
    (RQ /. (v : u | w | t | T) :→ vk) + (Q /. ui | wj → 0),
    e^-RQ DP_{ui→D𝓋, wj→Dω}[P][Λ[𝒪, tk, Tk, yk, ak, xk, 𝓋, ω, δ] e^RQ] /.
     {𝓋 → (∂_{𝓋i} Q /. ωj → 0), ω → (∂_{ωj} Q /. 𝓋i → 0), δ → ∂_{𝓋i,ωj} Q}
   ]];
```

𝔼

𝔼[L, Q, P] means $e^{\hbar(L+Q)} P$, where L is linear in the a's, Q is a combination of $x_i\, y_j$, and P is a perturbation polynomial. It should be interpreted via C𝕆[𝔼[...], $\{x_1,\ a_1,\ y_1\}_i$, ...] (with some default for direct interpretation), or likewise via Q𝕆[𝔼[...], $\{x_1,\ a_1,\ y_1\}_i$, ...]. In themselves, C𝕆 and Q𝕆 should have an interpretation in CU/QU by casting.

Alternative Algorithms

```
λalt[CU] := Module[{eq, d, b, c, so},
   eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
   {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
   Normal@Series[e^(-η y-ξ x+η ξ t+c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, $k}]];
```

```
{λalt[CU], HL[Simplify[λalt[CU] == λ[CU]]]}
```

$$\left\{1+\epsilon\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right),\ \text{True}\right\}$$