Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$p = 2; $k = 2; (* $k can't be ∞ at least because of Faddeev-Quesne. *)
If[$k == 0, ϵ = 0, ϵ /: ϵ^k_ /; k > $k := 0]; (* $k=0 fails in Series[..{ϵ,...}] *)
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ℏ t_i), T → e^(ℏ t)};
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
   Collect[Normal@Series[ℰ, {ℏ, 0, $p}], ℏ, Together] ];
SST[ℰ_] :=
  Block[{ℏ, ϵ}, Collect[Normal@Series[ℰ /. TRule, {ℏ, 0, $p}], ℏ, Together] ];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, Collect[Normal@Series[#, {ℏ, 0, $p}], ℏ, Expand] &];
SimpT[ℰ_] :=
  Collect[ℰ, _CU | _QU, Collect[Normal@Series[# /. TRule, {ℏ, 0, $p}], ℏ, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :> c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, CE, pow,
    gs = Generators /. {opts}, cs = Centrals /. {opts} },
   (#U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Thread[gs → Range@Length@gs]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   CE[ℰ_] := Collect[ℰ, _U, (Expand[#] /. ℏ^d_ /; d > $p ⧴ 0) &];
   U_i_[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ Replace[u, x_ ⧴ x_i, 1]};
   U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** 1_U := x; 1_U ** x_ := x;
   (a_. * x_U) ** (b_. * y_U) := If[a b === 0, 0, CE[a b (x ** y)]];
   U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
     U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List ⧴ l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
       ]] /. x_null ⧴ x];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
      CoefficientRules[ℰ, First /@ {ss}] /.
       (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   S_i_[c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i] **
      U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]]; ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
   Replace[ongs, (g_ → img_) ⧴ (m[U[g]] = img), {1}];
   m[1_U] = 1_V;
   m[U[g_i_]] := V_i[m[U@g]];
   m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
   m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]]; )
```

Meta-Operations

QLImplementation

```
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{0.859375,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y_1, a_1, x_1}},
 Table[HL@Simp[S_1[z1 ** z2] - S_1[z2] ** S_1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y_1, a_1, x_1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S_1@S_1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T e^(-2 e a h)) / h, {a, 0, 3}]
```

Aside

$$\frac{1-T}{h} + 2\,e\,T\,a - 2\,(e^2\,h\,T)\,a^2 + \frac{4}{3}\,e^3\,h^2\,T\,a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q = SS[e^(γ ϵ ℏ)];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] = (q - 1) QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T e^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU = 𝕆_QU[{a, y}, SS[-T^-1 e^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU = 𝕆_QU[{a, x}, SS[-e^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\Big\{\Big\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\,QU[y],\ \{QU[y], QU[x]\} \to \frac{(-1+T)\,QU[]}{\hbar} - 2\,T\,\epsilon\,QU[a] - \gamma\,\epsilon\,\hbar\,QU[y,x]\Big\},$$
$$\{\{QU[a], QU[y]\} \to -\gamma\,QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\,QU[x]\},$$
$$\Big\{\{QU[x], QU[y]\} \to \frac{(1-T)\,QU[]}{\hbar} + 2\,T\,\epsilon\,QU[a] + \gamma\,\epsilon\,\hbar\,QU[y,x],\ \{QU[x], QU[a]\} \to -\gamma\,QU[x],\ \{QU[x], QU[x]\} \to 0\Big\}\Big\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

$$\Big\{8.60938,\ \Big\{\Big(\frac{28\,\gamma^4 - 56\,T\,\gamma^4 + 28\,T^2\,\gamma^4}{\hbar^2} + \frac{82\,\gamma^5\,\epsilon - 280 \ll 3\gg + 198\,T^2\,\gamma^5\,\epsilon}{\hbar}\Big)\,QU[y,y,y,x,x] + \ll 18 \gg + (1 + 8\,\gamma\,\epsilon\,\hbar)\,QU[\ll 1 \gg],\ 0\Big\}\Big\}$$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{{QU[y1], QU[y1]} → 0, {QU[y1], QU[a1]} → 0, {QU[y1], QU[x1]} → 0},
 {{QU[a1], QU[y1]} → 0, {QU[a1], QU[a1]} → 0, {QU[a1], QU[x1]} → 0},
 {{QU[x1], QU[y1]} → 0, {QU[x1], QU[a1]} → 0, {QU[x1], QU[x1]} → 0}}
```

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$$\{13.0156, \{48\, t\, \gamma^5 \in \text{CU}[y,y,y,x,x] + \ll 77 \gg + \text{CU}[y,y,y,y,y,a,a,a,a,x,x,x,x],$$
$$2\left(\frac{4\gamma^5\in}{\hbar} - \frac{8\,T\,\gamma^5\in}{\hbar} + \frac{4\,T^2\gamma^5\in}{\hbar}\right)\text{QU}[y,y,y,x,x] +$$
$$\ll 217 \gg + 8\,\gamma\in\hbar\,\text{QU}[y,y,y,y,y,a,a,a,a,x,x,x,x],\ 0\}\}$$

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆_QU[{a, x}, SS[-T^(-1/2) e^(ℏ ∈ a) x]],
  a → -a_QU, x → 𝕆_QU[{a, y}, SS[-T^(-1/2) e^(ℏ ∈ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

```
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$$\left\{\text{QU}[y] \to -\frac{\text{QU}[x]}{\sqrt{T}} - \frac{\in\hbar\,\text{QU}[a,x]}{\sqrt{T}} \to \text{QU}[y],\ \text{QU}[a] \to -\text{QU}[a] \to \text{QU}[a],\right.$$
$$\left.\text{QU}[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\in\hbar}{\sqrt{T}}\right)\text{QU}[y] - \frac{\in\hbar\,\text{QU}[y,a]}{\sqrt{T}} \to \text{QU}[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\text{AD\$f} = \gamma\left(\left(\text{Cosh}\left[\hbar\left(a\,e + \frac{\gamma\,e}{2} - \frac{t}{2}\right)\right] - \text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,e}{2}\right)^2 + e\,\omega}\right]\right)\Big/\left(\hbar\,e^{\hbar\,((a+\gamma)\,e-t/2)}\,\text{Sinh}\left[\frac{\gamma\,e\,\hbar}{2}\right]\left(a^2\,e + a\,\gamma\,e - a\,t - \omega\right)\right)\right);$$

Scaling behaviour of AD$f:

```
HL@Simplify[AD$f == ((AD$f /. γ → 1) /. {e → γ e, a → γ^-1 a, ω → γ^-1 ω})]
True
```

```
HL@FullSimplify[
  AD$f == ((AD$f /. γ → 1) /. {ħ → γ^2 ħ, e → e / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
True
```

ADeq

```
AD$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[AD, QU → CU,
  {a → a_CU, x → CU@x, y → S_CU[SS[AD$f] /. e → ϵ, a → a_CU, ω → AD$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

```
SD$g = √(((2 γ (Cosh[ħ/2 √(t² + γ² e² + 4 e ϖ)] - Cosh[(t - e γ - 2 e a)/(2/ħ)]))/
    (Sinh[γ e ħ/2] (t (2 a + γ) - 2 a (a + γ) e + 2 ϖ) ħ)));
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ħ ((e-t)/2 + e a)] - Cosh[ħ √((t²+e²)/4 + e ϖ)])/(ħ Sinh[(-e ħ)/2] (ϖ - e a² + (t - e) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ² ħ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    √SD$Q /. c → a + 1/2 /. {ħ → γ² ħ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,e+\frac{e-t}{2}\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}\left(e^2+t^2\right)+e\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{e\,\hbar}{2}\right]\right)\Big/\left(\left(-a^2 e+\frac{t}{2}+a\,(-e+t)+\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\right.$$

$$\left(4\left(-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{e^2+t^2+4\,e\,\varpi}\;\hbar\right]+\mathrm{Cosh}\left[c\,e\,\hbar-\frac{t\,\hbar}{2}\right]\right)\mathrm{Csch}\left[\frac{e\,\hbar}{2}\right]\right)\Big/\left(\left(\left(-1+4\,c^2\right)e-4\,(c\,t+\varpi)\right)\hbar\right),$$

$$\left.\mathrm{True},\ \mathrm{True}\right\}$$

SDeq

```
SD$f = Simplify[e^(ħ (t/2-e a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x → 𝕊_CU[SS[SD$f] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y → 𝕊_CU[SS[SD$g] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

R in QU.

Faddeev-Quesne's formula:

Quesne

$$\mathbb{e}_{q\_,k\_}[x\_] := \mathbb{e}^\wedge\left(\sum_{j=1}^{k} \frac{(1-q)^j x^j}{j\,(1-q^j)}\right);\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
Table[Together@SeriesCoefficient[𝕖ρ,5[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1,\ 1,\ \frac{1}{1+\rho},\ \frac{1}{(1+\rho)\,(1+\rho+\rho^2)},\ \frac{1}{(1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)},\ 1\Big/\left((1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)\,(1+\rho+\rho^2+\rho^3+\rho^4)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, ρ] SeriesCoefficient[𝕖ρ,5[x], {x, 0, n}]], {n, 0, 5}]
```

```
{1, 1, 1, 1, 1, 1}
```

R

$$QU[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[\{y_1, a_1\}_i,\ \{a_2, x_2\}_j,\ SS\left[\mathbb{e}^{\hbar\, b_1 a_2}\, \mathbb{e}_q[\hbar\, y_1 x_2]\ /.\ b_1 \to \gamma^{-1}\,(\epsilon\, a_1 - t_i)\right]\right];$$
$$QU\left[R^{-1}_{i\_,j\_}\right] := S_j@QU[R_{i,j}];$$

```
QU[R3,4] // Short
```

$$QU[\,] + \frac{\epsilon\,\hbar\, QU[a_3, a_4]}{\gamma} + \hbar\, QU[y_3, x_4] + \frac{\ll 1\gg}{\gamma} + \ll 1\gg - \frac{\ll 1\gg}{\gamma} - \frac{\epsilon \ll 3\gg}{\gamma^2} - \frac{\hbar^2\, QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2\, QU[a_4, a_4]\, t_3^2}{2\,\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

$$QU\left[R_{1,2} ** R^{-1}_{1,2}\right]\ //\ Simp\ //\ HL\ //\ Timing$$

```
{0.09375, QU[]}
```

Verifying R3 (~156 secs @ $p=4, $k=2):

$$\{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]],\ HL@SimpT[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]\}\ //\ Timing$$

$$\left\{0.421875,\ \left\{QU[\,] + \frac{\epsilon\,\hbar\, QU[a_1, a_2]}{\gamma} + \ll 85\gg + QU[y_1, y_1, x_3, x_3]\left(\frac{\hbar^2}{2} - \hbar^2\, T_2 + \frac{1}{2}\,\hbar^2\, T_2^2\right),\ 0\right\}\right\}$$

The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = ( 0 0 ; ∊ 0 ); ρ@a_CU = ρ@a_QU = ( γ 0 ; 0 0 );
ρ@x_CU = ( 0 γ ; 0 0 ); ρ@x_QU = SS@( 0 (1 - e^(-γ∊ħ)) / (∊ ħ) ; 0 0 );
ρ[e^𝓔_] := MatrixExp[ρ[𝓔]];
ρ[𝓔_] :=
  (𝓔 /. {t → γ ∊, T → e^(ħ γ ∊)} /. (U : CU | QU)[u___] :→ Fold[Dot, ( 1 0 ; 0 1 ), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // SS // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

The Logoi Λ

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
With[{U = CU},
 Module[{G, F, fs, bs, e, b, es, sol},
  G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
  fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
  F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ∊^l U@{y^i, a^j, x^k})];
  es = Flatten[
    Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
  sol = Echo@First[F /. DSolve[es, fs, η]];
  Echo[sol /. {e- → 1, U → Times}];
  Collect[sol /. {e- → 1, U → Times}, ∊, Simplify]
 ]]
```

» $-t\,\xi\,CU[\,] + 2\,\epsilon\,\xi\,CU[a] - \gamma\,\epsilon\,\xi^2\,CU[x] + CU[y]$

» $\{f_{0,0,0,0}[\eta],\ f_{1,0,0,0}[\eta],\ f_{1,0,0,1}[\eta],\ f_{1,0,1,0}[\eta],\ f_{1,0,1,1}[\eta],\ f_{1,1,0,0}[\eta],\ f_{1,1,0,1}[\eta],\ f_{1,1,1,0}[\eta],\ f_{1,1,1,1}[\eta]\}$

» $CU[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,CU[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,CU[x]\,f_{1,0,0,1}[\eta] + \epsilon\,CU[a]\,f_{1,0,1,0}[\eta] + \epsilon\,CU[a, x]\,f_{1,0,1,1}[\eta] + \epsilon\,CU[y]\,f_{1,1,0,0}[\eta] + \epsilon\,CU[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,CU[y, a]\,f_{1,1,1,0}[\eta] + \epsilon\,CU[y, a, x]\,f_{1,1,1,1}[\eta]$

» $e^{-t\eta\xi}\,CU[\,] + \frac{1}{2} e^{-t\eta\xi}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,CU[\,] + 2\,e^{-t\eta\xi}\,\epsilon\,\eta\,\xi\,CU[a] - e^{-t\eta\xi}\,\gamma\,\epsilon\,\eta\,\xi^2\,CU[x] - e^{-t\eta\xi}\,\gamma\,\epsilon\,\eta^2\,\xi\,CU[y]$

» $1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2$

$1 + \frac{1}{2}\,\epsilon\,\eta\,\xi\,\left(4\,a + \gamma\,\left(-2\,y\,\eta - 2\,x\,\xi + t\,\eta\,\xi\right)\right)$

Logos

```
λ[U_] := Module[{G, F, fs, f, bs, e, b, es},
   G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
   fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, l}, {j, 0, l}, {k, 0, l}];
   F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ϵ^l U@{y^i, a^j, x^k});
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   F /. DSolve[es, fs, η][[1]] /. {e- → 1, U → Times}];
```

```
{tt = λ[CU], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\Big\{1+2\,a\,\epsilon\,\eta\,\xi-y\,\gamma\,\epsilon\,\eta^2\,\xi-x\,\gamma\,\epsilon\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2+2\,a^2\,\epsilon^2\,\eta^2\,\xi^2-2\,a\,y\,\gamma\,\epsilon^2\,\eta^3\,\xi^2+\frac{1}{2}\,y^2\,\gamma^2\,\epsilon^2\,\eta^4\,\xi^2-2\,a\,x\,\gamma\,\epsilon^2\,\eta^2\,\xi^3+x\,y\,\gamma^2\,\epsilon^2\,\eta^3\,\xi^3+\frac{1}{2}\,x^2\,\gamma^2\,\epsilon^2\,\eta^2\,\xi^4-\frac{1}{2}\,y\,\gamma^2\,\epsilon^2\,\eta^3\,\xi^2\,(-2+t\,\eta\,\xi)-\frac{1}{2}\,x\,\gamma^2\,\epsilon^2\,\eta^2\,\xi^3\,(-2+t\,\eta\,\xi)+a\,\gamma\,\epsilon^2\,\eta^2\,\xi^2\,(-1+t\,\eta\,\xi)+\frac{1}{24}\,t\,\gamma^2\,\epsilon^2\,\eta^3\,\xi^3\,(-8+3\,t\,\eta\,\xi),$$

$$\epsilon\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)+\epsilon^2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\Big\}$$

```
{tt = λ[QU], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\{1+2\,a\,T\,\epsilon\,\eta\,\xi-\frac{1}{2}(-1+3T)\,y\,\gamma\,\epsilon\,\eta^2\,\xi-\frac{1}{2}(-1+3T)\,x\,\gamma\,\epsilon\,\eta\,\xi^2-$$
$$a\,T\,y\,\gamma\,\epsilon^2\,\eta^2\,\xi\,(-\eta\,\xi+3\,T\,\eta\,\xi-3\,\hbar)-a\,T\,x\,\gamma\,\epsilon^2\,\eta\,\xi^2\,(-\eta\,\xi+3\,T\,\eta\,\xi-3\,\hbar)+$$
$$2\,a^2\,T\,\epsilon^2\,\eta\,\xi\,(T\,\eta\,\xi-\hbar)+\frac{(-1+T)\,(-1+3\,T)\,\gamma\,\epsilon\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\epsilon\,\eta\,\xi\,\hbar+2\,a\,T\,x\,y\,\gamma\,\epsilon^2\,\eta^2\,\xi^2\,\hbar-$$
$$\frac{1}{2}\,x\,y^2\,\gamma^2\,\epsilon^2\,\eta^2\,\xi\,(-\eta\,\xi+3\,T\,\eta\,\xi-\hbar)\,\hbar-\frac{1}{2}\,x^2\,y\,\gamma^2\,\epsilon^2\,\eta\,\xi^2\,(-\eta\,\xi+3\,T\,\eta\,\xi-\hbar)\,\hbar+$$
$$\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\epsilon^2\,\eta^2\,\xi^2\,\hbar^2+\frac{1}{24}\,y^2\,\gamma^2\,\epsilon^2\,\eta^3\,\xi\,(3\,\eta\,\xi-18\,T\,\eta\,\xi+27\,T^2\,\eta\,\xi+4\,\hbar-28\,T\,\hbar)+$$
$$\frac{1}{24}\,x^2\,\gamma^2\,\epsilon^2\,\eta\,\xi^3\,(3\,\eta\,\xi-18\,T\,\eta\,\xi+27\,T^2\,\eta\,\xi+4\,\hbar-28\,T\,\hbar)+$$
$$\frac{1}{2\,\hbar}a\,T\,\gamma\,\epsilon^2\,\eta^2\,\xi^2\,(\eta\,\xi-4\,T\,\eta\,\xi+3\,T^2\,\eta\,\xi+4\,\hbar-6\,T\,\hbar)+$$
$$\frac{1}{4}\,x\,y\,\gamma^2\,\epsilon^2\,\eta\,\xi\,(2\,\eta^2\,\xi^2-10\,T\,\eta^2\,\xi^2+12\,T^2\,\eta^2\,\xi^2+5\,\eta\,\xi\,\hbar-21\,T\,\eta\,\xi\,\hbar+2\,\hbar^2)-\frac{1}{24\,\hbar}y\,\gamma^2\,\epsilon^2\,\eta^2\,\xi$$
$$(-3\,\eta^2\,\xi^2+21\,T\,\eta^2\,\xi^2-45\,T^2\,\eta^2\,\xi^2+27\,T^3\,\eta^2\,\xi^2-10\,\eta\,\xi\,\hbar+68\,T\,\eta\,\xi\,\hbar-82\,T^2\,\eta\,\xi\,\hbar-6\,\hbar^2+30\,T\,\hbar^2)-$$
$$\frac{1}{24\,\hbar}x\,\gamma^2\,\epsilon^2\,\eta\,\xi^2\,(-3\,\eta^2\,\xi^2+21\,T\,\eta^2\,\xi^2-45\,T^2\,\eta^2\,\xi^2+27\,T^3\,\eta^2\,\xi^2-10\,\eta\,\xi\,\hbar+68\,T\,\eta\,\xi\,\hbar-$$
$$82\,T^2\,\eta\,\xi\,\hbar-6\,\hbar^2+30\,T\,\hbar^2)+\frac{1}{288\,\hbar^2}(-1+T)\,\gamma^2\,\epsilon^2\,\eta^2\,\xi^2\,(-9\,\eta^2\,\xi^2+63\,T\,\eta^2\,\xi^2-$$
$$135\,T^2\,\eta^2\,\xi^2+81\,T^3\,\eta^2\,\xi^2-40\,\eta\,\xi\,\hbar+272\,T\,\eta\,\xi\,\hbar-328\,T^2\,\eta\,\xi\,\hbar-36\,\hbar^2+180\,T\,\hbar^2),$$
$$\frac{1}{4\,\hbar}\epsilon\,(\gamma\,\eta^2\,\xi^2-4\,T\,\gamma\,\eta^2\,\xi^2+3\,T^2\,\gamma\,\eta^2\,\xi^2+8\,a\,T\,\eta\,\xi\,\hbar+2\,y\,\gamma\,\eta^2\,\xi\,\hbar-6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$2\,x\,\gamma\,\eta\,\xi^2\,\hbar-6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2)+\frac{1}{72\,\hbar}$$
$$\epsilon^2\,(10\,\gamma^2\,\eta^3\,\xi^3-78\,T\,\gamma^2\,\eta^3\,\xi^3+150\,T^2\,\gamma^2\,\eta^3\,\xi^3-82\,T^3\,\gamma^2\,\eta^3\,\xi^3+144\,a\,T\,\gamma\,\eta^2\,\xi^2\,\hbar-216\,a\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar+$$
$$9\,\gamma^2\,\eta^2\,\xi^2\,\hbar-54\,T\,\gamma^2\,\eta^2\,\xi^2\,\hbar+45\,T^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar+30\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar-204\,T\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar+$$
$$246\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2\,\hbar+30\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar-204\,T\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar+246\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3\,\hbar-144\,a^2\,T\,\eta\,\xi\,\hbar^2+$$
$$216\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^2+18\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^2-90\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar^2+12\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^2-84\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar^2+$$
$$216\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^2+18\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^2-90\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar^2+90\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2-378\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2+$$
$$12\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^2-84\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar^2+36\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^3+36\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^3+36\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^3)\}$$

Logos

```
wc[CU] = t; wc[QU] = (T - 1) / ħ;
Λ[U_] := Λ[U] = Module[{Q, w}, Q = (-w ξ η + η y + ξ x + δ y x) / (1 + w δ);
    Collect[(1 + w δ)^-1 e^-Q DP_{ξ→D_x, η→D_y}[λ[U]][e^Q] /. w → wc[U], ϵ, Simplify]];
Λ[U_, t1_, T1_, y1_, a1_, x1_, ξ1_, η1_, δ1_] :=
  Λ[U] /. {t → t1, T → T1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1, δ → δ1};
```

```
{tt = Λ[CU], Collect[Normal@Series[Log[tt], {ϵ, 0, $k}], ϵ, Simplify]} /. ϵ^k_ /; k ≥ 3 :→ 0
```

$$\{\frac{1}{1+t\,\delta}+\frac{1}{24\,(1+t\,\delta)^9}$$
$$\epsilon^2\,(48\,a^2\,(1+t\,\delta)^4\,(2\,\delta^2\,(1+t\,\delta)^2+4\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2)-$$
$$24\,a\,\gamma\,(1+t\,\delta)^4\,(2\,\delta^2\,(1+t\,\delta)^2+4\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2)-$$
$$48\,a\,y\,\gamma\,(1+t\,\delta)^3\,(x\,\delta+\eta)$$

$$\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)+24y\gamma^2(1+t\delta)^3$$
$$(x\delta+\eta)\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)-48ax\gamma$$
$$(1+t\delta)^3(y\delta+\xi)\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)+$$
$$24x\gamma^2(1+t\delta)^3(y\delta+\xi)\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.(x\delta+\eta)^2(y\delta+\xi)^2\right)+12y^2\gamma^2(1+t\delta)^2(x\delta+\eta)^2$$
$$\left(12\delta^2(1+t\delta)^2+8\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)+12x^2\gamma^2(1+t\delta)^2$$
$$(y\delta+\xi)^2\left(12\delta^2(1+t\delta)^2+8\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)+$$
$$24at\gamma(1+t\delta)^2\left(6\delta^3(1+t\delta)^3+18\delta^2(1+t\delta)^2(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.9\delta(1+t\delta)(x\delta+\eta)^2(y\delta+\xi)^2+(x\delta+\eta)^3(y\delta+\xi)^3\right)-$$
$$8t(\gamma+t\gamma\delta)^2\left(6\delta^3(1+t\delta)^3+18\delta^2(1+t\delta)^2(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.9\delta(1+t\delta)(x\delta+\eta)^2(y\delta+\xi)^2+(x\delta+\eta)^3(y\delta+\xi)^3\right)+$$
$$24xy(\gamma+t\gamma\delta)^2\left(6\delta^3(1+t\delta)^3+18\delta^2(1+t\delta)^2(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.9\delta(1+t\delta)(x\delta+\eta)^2(y\delta+\xi)^2+(x\delta+\eta)^3(y\delta+\xi)^3\right)-$$
$$12ty\gamma^2(1+t\delta)(x\delta+\eta)\left(24\delta^3(1+t\delta)^3+36\delta^2(1+t\delta)^2(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.12\delta(1+t\delta)(x\delta+\eta)^2(y\delta+\xi)^2+(x\delta+\eta)^3(y\delta+\xi)^3\right)-$$
$$12tx\gamma^2(1+t\delta)(y\delta+\xi)\left(24\delta^3(1+t\delta)^3+36\delta^2(1+t\delta)^2(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.12\delta(1+t\delta)(x\delta+\eta)^2(y\delta+\xi)^2+(x\delta+\eta)^3(y\delta+\xi)^3\right)+$$
$$3t^2\gamma^2\left(24\delta^4(1+t\delta)^4+96\delta^3(1+t\delta)^3(x\delta+\eta)(y\delta+\xi)+72\delta^2(1+t\delta)^2\right.$$
$$\left.\left.(x\delta+\eta)^2(y\delta+\xi)^2+16\delta(1+t\delta)(x\delta+\eta)^3(y\delta+\xi)^3+(x\delta+\eta)^4(y\delta+\xi)^4\right)\right)+$$
$$\frac{1}{2(1+t\delta)^5}\epsilon\left(4a(1+t\delta)^2\left((t+xy)\delta^2+\eta\xi+\delta(1+y\eta+x\xi)\right)+\right.$$
$$\gamma\left(2t^3\delta^4+4t^2\delta^2\left(\delta-xy\delta^2+\eta\xi\right)-2\left(y\eta\left(\delta(2+y\eta)+\eta\xi\right)+x^2\delta\left(2y^2\delta^2+3y\delta\xi+\xi^2\right)+\right.\right.$$
$$\left.\left.x\left(3y^2\delta^2\eta+4y\delta(\delta+\eta\xi)+\xi(2\delta+\eta\xi)\right)\right)-t\left(3x^2y^2\delta^4-4\delta\eta\xi-\eta^2\xi^2+\right.\right.$$
$$\left.\left.\left.\left.4xy\delta^3(3+y\eta+x\xi)+\delta^2\left(-2+y^2\eta^2+4x\xi+x^2\xi^2+4y(\eta+x\eta\xi)\right)\right)\right)\right)\right),$$
$$\epsilon\left(\frac{2a\left((t+xy)\delta^2+\eta\xi+\delta(1+y\eta+x\xi)\right)}{(1+t\delta)^2}+\frac{1}{2(1+t\delta)^4}\right.$$
$$\gamma\left(2t^3\delta^4+4t^2\delta^2\left(\delta-xy\delta^2+\eta\xi\right)-2\left(y\eta\left(\delta(2+y\eta)+\eta\xi\right)+x^2\delta\left(2y^2\delta^2+3y\delta\xi+\xi^2\right)+\right.\right.$$
$$\left.\left.x\left(3y^2\delta^2\eta+4y\delta(\delta+\eta\xi)+\xi(2\delta+\eta\xi)\right)\right)-t\left(3x^2y^2\delta^4-4\delta\eta\xi-\eta^2\xi^2+\right.\right.$$
$$\left.\left.\left.\left.4xy\delta^3(3+y\eta+x\xi)+\delta^2\left(-2+y^2\eta^2+4x\xi+x^2\xi^2+4y(\eta+x\eta\xi)\right)\right)\right)\right)+\frac{1}{24(1+t\delta)^8}$$
$$\epsilon^2\left(48a^2(1+t\delta)^4\left(2\delta^2(1+t\delta)^2+4\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)-\right.$$
$$24a\gamma(1+t\delta)^4\left(2\delta^2(1+t\delta)^2+4\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)-$$
$$48ay\gamma(1+t\delta)^3(x\delta+\eta)$$
$$\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)+24y\gamma^2(1+t\delta)^3$$
$$(x\delta+\eta)\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)-48ax\gamma$$
$$(1+t\delta)^3(y\delta+\xi)\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+(x\delta+\eta)^2(y\delta+\xi)^2\right)+$$
$$24x\gamma^2(1+t\delta)^3(y\delta+\xi)\left(6\delta^2(1+t\delta)^2+6\delta(1+t\delta)(x\delta+\eta)(y\delta+\xi)+\right.$$
$$\left.(x\delta+\eta)^2(y\delta+\xi)^2\right)+12y^2\gamma^2(1+t\delta)^2(x\delta+\eta)^2$$

$$\left(12\,\delta^2\,(1+t\,\delta)^2+8\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+12\,x^2\,\gamma^2\,(1+t\,\delta)^2$$
$$(y\,\delta+\xi)^2\left(12\,\delta^2\,(1+t\,\delta)^2+8\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+$$
$$24\,a\,t\,\gamma\,(1+t\,\delta)^2\left(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)-$$
$$8\,t\,(\gamma+t\,\gamma\,\delta)^2\left(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)+$$
$$24\,x\,y\,(\gamma+t\,\gamma\,\delta)^2\left(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)-$$
$$12\,t\,y\,\gamma^2\,(1+t\,\delta)\,(x\,\delta+\eta)\left(24\,\delta^3\,(1+t\,\delta)^3+36\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.12\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)-$$
$$12\,t\,x\,\gamma^2\,(1+t\,\delta)\,(y\,\delta+\xi)\left(24\,\delta^3\,(1+t\,\delta)^3+36\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.12\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)+$$
$$3\,t^2\,\gamma^2\left(24\,\delta^4\,(1+t\,\delta)^4+96\,\delta^3\,(1+t\,\delta)^3\,(x\,\delta+\eta)\,(y\,\delta+\xi)+72\,\delta^2\,(1+t\,\delta)^2\right.$$
$$\left.(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+16\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3+(x\,\delta+\eta)^4\,(y\,\delta+\xi)^4\right)-$$
$$3\left(4\,a\,(1+t\,\delta)^2\,\left((t+x\,y)\,\delta^2+\eta\,\xi+\delta\,(1+y\,\eta+x\,\xi)\right)+\gamma\,\left(2\,t^3\,\delta^4+4\,t^2\,\delta^2\,(\delta-x\,y\,\delta^2+\eta\,\xi)-\right.\right.$$
$$2\,\left(y\,\eta\,(\delta\,(2+y\,\eta)+\eta\,\xi)+x^2\,\delta\,(2\,y^2\,\delta^2+3\,y\,\delta\,\xi+\xi^2)+x\,\left(3\,y^2\,\delta^2\,\eta+4\,y\,\delta\,(\delta+\eta\,\xi)+\right.\right.$$
$$\left.\left.\xi\,(2\,\delta+\eta\,\xi)\right)\right)-t\,\left(3\,x^2\,y^2\,\delta^4-4\,\delta\,\eta\,\xi-\eta^2\,\xi^2+4\,x\,y\,\delta^3\,(3+y\,\eta+x\,\xi)+\right.$$
$$\left.\left.\left.\left.\delta^2\,\left(-2+y^2\,\eta^2+4\,x\,\xi+x^2\,\xi^2+4\,y\,(\eta+x\,\eta\,\xi)\right)\right)\right)\right)^2\right)+\mathrm{Log}\left[\frac{1}{1+t\,\delta}\right]\Big\}$$

```
{tt = Λ[QU], Collect[Normal@Series[Log[tt], {ϵ, 0, $k}], ϵ, Simplify]} /. ϵ^k_ /; k ≥ 2 ⧴ 0
```

$$
\begin{aligned}
&\Big\{\frac{\hbar}{(-1+T)\,\delta+\hbar}+\\
&\quad\frac{1}{4\left((-1+T)\,\delta+\hbar\right)^5}\in\hbar^2\Big(8\,a\,T\left((-1+T)\,\delta+\hbar\right)^2\left(\eta\,\xi\,\hbar+\delta\left(1+y\,\eta+x\,\xi\right)\hbar+\delta^2\left(-1+T+x\,y\,\hbar\right)\right)+\\
&\qquad\gamma\Big(\eta\,\xi\,\hbar^2\left((-1+3\,T)\,\eta\left((-1+T)\,\xi-2\,y\,\hbar\right)+2\,x\,\hbar\left(\xi-3\,T\,\xi+2\,y\,\hbar\right)\right)+\\
&\qquad\quad(-1+T)\,\delta^4\left(-2+6\,T^3-x^2\,y^2\,\hbar^2-2\,T^2\,(7+4\,x\,y\,\hbar)+T\left(10+8\,x\,y\,\hbar-5\,x^2\,y^2\,\hbar^2\right)\right)-\\
&\qquad\quad4\,\delta^3\,\hbar\Big(1-3\,T^3+x^2\,y^2\,\hbar^2+T^2\left(7+2\,x\,y\left(3+y\,\eta\right)\hbar+2\,x^2\,y\,\xi\,\hbar\right)+\\
&\qquad\qquad T\left(-5-2\,x\,y\left(3+y\,\eta\right)\hbar+x^2\,y\,\hbar\left(-2\,\xi+y\,\hbar\right)\right)\Big)+\\
&\qquad\quad2\,\delta\,\hbar^2\Big((1-3\,T)\,y^2\,\eta^2\,\hbar+2\,\eta\left(\xi+3\,T^2\,\xi-4\,T\,\xi\left(1+x\,y\,\hbar\right)+y\,\hbar\left(1-3\,T+x\,y\,\hbar\right)\right)+\\
&\qquad\qquad x\,\hbar\left((x-3\,T\,x)\,\xi^2+2\,y\,\hbar+\xi\left(2-6\,T+2\,x\,y\,\hbar\right)\right)\Big)-\\
&\qquad\quad\delta^2\,\hbar\Big(\left(1-4\,T+3\,T^2\right)y^2\,\eta^2\,\hbar+\hbar\Big(-2+3\,T^2\left(-2+4\,x\,\xi+x^2\,\xi^2\right)+4\,x\,(\xi+y\,\hbar)+x^2\\
&\qquad\qquad\qquad\left(\xi^2+2\,y\,\xi\,\hbar-4\,y^2\,\hbar^2\right)-2\,T\left(-4+x\left(8\,\xi-6\,y\,\hbar\right)+x^2\,\xi\left(2\,\xi-5\,y\,\hbar\right)\right)\Big)+2\,\eta\\
&\qquad\qquad\left(-2\,(-1+T)\,\xi\left(1+3\,T^2-2\,T\left(2+x\,y\,\hbar\right)\right)+y\,\hbar\left(2+6\,T^2+x\,y\,\hbar+T\left(-8+5\,x\,y\,\hbar\right)\right)\right)\Big)\Big)\Big),\\
&\frac{1}{4\left((-1+T)\,\delta+\hbar\right)^4}\in\hbar\Big(8\,a\,T\left((-1+T)\,\delta+\hbar\right)^2\left(\eta\,\xi\,\hbar+\delta\left(1+y\,\eta+x\,\xi\right)\hbar+\delta^2\left(-1+T+x\,y\,\hbar\right)\right)+\\
&\qquad\gamma\Big(\eta\,\xi\,\hbar^2\left((-1+3\,T)\,\eta\left((-1+T)\,\xi-2\,y\,\hbar\right)+2\,x\,\hbar\left(\xi-3\,T\,\xi+2\,y\,\hbar\right)\right)+\\
&\qquad\quad(-1+T)\,\delta^4\left(-2+6\,T^3-x^2\,y^2\,\hbar^2-2\,T^2\,(7+4\,x\,y\,\hbar)+T\left(10+8\,x\,y\,\hbar-5\,x^2\,y^2\,\hbar^2\right)\right)-\\
&\qquad\quad4\,\delta^3\,\hbar\Big(1-3\,T^3+x^2\,y^2\,\hbar^2+T^2\left(7+2\,x\,y\left(3+y\,\eta\right)\hbar+2\,x^2\,y\,\xi\,\hbar\right)+\\
&\qquad\qquad T\left(-5-2\,x\,y\left(3+y\,\eta\right)\hbar+x^2\,y\,\hbar\left(-2\,\xi+y\,\hbar\right)\right)\Big)+\\
&\qquad\quad2\,\delta\,\hbar^2\Big((1-3\,T)\,y^2\,\eta^2\,\hbar+2\,\eta\left(\xi+3\,T^2\,\xi-4\,T\,\xi\left(1+x\,y\,\hbar\right)+y\,\hbar\left(1-3\,T+x\,y\,\hbar\right)\right)+\\
&\qquad\qquad x\,\hbar\left((x-3\,T\,x)\,\xi^2+2\,y\,\hbar+\xi\left(2-6\,T+2\,x\,y\,\hbar\right)\right)\Big)-\\
&\qquad\quad\delta^2\,\hbar\Big(\left(1-4\,T+3\,T^2\right)y^2\,\eta^2\,\hbar+\hbar\Big(-2+3\,T^2\left(-2+4\,x\,\xi+x^2\,\xi^2\right)+4\,x\,(\xi+y\,\hbar)+x^2\\
&\qquad\qquad\qquad\left(\xi^2+2\,y\,\xi\,\hbar-4\,y^2\,\hbar^2\right)-2\,T\left(-4+x\left(8\,\xi-6\,y\,\hbar\right)+x^2\,\xi\left(2\,\xi-5\,y\,\hbar\right)\right)\Big)+\\
&\qquad\qquad2\,\eta\Big(-2\,(-1+T)\,\xi\left(1+3\,T^2-2\,T\left(2+x\,y\,\hbar\right)\right)+y\,\hbar\Big(2+6\,T^2+x\,y\,\hbar+\\
&\qquad\qquad\qquad T\left(-8+5\,x\,y\,\hbar\right)\Big)\Big)\Big)\Big)\Big)+\mathrm{Log}\Big[\frac{\hbar}{(-1+T)\,\delta+\hbar}\Big]\Big\}
\end{aligned}
$$

```
{Short[lhs = 𝕆_CU[{x, y}, SS[ⅇ^(ℏ (ξ x+η y+δ x y))]], 5], HL@Simp[lhs −
    𝕆_CU[{y, a, x}, SS[ⅇ^(ℏ (ξ x+η y+δ x y−t ℏ ξ η)/(1+ℏ t δ)) Λ[CU, t, T, y, a, x, ℏ ξ, ℏ η, ℏ δ]]], Together]}
```

$$
\begin{aligned}
&\Big\{\left(1-t\,\delta\,\hbar+t^2\,\delta^2\,\hbar^2+t\,\gamma\,\delta^2\in\hbar^2-t\,\eta\,\xi\,\hbar^2\right)\mathrm{CU}[\,]+\\
&\quad\left(2\,\delta\in\hbar-4\,t\,\delta^2\in\hbar^2+2\in\eta\,\xi\,\hbar^2\right)\mathrm{CU}[a]+\left(\xi\,\hbar-2\,t\,\delta\,\xi\,\hbar^2-2\,\gamma\,\delta\in\xi\,\hbar^2\right)\mathrm{CU}[x]+\\
&\quad\left(\eta\,\hbar-2\,t\,\delta\,\eta\,\hbar^2-2\,\gamma\,\delta\in\eta\,\hbar^2\right)\mathrm{CU}[y]+4\,\delta\in\xi\,\hbar^2\,\mathrm{CU}[a,x]+\frac{1}{2}\,\xi^2\,\hbar^2\,\mathrm{CU}[x,x]+\\
&\quad4\,\delta\in\eta\,\hbar^2\,\mathrm{CU}[y,a]+\left(\delta\,\hbar-2\,t\,\delta^2\,\hbar^2-4\,\gamma\,\delta^2\in\hbar^2+\eta\,\xi\,\hbar^2\right)\mathrm{CU}[y,x]+\frac{1}{2}\,\eta^2\,\hbar^2\,\mathrm{CU}[y,y]+\\
&\quad4\,\delta^2\in\hbar^2\,\mathrm{CU}[y,a,x]+\delta\,\xi\,\hbar^2\,\mathrm{CU}[y,x,x]+\delta\,\eta\,\hbar^2\,\mathrm{CU}[y,y,x]+\frac{1}{2}\,\delta^2\,\hbar^2\,\mathrm{CU}[y,y,x,x],\ 0\Big\}
\end{aligned}
$$

```
{Short[lhs = SimpT@𝕆QU[{x, y}, SS[ℯ^(ℏ (ξ x+η y+δ x y))]], 5],
 rhs = SimpT@𝕆QU[{y, a, x},
    SS[ℯ^(ℏ ν (ξ x+η y+δ x y-(T-1) ξ η)) Λ[QU, t, T, y, a, x, ℏ ξ, ℏ η, ℏ δ] /. ν → (1 + (T - 1) δ)^-1]];
 HL[Simplify[lhs == rhs]]}
```

$\{(1 - t\delta\hbar + (-\frac{t^2\delta}{2} + t^2\delta^2 + t\gamma\delta^2\epsilon - t\eta\xi)\hbar^2)\,QU[\,] +$
$(2\delta\epsilon\hbar + (2t\delta\epsilon - 4t\delta^2\epsilon + 2\epsilon\eta\xi)\hbar^2)\,QU[a] + (\xi\hbar + (-2t\delta\xi - 2\gamma\delta\epsilon\xi)\hbar^2)\,QU[x] +$
$(\eta\hbar + (-2t\delta\eta - 2\gamma\delta\epsilon\eta)\hbar^2)\,QU[y] + 4\delta\epsilon\xi\hbar^2\,QU[a, x] + \frac{1}{2}\xi^2\hbar^2\,QU[x, x] +$
$4\delta\epsilon\eta\hbar^2\,QU[y, a] + (\delta\hbar + (-2t\delta^2 + \gamma\delta\epsilon - 4\gamma\delta^2\epsilon + \eta\xi)\hbar^2)\,QU[y, x] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y] +$
$4\delta^2\epsilon\hbar^2\,QU[y, a, x] + \delta\xi\hbar^2\,QU[y, x, x] + \delta\eta\hbar^2\,QU[y, y, x] + \frac{1}{2}\delta^2\hbar^2\,QU[y, y, x, x]$, True}

Œ and Swaps

Swaps from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
CU@ŒCU[specs___, Q_, P_] := 𝕆CU[specs, SS[ℯ^Q P]];
QU@ŒQU[specs___, Q_, P_] := 𝕆QU[specs, SS[ℯ^Q P]];
```

```
CU@ŒCU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (ℯ^t1 - 1) y1 x2, 1 + ϵ x1 y2] // Short
```

$CU[\,] + \ll 13 \gg + CU[y_1, x_1]\left(-\gamma\epsilon\hbar^2 t_2 + e^{t_1}\gamma\epsilon\hbar^2 t_2 + \frac{\epsilon\hbar t_2}{t_1} - \frac{e^{t_1}\epsilon\hbar t_2}{t_1}\right)$

```
HL[ρ[ℯ^(ξ CU@x)].ρ[ℯ^(α CU@a)] == ρ[ℯ^(α CU@a)].ρ[ℯ^(ℯ^(-γ α) ξ CU@x)]]
```

True

SW

```
SW_{x_i_,a_j_}[Œ_U_[OrderlessPatternSequence[{lh___, x_i_, a_j_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = ℯ^(-γ α) ξ x_i + α a_j},
  Œ_U[{lh, a_j, x_i, rh}_s, more, ℯ^(-γ α) ξ x_i + (Q /. x_i → 0), ℯ^-q DP_{x_i→D_ξ, a_j→D_α}[P][ℯ^q]] /.
   {α → ∂_{a_j} Q, ξ → ∂_{x_i} Q}]
```

```
co = ŒCU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (ℯ^t1 - 1) y1 x2, 1 + ϵ x1 y2]
```

$Œ_{CU}\left[\{y_1, x_1\}_1, \{x_2, a_2, y_2\}_2, \hbar a_2 t_1 + \frac{(-1 + e^{t_1})\hbar x_2 y_1}{t_1}, 1 + \epsilon x_1 y_2\right]$

```
SW_{x2,a2}[co]
```

$Œ_{CU}\left[\{a_2, x_2, y_2\}_2, \{y_1, x_1\}_1, \hbar a_2 t_1 + \frac{e^{-\gamma\hbar t_1}(-1 + e^{t_1})\hbar x_2 y_1}{t_1}, 1 + \epsilon x_1 y_2\right]$

```
With[{c𝕆 = 𝔼CU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (𝕖^t1 - 1) y1 x2, 1 + ϵ x1 y2]},
 HL[CU[c𝕆] == CU[c𝕆 // SW_{x2,a2}]]]
```

True

```
With[{c𝕆 = 𝔼CU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {CU[c𝕆] // Short, HL[CU[c𝕆] == CU[c𝕆 // SW_{x2,a2}]]}
]
```

$\{CU[a_1, a_1, a_1]\left(\frac{1}{2}\epsilon\hbar^2 l_1 l_{11}^2 t_1^2 + \epsilon\hbar^2 l_1 l_{11} l_{21} t_1 t_2 + \frac{1}{2}\epsilon\hbar^2 l_1 l_{21}^2 t_2^2\right) + \ll 75 \gg + CU[\,]\,(\ll 1 \gg),$

True}

```
With[{q𝕆 = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {QU[q𝕆] // Short, HL[QU[q𝕆] == QU[q𝕆 // SW_{x2,a2}]]}
]
```

$\{QU[a_1, a_1, a_1]\left(\frac{1}{2}\epsilon\hbar^2 l_1 l_{11}^2 t_1^2 + \epsilon\hbar^2 l_1 l_{11} l_{21} t_1 t_2 + \frac{1}{2}\epsilon\hbar^2 l_1 l_{21}^2 t_2^2\right) + \ll 75 \gg + QU[\,]\,(\ll 1 \gg),$

True}

SW

```
SW_{a_j_, y_i_}[𝔼_U_[OrderlessPatternSequence[{lh___, a_j_, y_i_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = 𝕖^(-γ α) η y_i + α a_j},
  𝔼_U[{lh, y_i, a_j, rh}_s, more, 𝕖^(-γ α) η y_i + (Q /. y_i → 0), 𝕖^-q DP_{y_i→D_η, a_j→D_α}[P][𝕖^q]] /.
   {α → ∂_{a_j} Q, η → ∂_{y_i} Q}]
```

```
With[{q𝕆 = 𝔼QU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {QU[q𝕆] // Short, HL@Simp[QU[q𝕆] - QU[q𝕆 // SW_{a2,y2}]]}
]
```

$\{QU[a_1, a_1, a_1]\left(\frac{1}{2}\epsilon\hbar^2 l_1 l_{11}^2 t_1^2 + \epsilon\hbar^2 l_1 l_{11} l_{21} t_1 t_2 + \frac{1}{2}\epsilon\hbar^2 l_1 l_{21}^2 t_2^2\right) + \ll 75 \gg + QU[\,]\,(\ll 1 \gg), 0\}$

SW

```
SW_{x_i_, y_j_ → k_}[𝔼CU[OrderlessPatternSequence[{lh___, x_i_, y_j_, rh___}_s_, more___], Q_, P_]] :=
 With[{q = ν (ξ x_k + η y_k + δ x_k y_k - t_k ξ η)},
  𝔼CU[{lh, y_k, a_k, x_k, rh}_s, more, q + (Q /. x_i | y_j → 0),
     𝕖^-q DP_{x_i→D_ξ, y_j→D_η}[P][Λ[CU, t_k, T_k, y_k, a_k, x_k, ξ, η, δ] 𝕖^q]]
   /. ν → (1 + t_k δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]
```

```
With[{c𝕆 = 𝔈CU[{x₁, y₁}₁, {x₂, a₂, y₂}₂,
    ℏ (l₁₂ t₁ a₂ + l₂₂ t₂ a₂ + γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
    1 + ϵ (l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)]},
  {CU[c𝕆] // Short, HL[CU[c𝕆] == CU[c𝕆 // SW_{x₁,y₁→1}]]}
]
```

$\{\mathrm{CU}[a_2, a_2, a_2] \left(\frac{1}{2} \epsilon \hbar^2 l_2 l_{12}^2 t_1^2 + \epsilon \hbar^2 l_2 l_{12} l_{22} t_1 t_2 + \frac{1}{2} \epsilon \hbar^2 l_2 l_{22}^2 t_2^2\right) + \ll 54 \gg + \mathrm{CU}[\,] (\ll 1 \gg),$

True}

SW

```
SW_{x_i_,y_j_→k_}[𝔈QU[OrderlessPatternSequence[{lh___, x_i_, y_j_, rh___}_s_, more___], Q_, P_]] :=
  With[{q = ν (ξ x_k + η y_k + δ x_k y_k - ℏ^-1 (T_k - 1) ξ η)},
   𝔈QU[{lh, y_k, a_k, x_k, rh}_s, more, q + (Q /. x_i | y_j → 0),
     𝕖^-q DP_{x_i→D_ξ, y_j→D_η}[P][Λ[QU, t_k, T_k, y_k, a_k, x_k, ξ, η, δ] 𝕖^q]]
     /. ν → (1 + ℏ^-1 (T_k - 1) δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]
```

```
With[
  {q𝕆 = 𝔈QU[{x₁, y₁}₁, {x₂, a₂, y₂}₂, ℏ (l₁₂ t₁ a₂ + l₂₂ t₂ a₂ + γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
    1 + ϵ (l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)]},
  {QU[q𝕆] // Short, HL[err = SimpT[QU[q𝕆] - QU[q𝕆 // SW_{x₁,y₁→1}]]]}
]
```

$\{\mathrm{QU}[a_2, a_2, a_2] \left(\frac{1}{2} \epsilon \hbar^2 l_2 l_{12}^2 t_1^2 + \epsilon \hbar^2 l_2 l_{12} l_{22} t_1 t_2 + \frac{1}{2} \epsilon \hbar^2 l_2 l_{22}^2 t_2^2\right) + \ll 54 \gg + \mathrm{QU}[\,] (\ll 1 \gg), 0\}$

Rewrite Rules

RR: Rewrite Rule. RQ: Revised Quadratic.

RR

```
RR[{u_i_, w_j_} → {vs__, k_}, {𝓊_, ω_}, RQ_, λ_][(𝒪 : C𝕆 | Q𝕆)[
    OrderlessPatternSequence[{lh___, u_i_, w_j_, rh___}_s_, more___, 𝔼[Q_, P_]]]] :=
  𝒪[{lh, Sequence @@ (#_k & /@ {vs}), rh}_s, more, 𝔼[
    (RQ /. (v : u | w | t | T) :→ v_k) + (Q /. u_i | w_j → 0),
    𝕖^-RQ DP_{u_i→D_𝓊, w_j→D_ω}[P][Λ[𝒪, t_k, T_k, y_k, a_k, x_k, 𝓊, ω, δ] 𝕖^RQ] /.
     {𝓊 → (∂_{𝓊_i} Q /. ω_j → 0), ω → (∂_{ω_j} Q /. 𝓊_i → 0), δ → ∂_{𝓊_i,ω_j} Q}
   ]];
```

𝔼

𝔼[L, Q, P] means $e^{\hbar(L+Q)} P$, where L is linear in the a's, Q is a combination of $x_i y_j$, and P is a perturbation polynomial. It should be interpreted via C𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...] (with some default for direct interpretation), or likewise via Q𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...]. In themselves, C𝕆 and Q𝕆 should have an interpretation in CU/QU by casting.

Alternative Algorithms

Logos

```
λalt[CU] := Module[{eq, d, b, c, so},
    eq = ρ@𝕖^(ξ x_CU).ρ@𝕖^(η y_CU) == ρ@𝕖^(d y_CU).ρ@𝕖^(c (t 1_CU - 2 ϵ a_CU)).ρ@𝕖^(b x_CU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Normal@Series[𝕖^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, $k}]];
```

```
{λalt[CU], HL[Simplify[λalt[CU] == λ[CU]]]}
```

$$\left\{1+\epsilon\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right),\ \text{True}\right\}$$