Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$p = 3; $k = 1; ϵ /: ϵ^k_· /; k > $k := 0;
(* $k can't be ∞ at least because of Quesne. Can't be ≤
 1 at least because of the explicit ϵ^2 in S𝔻$g. *)
SetAttributes[{SS, SST}, HoldAll];
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
   Collect[Normal@Series[ℰ, {ℏ, 0, $p}], ℏ, Together] ];
SST[ℰ_] := Block[{ℏ, ϵ},
   Collect[Normal@Series[ℰ /. {T_i_ → e^(ℏ t_i/2), T → e^(ℏ t/2)}, {ℏ, 0, $p}], ℏ, Together] ];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, Collect[Normal@Series[#, {ℏ, 0, $p}], ℏ, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU,
   Collect[Normal@Series[# /. {T_i_ → e^(ℏ t_i/2), T → e^(ℏ t/2)}, {ℏ, 0, $p}], ℏ, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, CE, pow,
    gs = Generators /. {opts}, cs = Centrals /. {opts} },
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gen's pattern *)
  sr = Thread[gs → Range@Length@gs]; (* sorting rule *)
  cp = Alternatives @@ cs; (* cent's pattern *)
  CE[ℰ_] := Collect[ℰ, _U, (Expand[#] /. ℏ^d_ /; d > $p :> 0) &];
  U_i_[ℰ_] := ℰ /. {t : cp :> t_i, u_U :> Replace[u, x_ :> x_i, 1]};
  U_i_[NCM[]] := U[];
  B[U@(x_)_i_, U@(y_)_i_] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** U[] := x; U[] ** x_ := x;
  (a_. * x_U) ** (b_. * y_U) := If[a b === 0, 0, CE[a b (x ** y)]];
  U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
    U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{} = U[];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[poly_, specs___] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List :> l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ :> (l /. x_i_ :> x_s));
    CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
      ]] /. x_null :> x
  ];
  pow[ℰ_, 0] = U[]; pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
    CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) :> c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  S_i_[c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i] **
      U_i[NCM @@ Reverse@Cases[u, x_i :> S@U@x]]]; ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) :→ (m[U[g]] = img), {1}];
  m[U[]] = V[];
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U :→ m[u]];)
```

Meta-Operations

QLImplementation

```
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing $sl_2^{\gamma\epsilon}$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ∈ a_CU - t CU[];
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{0.71875, {(28 t^2 γ^4 + 116 t γ^5 ∈ + 120 γ^6 ∈^2) CU[y, y, y, x, x] +
   <<23>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y_1, a_1, x_1}},
 Table[HL@Simp[S_1[z1 ** z2] - S_1[z2] ** S_1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Implementing $\mathcal{U}_{\gamma\epsilon;\hbar}$

With $q = e^{\hbar\gamma\epsilon}$, $A = e^{-\hbar\epsilon a}$, $T = e^{\hbar t/2}$, and $[f, g]_q := fg - qgf$, our algebra is $\mathcal{U}_{\gamma\epsilon;\hbar} = \langle t, y, a, x\rangle / \mathcal{R}$, where $\mathcal{R} = \left([t, *] = 0,\ [a, y] = -\gamma y,\ [x, y]_q = \left(1 - T^2 A^2\right)/\hbar,\ [x, a] = -\gamma x\right)$.

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q = SS[e^(γ ϵ ħ)]; (*T=SS[e^(ħ t/2)];*)
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] = (q - 1) QU@{y, x} + O_QU[SS[(1 - T^2 e^(-2 ϵ a ħ)) / ħ], {a}];
(S@y_QU = O_QU[SS[-T^-2 e^(ħ ϵ a) y], {a, y}]; S@a_QU = -a_QU; S@x_QU = O_QU[SS[-e^(ħ ϵ a) x], {a, x}];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\Big\{\Big\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\, QU[y],\ \{QU[y], QU[x]\} \to \frac{(-1 + T^2)\, QU[\,]}{\hbar} - 2 T^2 \epsilon\, QU[a] + 2 T^2 \epsilon^2 \hbar\, QU[a, a] + \left(-\gamma \epsilon \hbar - \frac{1}{2} \gamma^2 \epsilon^2 \hbar^2\right) QU[y, x]\Big\},$$
$$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$$
$$\Big\{\{QU[x], QU[y]\} \to \frac{(1 - T^2)\, QU[\,]}{\hbar} + 2 T^2 \epsilon\, QU[a] - 2 T^2 \epsilon^2 \hbar\, QU[a, a] + \left(\gamma \epsilon \hbar + \frac{1}{2} \gamma^2 \epsilon^2 \hbar^2\right) QU[y, x],$$
$$\{QU[x], QU[a]\} \to -\gamma\, QU[x],\ \{QU[x], QU[x]\} \to 0\Big\}\Big\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

$\{37.6406,$
$\left\{\left(135\,\gamma^6\,\epsilon^2 - 730\,T^2\,\gamma^6\,\epsilon^2 + 715\,T^4\,\gamma^6\,\epsilon^2 + \frac{28\,\gamma^4 - 56\,T^2\,\gamma^4 + 28\,T^4\,\gamma^4}{\hbar^2} + \frac{82\,\gamma^5\,\epsilon - 280\,T^2\,\gamma^5\,\epsilon + 198\,T^4\,\gamma^5\,\epsilon}{\hbar}\right)\right.$
$\text{QU}[y, y, y, x, x] + \ll 22 \gg + (\ll 1 \gg)\,\text{QU}[y, y, \ll 10 \gg, x], 0\}\}$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas}]]
```

$\{\{\{\text{QU}[y_1], \text{QU}[y_1]\} \to 0, \{\text{QU}[y_1], \text{QU}[a_1]\} \to 0, \{\text{QU}[y_1], \text{QU}[x_1]\} \to 0\},$
$\{\{\text{QU}[a_1], \text{QU}[y_1]\} \to 0, \{\text{QU}[a_1], \text{QU}[a_1]\} \to 0, \{\text{QU}[a_1], \text{QU}[x_1]\} \to 0\},$
$\{\{\text{QU}[x_1], \text{QU}[y_1]\} \to 0, \{\text{QU}[x_1], \text{QU}[a_1]\} \to 0, \{\text{QU}[x_1], \text{QU}[x_1]\} \to 0\}\}$

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. {QU → CU, T → e^(ℏ t/2)}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{37.6719,$
$\{2\,(8\,t^2\,\gamma^4 + 16\,t\,\gamma^5\,\epsilon)\,\text{CU}[y, y, y, x, x] + (8\,t\,\gamma^5\,\epsilon + 16\,\gamma^6\,\epsilon^2)\,\text{CU}[y, y, y, x, x] + \ll 106 \gg +$
$\text{CU}[y, y, y, y, y, a, a, a, a, x, x, x, x], (-8\,T^2\,\gamma^6\,\epsilon^2 + 8\,T^4\,\gamma^6\,\epsilon^2)\,\text{QU}[y, y, y, x, x] + \ll 489 \gg +$
$\left(\gamma\,\epsilon\,\hbar + \frac{15}{2}\,\gamma^2\,\epsilon^2\,\hbar^2\right)\text{QU}[y, y, y, y, y, a, a, a, a, x, x, x, x], 0\}\}$

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆_QU[SS[-T^-1 e^(ℏ ϵ a) x], {a, x}],
   a → -a_QU, x → 𝕆_QU[SS[-T^-1 e^(ℏ ϵ a) y], {a, y}]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}]]
```

$\{\text{CU}[y] \to -\text{CU}[x] \to \text{CU}[y], \text{CU}[a] \to -\text{CU}[a] \to \text{CU}[a], \text{CU}[x] \to -\text{CU}[y] \to \text{CU}[x]\}$

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Qθ[Qθ[z]]], {z, bas}] ]
```

$$\left\{QU[y] \to -\frac{QU[x]}{T} - \frac{\epsilon\hbar\, QU[a,x]}{T} - \frac{\epsilon^2\hbar^2\, QU[a,a,x]}{2T} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$$
$$\left. QU[x] \to \left(-\frac{1}{T} + \frac{\gamma\epsilon\hbar}{T} - \frac{\gamma^2\epsilon^2\hbar^2}{2T}\right) QU[y] + \left(-\frac{\epsilon\hbar}{T} + \frac{\gamma\epsilon^2\hbar^2}{T}\right) QU[y,a] - \frac{\epsilon^2\hbar^2\, QU[y,a,a]}{2T} \to QU[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\mathbb{AD}\$f = \frac{\gamma}{\hbar} e^{\hbar\left(\frac{t}{2}-(a+\gamma)\epsilon\right)} \frac{\text{Cosh}\left[\hbar\left(a\epsilon + \frac{\gamma\epsilon}{2} - \frac{t}{2}\right)\right] - \text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\epsilon}{2}\right)^2 + \epsilon\omega}\right]}{\text{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(a^2\epsilon + a\gamma\epsilon - a t - \omega\right)};$$

Scaling behaviour of A𝔻$f:

$$\text{HL@Simplify}\left[\mathbb{AD}\$f == \left(\left(\mathbb{AD}\$f\ /.\ \gamma \to 1\right)\ /.\ \left\{\epsilon \to \gamma\epsilon,\ a \to \gamma^{-1} a,\ \omega \to \gamma^{-1}\omega\right\}\right)\right]$$

True

$$\text{HL@FullSimplify}\left[\mathbb{AD}\$f == \left(\left(\mathbb{AD}\$f\ /.\ \gamma \to 1\right)\ /.\ \left\{\hbar \to \gamma^2\hbar,\ \epsilon \to \epsilon/\gamma,\ a \to a/\gamma,\ t \to \gamma^{-2} t,\ \omega \to \gamma^{-3}\omega\right\}\right)\right]$$

True

ADeq

$$\mathbb{AD}\$\omega = \gamma\, CU[y, x] + \epsilon\, CU[a, a] - (t - \gamma\epsilon)\, CU[a];$$

ADeq

$$\text{DeclareMorphism}[\mathbb{AD},\ QU \to CU,\ \{a \to a_{CU},\ x \to CU@x,\ y \to \mathbb{S}_{CU}[SS[\mathbb{AD}\$f],\ a \to a_{CU},\ \omega \to \mathbb{AD}\$\omega] ** y_{CU}\}]$$

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$\mathbb{SD}\$g = \sqrt{\frac{\text{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\epsilon^2+4\epsilon\varpi}\right]-\text{Cosh}\left[\frac{\hbar}{2}\left(t-(2a+\gamma)\epsilon\right)\right]}{\text{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(t(2a+\gamma)-2a(a+\gamma)\epsilon+2\varpi\right)\hbar/(2\gamma)}};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

$$\Big\{\mathbb{SD}\$P = \frac{\text{Cosh}\left[\hbar\left(\frac{\epsilon-t}{2}+\epsilon a\right)\right]-\text{Cosh}\left[\hbar\sqrt{\frac{t^2+\epsilon^2}{4}+\epsilon\varpi}\right]}{\hbar\,\text{Sinh}\left[\frac{-\epsilon\hbar}{2}\right]\left(\varpi-\epsilon a^2+(t-\epsilon)a+t/2\right)},$$

```
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
     SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 Simplify[SD$g == FullSimplify[
     Sqrt[SD$Q] /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\Big\{-\left(\left(\left(\text{Cosh}\left[\left(a\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\text{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\varpi}\;\hbar\right]\right)\text{Csch}\left[\frac{\epsilon\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a(t-\epsilon)-a^2\epsilon+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},$$

$$-\left(\left(4\left(\text{Cosh}\left[\frac{1}{2}(t-2c\epsilon)\hbar\right]-\text{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\epsilon\varpi}\;\hbar\right]\right)\text{Csch}\left[\frac{\epsilon\hbar}{2}\right]\right)\Big/\left((4ct+\epsilon-4c^2\epsilon+4\varpi)\hbar\right)\right),\ \text{True},\ \text{True}\Big\}$$

SDeq

$$\mathbb{SD}\$f = \text{FullSimplify}\left[e^{\hbar(t/2-\epsilon a)}\left(\mathbb{SD}\$g\ /.\ \{a\to -a,\ t\to -t\}\right)\right];$$

SDeq

$$\mathbb{SD}\$\varpi = \gamma\,\text{CU}[y, x] + \epsilon\,\text{CU}[a, a] - (t-\gamma\epsilon)\,\text{CU}[a] - t\gamma\,\text{CU}[\,]/2;$$

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
   x → S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
   y → S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

R in QU.

Quesne’s formula:

Quesne

$$\mathbb{e}_{q\_,k\_}[x\_] := \mathbb{e}\verb|^|\left(\sum_{j=1}^{k} \frac{(1-q)^j x^j}{j\,(1-q^j)}\right);\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
Table[Together@SeriesCoefficient[e_{ρ,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1,1,\frac{1}{1+\rho},\frac{1}{(1+\rho)\,(1+\rho+\rho^2)},\frac{1}{(1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)},\frac{1}{(1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)\,(1+\rho+\rho^2+\rho^3+\rho^4)}\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, ρ] SeriesCoefficient[e_{ρ,5}[x], {x, 0, n}]], {n, 0, 5}]
```

```
{1, 1, 1, 1, 1, 1}
```

R

$$\mathrm{QU}[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[SS\left[\mathbb{e}^{\hbar b_1 a_2}\,\mathbb{e}_q[\hbar y_1 x_2]\ /.\ b_1 \to \gamma^{-1}\left(\epsilon a_1 - t_i\right)\right],\ \{y_1, a_1\}_i,\ \{a_2, x_2\}_j\right];$$
$$\mathrm{QU}\left[R^{-1}_{i\_,j\_}\right] := S_j @ \mathrm{QU}[R_{i,j}];$$

```
QU[R_{3,4}] // Short
```

$$\mathrm{QU}[\,] + \frac{\epsilon\,\hbar\,\mathrm{QU}[a_3, a_4]}{\gamma} + \hbar\,\mathrm{QU}[y_3, x_4] + \ll 20 \gg + \frac{\hbar^3\,\mathrm{QU}[y_3, a_4, a_4, x_4]\,t_3^2}{2\,\gamma^2} - \frac{\hbar^3\,\mathrm{QU}[a_4, a_4, a_4]\,t_3^3}{6\,\gamma^3}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^{-1}_{1,2}] // Simp // HL // Timing
```

```
{0.515625, QU[]}
```

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$\{11.75, \{QU[] + \frac{\epsilon\hbar\, QU[a_1, a_2]}{\gamma} + \ll 455 \gg + QU[y_1, y_1, y_1, x_3, x_3, x_3]\left(\frac{\hbar^3}{6} - \frac{1}{2}\hbar^3 T_2^2 + \frac{1}{2}\hbar^3 T_2^4 - \frac{1}{6}\hbar^3 T_2^6\right), 0\}\}$

The representation ρ

rho

ρ@y_{CU} = ρ@y_{QU} = $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$; ρ@a_{CU} = ρ@a_{QU} = $\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$;

ρ@x_{CU} = $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$; ρ@x_{QU} = SS@$\begin{pmatrix} 0 & (1 - e^{-\gamma\epsilon\hbar}) / (\epsilon\hbar) \\ 0 & 0 \end{pmatrix}$;

$\rho[e^{\mathcal{E}\_}]$:= MatrixExp[$\rho[\mathcal{E}]$];

$\rho[\mathcal{E}\_]$:=

$(\mathcal{E}$ /. $\{t \to \gamma\epsilon, T \to e^{\hbar\gamma\epsilon/2}\}$ /. $(U : CU \mid QU)[u\_\_\_] :\to$ Fold[Dot, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, ρ /@ U /@ {u}])

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // SS // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

The Classical Logos CΛ

Lemma 3C. To degree k,
$\mathbb{O}_{CU}(e^{\eta y+\xi x+\delta y x} \mid x y) = \mathbb{O}_{CU}(v\, e^{v(-t\xi\eta+\eta y+\xi x+\delta y x)}\, C\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta) \mid y\, a\, x)$, with $v = (1 + t\,\delta)^{-1}$ and where $C\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta)$ is a fixed polynomial of degree at most $4\,k$ in y, $\sqrt{a}$, x, η, ξ, with scalar coefficients.

Comment. Even better, $\log(C\Lambda_k)$ is of degree at most $2\,k + 2$ in said variables.

$\text{eqn} = \rho[e^{\xi x_{CU}}].\rho[e^{\eta y_{CU}}] == \rho[e^{d\, y_{CU}}].\rho[e^{c\,(t\, CU[] - 2\epsilon\, a_{CU})}].\rho[e^{b\, x_{CU}}]$

$\{\{1 + \gamma\epsilon\eta\xi, \gamma\xi\}, \{\epsilon\eta, 1\}\} == \{\{e^{-c\gamma\epsilon}, b\, e^{-c\gamma\epsilon}\gamma\}, \{d\, e^{-c\gamma\epsilon}\epsilon, e^{c\gamma\epsilon} + b\, d\, e^{-c\gamma\epsilon}\gamma\epsilon\}\}$

```
sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0
```

$\left\{d \to \frac{\eta}{1 + \gamma\epsilon\eta\xi}, b \to \frac{\xi}{1 + \gamma\epsilon\eta\xi}, c \to \frac{\text{Log}\left[\frac{1}{1+\gamma\epsilon\eta\xi}\right]}{\gamma\epsilon}\right\}$

Proof of Lemma 3C. We know that $\mathbb{O}_{CU}(e^{\xi x + \eta y} \mid x\, y) = \mathbb{O}_{CU}(e^{c\,t + a\,y - 2\epsilon\, c\, a + b\, x} \mid y\, a\, x)$, with $\left\{d \to \frac{\eta}{1+\gamma\epsilon\eta\xi}, b \to \frac{\xi}{1+\gamma\epsilon\eta\xi}, c \to \frac{\text{Log}[1+\gamma\epsilon\eta\xi]}{-\gamma\epsilon}\right\}$. Expanding in ϵ we get
$\mathbb{O}_{CU}(e^{\xi x + \eta y} \mid x\, y) = \mathbb{O}_{CU}(\lambda_\epsilon(\xi, \eta)\, e^{\eta y + \xi x - \eta\xi t} \mid y\, a\, x)$ and so
$\mathbb{O}_{CU}(e^{\eta y + \xi x + \delta y x} \mid x\, y) = \mathbb{O}(\lambda_\epsilon(\partial_x, \partial_y)\, e^{\delta\partial_\eta\partial_\xi}\, e^{\eta y + \xi x - \eta\xi t} \mid y\, a\, x) = \mathbb{O}(\lambda_\epsilon(\partial_x, \partial_y)\, v\, e^{v(-t\xi\eta+\eta y+\xi x+\delta y x)} \mid y\, a\, x)$.

Logos

```
SSϵ[ℰ_] := Block[{ϵ}, Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, Together]];
(* Shielded ϵ-Series *)
CΛ[t1_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
   {eqn, d, b, c, sol, λ, q, ν, ξ, η},
   eqn = ρ[ⅇ^(ξ x_CU)].ρ[ⅇ^(η y_CU)] == ρ[ⅇ^(d y_CU)].ρ[ⅇ^(c (t CU[] - 2 ϵ a_CU))].ρ[ⅇ^(b x_CU)];
   sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0;
   λ = Simplify[ⅇ^(-η y-ξ x+η ξ t) SSϵ[ⅇ^(c t + d y - 2 ϵ c a + b x) /. sol]];
   q = ⅇ^(ν (-t ξ η+η y+ξ x+δ y x));
   Collect[ν q^-1 DP_(ξ→D_x, η→D_y)[λ][q] /. ν → (1 + t δ)^-1, ϵ, Simplify] /.
    {t → t1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1} ];
```

CΛ[t, y, a, x, ξ, η, δ]

$$\frac{1}{1+t\,\delta}+\frac{1}{24\left(1+t\,\delta\right)^9}\epsilon^2\Big(48\,a^2\left(1+t\,\delta\right)^4\left(2\,\delta^2\left(1+t\,\delta\right)^2+4\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)-$$
$$24\,a\,\gamma\left(1+t\,\delta\right)^4\left(2\,\delta^2\left(1+t\,\delta\right)^2+4\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)-$$
$$48\,a\,y\,\gamma\left(1+t\,\delta\right)^3(x\,\delta+\eta)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)+24\,y\,\gamma^2\left(1+t\,\delta\right)^3$$
$$(x\,\delta+\eta)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)-48\,a\,x\,\gamma$$
$$\left(1+t\,\delta\right)^3(y\,\delta+\xi)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)+$$
$$24\,x\,\gamma^2\left(1+t\,\delta\right)^3(y\,\delta+\xi)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)+12\,y^2\,\gamma^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)^2$$
$$\left(12\,\delta^2\left(1+t\,\delta\right)^2+8\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)+12\,x^2\,\gamma^2$$
$$\left(1+t\,\delta\right)^2(y\,\delta+\xi)^2\left(12\,\delta^2\left(1+t\,\delta\right)^2+8\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)(y\,\delta+\xi)+(x\,\delta+\eta)^2(y\,\delta+\xi)^2\right)+$$
$$24\,a\,t\,\gamma\left(1+t\,\delta\right)^2\left(6\,\delta^3\left(1+t\,\delta\right)^3+18\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)(y\,\delta+\xi)+9\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2(y\,\delta+\xi)^2+(x\,\delta+\eta)^3(y\,\delta+\xi)^3\right)-$$
$$8\,t\left(\gamma+t\,\gamma\,\delta\right)^2\left(6\,\delta^3\left(1+t\,\delta\right)^3+18\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)(y\,\delta+\xi)+9\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2(y\,\delta+\xi)^2+(x\,\delta+\eta)^3(y\,\delta+\xi)^3\right)+$$
$$24\,x\,y\left(\gamma+t\,\gamma\,\delta\right)^2\left(6\,\delta^3\left(1+t\,\delta\right)^3+18\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)(y\,\delta+\xi)+9\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2(y\,\delta+\xi)^2+(x\,\delta+\eta)^3(y\,\delta+\xi)^3\right)-$$
$$12\,t\,y\,\gamma^2\left(1+t\,\delta\right)(x\,\delta+\eta)\left(24\,\delta^3\left(1+t\,\delta\right)^3+36\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)(y\,\delta+\xi)+12\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2(y\,\delta+\xi)^2+(x\,\delta+\eta)^3(y\,\delta+\xi)^3\right)-$$
$$12\,t\,x\,\gamma^2\left(1+t\,\delta\right)(y\,\delta+\xi)\left(24\,\delta^3\left(1+t\,\delta\right)^3+36\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)(y\,\delta+\xi)+12\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2(y\,\delta+\xi)^2+(x\,\delta+\eta)^3(y\,\delta+\xi)^3\right)+$$
$$3\,t^2\,\gamma^2\left(24\,\delta^4\left(1+t\,\delta\right)^4+96\,\delta^3\left(1+t\,\delta\right)^3(x\,\delta+\eta)(y\,\delta+\xi)+72\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)^2(y\,\delta+\xi)^2+16\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^3(y\,\delta+\xi)^3+(x\,\delta+\eta)^4(y\,\delta+\xi)^4\right)\Big)+$$
$$\frac{1}{2\left(1+t\,\delta\right)^5}\epsilon\Big(4\,a\left(1+t\,\delta\right)^2\left(\left(t+x\,y\right)\delta^2+\eta\,\xi+\delta\left(1+y\,\eta+x\,\xi\right)\right)+$$
$$\gamma\Big(2\,t^3\,\delta^4+4\,t^2\,\delta^2\left(\delta-x\,y\,\delta^2+\eta\,\xi\right)-2\left(y\,\eta\left(\delta\left(2+y\,\eta\right)+\eta\,\xi\right)+x^2\,\delta\left(2\,y^2\,\delta^2+3\,y\,\delta\,\xi+\xi^2\right)+x\left(3\,y^2\,\delta^2\,\eta+4\,y\,\delta\,(\delta+\eta\,\xi)+\xi\left(2\,\delta+\eta\,\xi\right)\right)\right)-t\left(3\,x^2\,y^2\,\delta^4-4\,\delta\,\eta\,\xi-\eta^2\,\xi^2+4\,x\,y\,\delta^3\left(3+y\,\eta+x\,\xi\right)+\delta^2\left(-2+y^2\,\eta^2+4\,x\,\xi+x^2\,\xi^2+4\,y\,(\eta+x\,\eta\,\xi)\right)\right)\Big)\Big)$$

```
{Short[lhs = 𝕆CU[SS[ℯ^(ℏ (ξ x+η y+δ x y))], {x, y}], 5], HL[lhs ==
   𝕆CU[SS[ℯ^(ℏ ν (ξ x+η y+δ x y-t ℏ ξ η)) CΛ[t, y, a, x, ℏ ξ, ℏ η, ℏ δ] /. ν → (1 + ℏ t δ)^-1], {y, a, x}]]}
```

$\{(1 - t\,\delta\,\hbar + t^2\,\delta^2\,\hbar^2 + t\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - t\,\eta\,\xi\,\hbar^2 - t^3\,\delta^3\,\hbar^3 - 3\,t^2\,\gamma\,\delta^3\,\epsilon\,\hbar^3 - 2\,t\,\gamma^2\,\delta^3\,\epsilon^2\,\hbar^3 + 2\,t^2\,\delta\,\eta\,\xi\,\hbar^3 + 2\,t\,\gamma\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3)\,\text{CU}[\,] + (2\,\delta\,\epsilon\,\hbar - 4\,t\,\delta^2\,\epsilon\,\hbar^2 - 2\,\gamma\,\delta^2\,\epsilon^2\,\hbar^2 + 2\,\epsilon\,\eta\,\xi\,\hbar^2 + 6\,t^2\,\delta^3\,\epsilon\,\hbar^3 + 12\,t\,\gamma\,\delta^3\,\epsilon^2\,\hbar^3 - 8\,t\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3 - 4\,\gamma\,\delta\,\epsilon^2\,\eta\,\xi\,\hbar^3)\,\text{CU}[a] + (\xi\,\hbar - 2\,t\,\delta\,\xi\,\hbar^2 - 2\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2 + 3\,t^2\,\delta^2\,\xi\,\hbar^3 + 9\,t\,\gamma\,\delta^2\,\epsilon\,\xi\,\hbar^3 + 6\,\gamma^2\,\delta^2\,\epsilon^2\,\xi\,\hbar^3 - t\,\eta\,\xi^2\,\hbar^3 - \gamma\,\epsilon\,\eta\,\xi^2\,\hbar^3)\,\text{CU}[x] + \ll 23 \gg + \frac{1}{2}\,\delta^2\,\xi\,\hbar^3\,\text{CU}[y, y, x, x, x] + \frac{1}{2}\,\delta^2\,\eta\,\hbar^3\,\text{CU}[y, y, y, x, x] + \frac{1}{6}\,\delta^3\,\hbar^3\,\text{CU}[y, y, y, x, x, x], \text{True}\}$

The Quantum Logos QΛ

Goal 1: In QU, compute $F = e^{-\eta y}\, e^{\xi x}\, e^{\eta y}\, e^{-\xi x}$.

First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum.

```
adx[ℰ_] := Simp[QU@x ** ℰ - ℰ ** QU@x];
G = Simp[NestList[adx, QU@y, $k + 1].Table[ξ^k / k!, {k, 0, $k + 1}]]
```

$\frac{(\xi - T^2\,\xi)\,\text{QU}[\,]}{\hbar} + 2\,T^2\,\epsilon\,\xi\,\text{QU}[a] + \left(\frac{1}{2}\,\gamma\,\epsilon\,\xi^2 - \frac{3}{2}\,T^2\,\gamma\,\epsilon\,\xi^2 + \left(\frac{1}{4}\,\gamma^2\,\epsilon^2\,\xi^2 - \frac{5}{4}\,T^2\,\gamma^2\,\epsilon^2\,\xi^2\right)\hbar\right)\text{QU}[x] + \text{QU}[y] - 2\,T^2\,\epsilon^2\,\xi\,\hbar\,\text{QU}[a, a] + 3\,T^2\,\gamma\,\epsilon^2\,\xi^2\,\hbar\,\text{QU}[a, x] + \left(\frac{1}{6}\,\gamma^2\,\epsilon^2\,\xi^3 - \frac{7}{6}\,T^2\,\gamma^2\,\epsilon^2\,\xi^3\right)\hbar\,\text{QU}[x, x] + \left(\gamma\,\epsilon\,\xi\,\hbar + \frac{1}{2}\,\gamma^2\,\epsilon^2\,\xi\,\hbar^2\right)\text{QU}[y, x] + \frac{1}{2}\,\gamma^2\,\epsilon^2\,\xi^2\,\hbar^2\,\text{QU}[y, x, x]$

```
G /. ϵ → 0
```

$\frac{(\xi - T^2\,\xi)\,\text{QU}[\,]}{\hbar} + \text{QU}[y]$

Now F satisfies the ODE $\partial_\eta F = \partial_\eta\left(e^{-\eta y}\, e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. We set it up and solve:

```
F = Sum[f_{l,i,j,k}[η] ϵ^l QU@{y^i, a^j, x^k},
  {l, 0, $k}, {i, 0, l}, {j, 0, l}, {k, 0, Min[l, 2 l - i - j]}]
```

$\text{QU}[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,\text{QU}[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,\text{QU}[x]\,f_{1,0,0,1}[\eta] + \epsilon\,\text{QU}[a]\,f_{1,0,1,0}[\eta] + \epsilon\,\text{QU}[a, x]\,f_{1,0,1,1}[\eta] + \epsilon\,\text{QU}[y]\,f_{1,1,0,0}[\eta] + \epsilon\,\text{QU}[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,\text{QU}[y, a]\,f_{1,1,1,0}[\eta] + \epsilon^2\,\text{QU}[\,]\,f_{2,0,0,0}[\eta] + \epsilon^2\,\text{QU}[x]\,f_{2,0,0,1}[\eta] + \epsilon^2\,\text{QU}[x, x]\,f_{2,0,0,2}[\eta] + \epsilon^2\,\text{QU}[a]\,f_{2,0,1,0}[\eta] + \epsilon^2\,\text{QU}[a, x]\,f_{2,0,1,1}[\eta] + \epsilon^2\,\text{QU}[a, x, x]\,f_{2,0,1,2}[\eta] + \epsilon^2\,\text{QU}[a, a]\,f_{2,0,2,0}[\eta] + \epsilon^2\,\text{QU}[a, a, x]\,f_{2,0,2,1}[\eta] + \epsilon^2\,\text{QU}[a, a, x, x]\,f_{2,0,2,2}[\eta] + \epsilon^2\,\text{QU}[y]\,f_{2,1,0,0}[\eta] + \epsilon^2\,\text{QU}[y, x]\,f_{2,1,0,1}[\eta] + \epsilon^2\,\text{QU}[y, x, x]\,f_{2,1,0,2}[\eta] + \epsilon^2\,\text{QU}[y, a]\,f_{2,1,1,0}[\eta] + \epsilon^2\,\text{QU}[y, a, x]\,f_{2,1,1,1}[\eta] + \epsilon^2\,\text{QU}[y, a, x, x]\,f_{2,1,1,2}[\eta] + \epsilon^2\,\text{QU}[y, a, a]\,f_{2,1,2,0}[\eta] + \epsilon^2\,\text{QU}[y, a, a, x]\,f_{2,1,2,1}[\eta] + \epsilon^2\,\text{QU}[y, y]\,f_{2,2,0,0}[\eta] + \epsilon^2\,\text{QU}[y, y, x]\,f_{2,2,0,1}[\eta] + \epsilon^2\,\text{QU}[y, y, x, x]\,f_{2,2,0,2}[\eta] + \epsilon^2\,\text{QU}[y, y, a]\,f_{2,2,1,0}[\eta] + \epsilon^2\,\text{QU}[y, y, a, x]\,f_{2,2,1,1}[\eta] + \epsilon^2\,\text{QU}[y, y, a, a]\,f_{2,2,2,0}[\eta]$

```
unowns = Cases[F, f___[η], ∞]
```

$\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta], f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{2,0,0,0}[\eta], f_{2,0,0,1}[\eta], f_{2,0,0,2}[\eta], f_{2,0,1,0}[\eta], f_{2,0,1,1}[\eta], f_{2,0,1,2}[\eta], f_{2,0,2,0}[\eta], f_{2,0,2,1}[\eta], f_{2,0,2,2}[\eta], f_{2,1,0,0}[\eta], f_{2,1,0,1}[\eta], f_{2,1,0,2}[\eta], f_{2,1,1,0}[\eta], f_{2,1,1,1}[\eta], f_{2,1,1,2}[\eta], f_{2,1,2,0}[\eta], f_{2,1,2,1}[\eta], f_{2,2,0,0}[\eta], f_{2,2,0,1}[\eta], f_{2,2,0,2}[\eta], f_{2,2,1,0}[\eta], f_{2,2,1,1}[\eta], f_{2,2,2,0}[\eta]\}$

```
bas = Union @@ Table[ϵ^l Cases[Coefficient[F, ϵ, l], _QU, ∞], {l, 0, $k}]
```

$\{QU[], \epsilon\, QU[], \epsilon^2 QU[], \epsilon\, QU[a], \epsilon^2 QU[a], \epsilon\, QU[x], \epsilon^2 QU[x], \epsilon\, QU[y], \epsilon^2 QU[y], \epsilon^2 QU[a, a], \epsilon\, QU[a, x], \epsilon^2 QU[a, x], \epsilon^2 QU[x, x], \epsilon\, QU[y, a], \epsilon^2 QU[y, a], \epsilon\, QU[y, x], \epsilon^2 QU[y, x], \epsilon^2 QU[y, y], \epsilon^2 QU[a, a, x], \epsilon^2 QU[a, x, x], \epsilon^2 QU[y, a, a], \epsilon^2 QU[y, a, x], \epsilon^2 QU[y, x, x], \epsilon^2 QU[y, y, a], \epsilon^2 QU[y, y, x], \epsilon^2 QU[a, a, x, x], \epsilon^2 QU[y, a, a, x], \epsilon^2 QU[y, a, x, x], \epsilon^2 QU[y, y, a, a], \epsilon^2 QU[y, y, a, x], \epsilon^2 QU[y, y, x, x]\}$

```
Short[eqns = Flatten[{(Coefficient[F - QU[], #] /. η → 0) == 0,
     Expand[Coefficient[Simp[F ** G - QU[y] ** F - ∂_η F], #]] == 0} & /@ bas], 8]
```

$$\Big\{-1 + f_{0,0,0,0}[0] + \epsilon\, f_{1,0,0,0}[0] + \epsilon^2 f_{2,0,0,0}[0] == 0,\ \frac{\xi f_{0,0,0,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{0,0,0,0}[\eta]}{\hbar} + \frac{\epsilon \xi f_{1,0,0,0}[\eta]}{\hbar} - \frac{T^2 \epsilon \xi f_{1,0,0,0}[\eta]}{\hbar} + \frac{\epsilon f_{1,0,0,1}[\eta]}{\hbar} - \frac{T^2 \epsilon f_{1,0,0,1}[\eta]}{\hbar} + \frac{\epsilon^2 \xi f_{2,0,0,0}[\eta]}{\hbar} - \frac{T^2 \epsilon^2 \xi f_{2,0,0,0}[\eta]}{\hbar} + \frac{\epsilon^2 f_{2,0,0,1}[\eta]}{\hbar} - \frac{T^2 \epsilon^2 f_{2,0,0,1}[\eta]}{\hbar} - f_{0,0,0,0}'[\eta] - \epsilon\, f_{1,0,0,0}'[\eta] - \epsilon^2 f_{2,0,0,0}'[\eta] == 0,\ f_{1,0,0,0}[0] == 0,$$

$$\frac{\xi f_{1,0,0,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{1,0,0,0}[\eta]}{\hbar} + \frac{f_{1,0,0,1}[\eta]}{\hbar} - \frac{T^2 f_{1,0,0,1}[\eta]}{\hbar} - f_{1,0,0,0}'[\eta] == 0,$$

$$f_{2,0,0,0}[0] == 0,\ \ll 52 \gg,\ \frac{\xi f_{2,2,2,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,2,0}[\eta]}{\hbar} - f_{2,2,2,0}'[\eta] == 0,\ f_{2,2,1,1}[0] == 0,$$

$$\gamma \xi \hbar f_{1,1,1,0}[\eta] - 2 \gamma f_{2,1,2,1}[\eta] + \frac{\xi f_{2,2,1,1}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,1,1}[\eta]}{\hbar} - f_{2,2,1,1}'[\eta] == 0,$$

$$f_{2,2,0,2}[0] == 0,\ \gamma \xi \hbar f_{1,1,0,1}[\eta] - \gamma f_{2,1,1,2}[\eta] + \frac{\xi f_{2,2,0,2}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,0,2}[\eta]}{\hbar} - f_{2,2,0,2}'[\eta] == 0\Big\}$$

```
Short[{sol} = DSolve[eqns, unowns, η], 8]
```

$$\left\{\left\{f_{0,0,0,0}[\eta] \to e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}},\ f_{1,0,0,0}[\eta] \to \frac{e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+T^2\right)\left(-1+3T^2\right)\gamma\eta^2\xi^2}{4\hbar},\right.\right.$$
$$f_{1,0,0,1}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+3T^2\right)\gamma\eta\xi^2,\ f_{1,0,1,0}[\eta] \to 2e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\eta\xi,$$
$$f_{1,0,1,1}[\eta] \to 0,\ f_{1,1,0,0}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+3T^2\right)\gamma\eta^2\xi,$$
$$f_{1,1,0,1}[\eta] \to e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma\eta\xi\hbar,\ f_{1,1,1,0}[\eta] \to 0,\ \ll 15 \gg,\ f_{2,1,2,0}[\eta] \to 0,\ f_{2,1,2,1}[\eta] \to 0,$$
$$f_{2,2,0,0}[\eta] \to \frac{1}{24}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\eta^3\xi\left(3\eta\xi-18T^2\eta\xi+27T^4\eta\xi+4\hbar-28T^2\hbar\right),$$
$$f_{2,2,0,1}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\eta^2\xi\left(-\eta\xi+3T^2\eta\xi-\hbar\right)\hbar,$$
$$\left.\left.f_{2,2,0,2}[\eta] \to \frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\eta^2\xi^2\hbar^2,\ f_{2,2,1,0}[\eta] \to 0,\ f_{2,2,1,1}[\eta] \to 0,\ f_{2,2,2,0}[\eta] \to 0\right\}\right\}$$

```
Union@Cases[sol, e-, ∞]
```

$$\left\{e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\right\}$$

```
FF = Collect[F /. sol /. {e- → 1, QU → Times}, ϵ, Simplify]
```

$$1+\frac{1}{4\hbar}\epsilon\eta\xi\left(8aT^2\hbar+\left(-1+3T^2\right)\gamma\eta\left(\left(-1+T^2\right)\xi-2y\hbar\right)+2x\gamma\hbar\left(\xi-3T^2\xi+2y\hbar\right)\right)+$$
$$\frac{1}{288\hbar^2}\epsilon^2\eta\xi\left(576a^2T^2\left(T^2\eta\xi-\hbar\right)\hbar^2+144aT^2\gamma\hbar\left(6x\xi\hbar^2+\right.\right.$$
$$\left.\left(-1+3T^2\right)\eta^2\xi\left(\left(-1+T^2\right)\xi-2y\hbar\right)+2\eta\hbar\left(\left(x-3T^2x\right)\xi^2+3y\hbar+\xi\left(2-3T^2+2xy\hbar\right)\right)\right)+$$
$$\gamma^2\left(9\left(1-3T^2\right)^2\eta^3\xi\left(\xi-T^2\xi+2y\hbar\right)^2+24x\hbar^3\left(2\left(1-7T^2\right)x\xi^2+6y\hbar+\xi\left(3-15T^2+6xy\hbar\right)\right)+\right.$$
$$12\eta\hbar^2\left(3\left(1-3T^2\right)^2x^2\xi^3+6y\hbar\left(1-5T^2+2xy\hbar\right)+\right.$$
$$3\xi\left(1+5T^4+10xy\hbar+4x^2y^2\hbar^2-6T^2\left(1+7xy\hbar\right)\right)+$$
$$\left.2x\xi^2\left(5+41T^4+6xy\hbar-2T^2\left(17+9xy\hbar\right)\right)\right)-4\eta^2\hbar\left(9\left(1-3T^2\right)^2\left(-1+T^2\right)x\xi^3+\right.$$
$$12\left(-1+7T^2\right)y^2\hbar^2+6y\xi\hbar\left(-5-41T^4-6xy\hbar+2T^2\left(17+9xy\hbar\right)\right)+$$
$$\left.\left.\left.2\xi^2\left(-5+41T^6-18xy\hbar-3T^4\left(25+36xy\hbar\right)+T^2\left(39+90xy\hbar\right)\right)\right)\right)\right)$$

```
{Short[lhs = SimpT@𝕆QU[SS[e^(ℏ (ξ x+η y))], {x, y}], 5],
 HL[lhs == SimpT@𝕆QU[SS[e^(ℏ (ξ x+η y+(1-T^2) ξ η)) (FF /. {ξ → ℏ ξ, η → ℏ η})], {y, a, x}]]}
```

$$\left\{\left(1-t\eta\xi\hbar^2-\frac{1}{2}t^2\eta\xi\hbar^3\right)QU[\,]+\left(2\epsilon\eta\xi\hbar^2+2t\epsilon\eta\xi\hbar^3\right)QU[a]+\right.$$
$$\left(\xi\hbar+\left(-t\eta\xi^2-\gamma\epsilon\eta\xi^2\right)\hbar^3\right)QU[x]+\left(\eta\hbar+\left(-t\eta^2\xi-\gamma\epsilon\eta^2\xi\right)\hbar^3\right)QU[y]-$$
$$2\epsilon^2\eta\xi\hbar^3QU[a,a]+2\epsilon\eta\xi^2\hbar^3QU[a,x]+\frac{1}{2}\xi^2\hbar^2QU[x,x]+2\epsilon\eta^2\xi\hbar^3QU[y,a]+$$
$$\left(\eta\xi\hbar^2+\gamma\epsilon\eta\xi\hbar^3\right)QU[y,x]+\frac{1}{2}\eta^2\hbar^2QU[y,y]+\frac{1}{6}\xi^3\hbar^3QU[x,x,x]+$$
$$\left.\frac{1}{2}\eta\xi^2\hbar^3QU[y,x,x]+\frac{1}{2}\eta^2\xi\hbar^3QU[y,y,x]+\frac{1}{6}\eta^3\hbar^3QU[y,y,y],\ \text{True}\right\}$$

Logos

```
QΛ[T1_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
  {G, F, fs, f, bs, e, b, es, λ, q, ν, ξ, η, t},
  G = Simp[
    Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[x_QU ** # - # ** x_QU] &, y_QU, $k + 1]];
  fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
  F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l QU@{y^i, a^j, x^k});
  es = Flatten[Table[Coefficient[e, b] == 0,
     {e, {F - QU[] /. η → 0, F ** G - y_QU ** F - ∂_η F}}, {b, bs}]];
  {λ} = F /. DSolve[es, fs, η] /. {e- → 1, QU → Times};
  q = e^(ν (-t ξ η + η y + ξ x + δ y x));
  Collect[ν q^-1 DP_{ξ→D_x, η→D_y}[λ][q] /. ν → (1 + t δ)^-1 /. t → (T^2 - 1) / ħ, ϵ, Simplify] /.
   {T → T1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1} ];
```

```
QΛ[T, y, a, x, ξ, η, δ]
```

$$\frac{\hbar}{\left(-1+T^2\right)\delta+\hbar}+\frac{1}{4\left(\left(-1+T^2\right)\delta+\hbar\right)^5}\in\hbar^2\Big(8\,a\,T^2\left(\left(-1+T^2\right)\delta+\hbar\right)^2\left(\eta\,\xi\,\hbar+\delta\left(1+y\,\eta+x\,\xi\right)\hbar+\delta^2\left(-1+T^2+x\,y\,\hbar\right)\right)+$$
$$\gamma\Big(\eta\,\xi\,\hbar^2\left(\left(-1+3\,T^2\right)\eta\left(\left(-1+T^2\right)\xi-2\,y\,\hbar\right)+2\,x\,\hbar\left(\xi-3\,T^2\,\xi+2\,y\,\hbar\right)\right)+$$
$$\left(-1+T^2\right)\delta^4\left(-2+6\,T^6-x^2\,y^2\,\hbar^2-2\,T^4\,(7+4\,x\,y\,\hbar)+T^2\left(10+8\,x\,y\,\hbar-5\,x^2\,y^2\,\hbar^2\right)\right)-$$
$$4\,\delta^3\,\hbar\Big(1-3\,T^6+x^2\,y^2\,\hbar^2+T^4\left(7+2\,x\,y\left(3+y\,\eta\right)\hbar+2\,x^2\,y\,\xi\,\hbar\right)+$$
$$T^2\left(-5-2\,x\,y\left(3+y\,\eta\right)\hbar+x^2\,y\,\hbar\left(-2\,\xi+y\,\hbar\right)\right)\Big)+$$
$$2\,\delta\,\hbar^2\Big(\left(1-3\,T^2\right)y^2\,\eta^2\,\hbar+2\,\eta\left(\xi+3\,T^4\,\xi-4\,T^2\,\xi\left(1+x\,y\,\hbar\right)+y\,\hbar\left(1-3\,T^2+x\,y\,\hbar\right)\right)+$$
$$x\,\hbar\left(\left(x-3\,T^2\,x\right)\xi^2+2\,y\,\hbar+\xi\left(2-6\,T^2+2\,x\,y\,\hbar\right)\right)\Big)-$$
$$\delta^2\,\hbar\Big(\left(1-4\,T^2+3\,T^4\right)y^2\,\eta^2\,\hbar+\hbar\Big(-2+3\,T^4\left(-2+4\,x\,\xi+x^2\,\xi^2\right)+4\,x\,(\xi+y\,\hbar)+$$
$$x^2\left(\xi^2+2\,y\,\xi\,\hbar-4\,y^2\,\hbar^2\right)-2\,T^2\left(-4+x\left(8\,\xi-6\,y\,\hbar\right)+x^2\,\xi\left(2\,\xi-5\,y\,\hbar\right)\right)\Big)+2\,\eta$$
$$\left(-2\left(-1+T^2\right)\xi\left(1+3\,T^4-2\,T^2\left(2+x\,y\,\hbar\right)\right)+y\,\hbar\left(2+6\,T^4+x\,y\,\hbar+T^2\left(-8+5\,x\,y\,\hbar\right)\right)\right)\Big)\Big)\Big)$$

```
{Short[lhs = SimpT@𝕆QU[SS[ⅇ^(ℏ (ξ x+η y+δ x y))], {x, y}], 5],
 rhs = SimpT@𝕆QU[SS[
     ⅇ^(ℏ ν (ξ x+η y+δ x y-(T^2-1) ξ η)) QΛ[T, y, a, x, ℏ ξ, ℏ η, ℏ δ] /. ν → (1 + (T^2 - 1) δ)^-1], {y, a, x}];
 HL[Simplify[lhs == rhs]]}
```

$\{(1 - t\delta\hbar + (-\frac{t^2\delta}{2} + t^2\delta^2 + t\gamma\delta^2\epsilon - t\eta\xi)\hbar^2 + (-\frac{t^3\delta}{6} + t^3\delta^2 - t^3\delta^3 + 2t^2\gamma\delta^2\epsilon - 3t^2\gamma\delta^3\epsilon + t\gamma^2\delta^2\epsilon^2 - 2t\gamma^2\delta^3\epsilon^2 - \frac{1}{2}t^2\eta\xi + 2t^2\delta\eta\xi + 2t\gamma\delta\epsilon\eta\xi)\hbar^3)$
$QU[] + (2\delta\epsilon\hbar + (2t\delta\epsilon - 4t\delta^2\epsilon - 2\gamma\delta^2\epsilon^2 + 2\epsilon\eta\xi)\hbar^2 + (t^2\delta\epsilon - 6t^2\delta^2\epsilon + 6t^2\delta^3\epsilon - 8t\gamma\delta^2\epsilon^2 + 12t\gamma\delta^3\epsilon^2 + 2t\epsilon\eta\xi - 8t\delta\epsilon\eta\xi - 4\gamma\delta\epsilon^2\eta\xi)\hbar^3)$
$QU[a] + (\xi\hbar + (-2t\delta\xi - 2\gamma\delta\epsilon\xi)\hbar^2 + (-t^2\delta\xi + 3t^2\delta^2\xi - 3t\gamma\delta\epsilon\xi + 9t\gamma\delta^2\epsilon\xi - 2\gamma^2\delta\epsilon^2\xi + 6\gamma^2\delta^2\epsilon^2\xi - t\eta\xi^2 - \gamma\epsilon\eta\xi^2)\hbar^3)$
$QU[x] + (\eta\hbar + (-2t\delta\eta - 2\gamma\delta\epsilon\eta)\hbar^2 + (-t^2\delta\eta + 3t^2\delta^2\eta - 3t\gamma\delta\epsilon\eta + 9t\gamma\delta^2\epsilon\eta - 2\gamma^2\delta\epsilon^2\eta + 6\gamma^2\delta^2\epsilon^2\eta - t\eta^2\xi - \gamma\epsilon\eta^2\xi)\hbar^3)$
$QU[y] + \ll 21 \gg + 3\delta^3\epsilon\hbar^3 QU[y, y, a, x, x] + \frac{1}{2}\delta^2\xi\hbar^3 QU[y, y, x, x, x] + \frac{1}{2}\delta^2\eta\hbar^3 QU[y, y, y, x, x] + \frac{1}{6}\delta^3\hbar^3 QU[y, y, y, x, x, x]$, True}

C𝕆, Q𝕆, and Swaps

Swaps from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
SetAttributes[{C𝕆, Q𝕆}, Orderless];
CU@C𝕆[specs___, 𝔼[L_, Q_, P_]] := 𝕆CU[SS[ⅇ^(L+Q) P], specs];
QU@Q𝕆[specs___, 𝔼[L_, Q_, P_]] := 𝕆QU[SS[ⅇ^(L+Q) P], specs];
```

```
CU@C𝕆[𝔼[ℏ t1 a2, ℏ t1^-1 (ⅇ^t1 - 1) y1 x2, 1 + ϵ x1 y2], {y1, x1}1, {x2, a2, y2}2] // Short
```

$CU[] + \ll 27 \gg + CU[y_1, x_1]\left(-\gamma\epsilon\hbar^2 t_2 + e^{t_1}\gamma\epsilon\hbar^2 t_2 + \frac{\epsilon\hbar t_2}{t_1} - \frac{\ll 1 \gg}{t_1} + \frac{1}{2}\gamma^2\epsilon\hbar^3 t_1 t_2 - \frac{1}{2}e^{t_1}\gamma^2\epsilon\hbar^3 t_1 t_2\right)$

```
HL[ρ[ⅇ^(ξ CU@x)].ρ[ⅇ^(α CU@a)] == ρ[ⅇ^(α CU@a)].ρ[ⅇ^(ⅇ^(-γ α) ξ CU@x)]]
```

True

SW

```
SW_{x_i_, a_j_}[C𝕆[{lh___, x_i_, a_j_, rh___}_s_, more___, 𝔼[L_, Q_, P_]]] :=
 C𝕆[{lh, a_j, x_i, rh}_s, more,
  With[{q = ⅇ^(-γ α) ξ x_i + α a_j},
   𝔼[L, ⅇ^(-γ α) ξ x_i + (Q /. x_i → 0), ⅇ^-q DP_{x_i→D_ξ, a_j→D_α}[P][ⅇ^q]] /. {α → ∂_{a_j} L, ξ → ∂_{x_i} Q}]]
```

```
c𝕠 = C𝕆[𝔼[ℏ t₁ a₂, ℏ t₁^-1 (e^t₁ - 1) y₁ x₂, 1 + ∈ x₁ y₂], {y₁, x₁}₁, {x₂, a₂, y₂}₂]
```

$C\mathbb{O}\left[\{y_1, x_1\}_1, \{x_2, a_2, y_2\}_2, \mathbb{E}\left[\hbar a_2 t_1, \frac{(-1+e^{t_1})\hbar x_2 y_1}{t_1}, 1+\in x_1 y_2\right]\right]$

```
SW_{x₂,a₂}[c𝕠]
```

$C\mathbb{O}\left[\{y_1, x_1\}_1, \{a_2, x_2, y_2\}_2, \mathbb{E}\left[\hbar a_2 t_1, \frac{e^{-\gamma\hbar t_1}(-1+e^{t_1})\hbar x_2 y_1}{t_1}, 1+\in x_1 y_2\right]\right]$

```
With[{c𝕠 = C𝕆[{y₁, x₁}₁, {x₂, a₂, y₂}₂, 𝔼[ℏ t₁ a₂, ℏ t₁^-1 (e^t₁ - 1) y₁ x₂, 1 + ∈ x₁ y₂]]},
 HL[CU[c𝕠] == CU[c𝕠 // SW_{x₂,a₂}]]]
```

True

```
With[{c𝕠 = C𝕆[{y₁, a₁, x₁}₁, {x₂, a₂, y₂}₂,
    𝔼[ℏ (l₁₁ t₁ a₁ + l₁₂ t₁ a₂ + l₂₁ t₂ a₁ + l₂₂ t₂ a₂), ℏ (γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
     1 + ∈ (l₁ a₁ + l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)]]},
 {CU[c𝕠] // Short, HL[CU[c𝕠] == CU[c𝕠 // SW_{x₂,a₂}]]}
]
```

$\{CU[a_1, a_1, a_1, a_1]\,(\ll 1 \gg) + \ll 180 \gg +$
$CU[]\,(1 - \in p_{22} t_2 + \ll 80 \gg + 24\gamma \in^2 \hbar^3 p_{22} t_2^3 \gamma_{22}^3 + 4 \in \hbar^3 p_{22} t_2^4 \gamma_{22}^3)$, True}

SW

```
SW_{x_i_,y_j_→k_}[C𝕆[{lh___, x_i_, y_j_, rh___}_s_, more___, 𝔼[L_, Q_, P_]]] :=
 C𝕆[{lh, y_k, a_k, x_k, rh}_s, more,
  With[{q = ν (ξ x_k + η y_k + δ x_k y_k - t_k ξ η)},
   𝔼[L, q + (Q /. x_i | y_j → 0), e^-q DP_{x_i→D_ξ, y_j→D_η}[P][CΛ[t_k, y_k, a_k, x_k, ξ, η, δ] e^q]] /.
     ν → (1 + t_k δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]]
```

```
With[{c𝕠 = C𝕆[{x₁, y₁}₁, {x₂, a₂, y₂}₂,
    𝔼[ℏ (l₁₂ t₁ a₂ + l₂₂ t₂ a₂), ℏ (γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
     1 + ∈ (l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)]]},
 {CU[c𝕠] // Short, HL[CU[c𝕠] == CU[c𝕠 // SW_{x₁,y₁→1}]]}
]
```

$\{12 \in^2 \hbar^3 CU[y_1, a_1, a_1, x_1]\, \gamma_{11}^3 + \ll 159 \gg +$
$CU[]\,(1 - \in p_{11} t_1 - \in p_{22} t_2 + \ll 296 \gg + \in \hbar^3 p_{11} t_1 t_2^3 \gamma_{22}^3 + 4 \in \hbar^3 p_{22} t_2^4 \gamma_{22}^3)$, True}

SW

```
SW_{x_i_,y_j_→k_}[Q𝕆[{lh___, x_i_, y_j_, rh___}_s_, more___, 𝔼[L_, Q_, P_]]] :=
 Q𝕆[{lh, y_k, a_k, x_k, rh}_s, more,
  With[{q = ν (ξ x_k + η y_k + δ x_k y_k - ℏ^-1 (T_k^2 - 1) ξ η)},
   𝔼[L, q + (Q /. x_i | y_j → 0), e^-q DP_{x_i→D_ξ, y_j→D_η}[P][QΛ[T_k, y_k, a_k, x_k, ξ, η, δ] e^q]] /.
     ν → (1 + ℏ^-1 (T_k^2 - 1) δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]]
```

```
With[{q⊕ = Q⊕[{x1, y1}1, {x2, a2, y2}2,
    𝔼[ℏ (l12 t1 a2 + l22 t2 a2), ℏ (γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]]},
 {QU[q⊕] // Short, HL[err = SimpT[QU[q⊕] - QU[q⊕ // SW_{x1,y1→1}]]]}
]
```

$\{QU[a_2, a_2, a_2] \left(\frac{1}{2} \epsilon \hbar^2 l_2 l_{12}^2 t_1^2 + \epsilon \hbar^2 l_2 l_{12} l_{22} t_1 t_2 + \frac{1}{2} \epsilon \hbar^2 l_2 l_{22}^2 t_2^2\right) + \ll 54 \gg + QU[] \left(1 + \frac{\epsilon p_{11}}{\hbar} + \frac{\epsilon p_{22}}{\hbar} - \gamma \epsilon l_{12} p_{22} t_1 + \ll 269 \gg + \ll 1 \gg + \frac{9 \ll 3 \gg \gamma^2_{\ll 2 \gg}}{\hbar} - \frac{\epsilon p_{11} T_1^2 T_2^4 \gamma_{22}^2}{\hbar} - \frac{3 \epsilon p_{22} T_2^6 \gamma_{22}^2}{\hbar}\right), 0\}$

Stitching Direct

```
MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].MatrixExp[η2 ρ[CU@y]].
  MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\gamma (\alpha_1+\alpha_2)} (1 + \gamma \epsilon \eta_2 \xi_1) & e^{\gamma \alpha_1} \gamma \left(e^{\gamma \alpha_2} \xi_2 + \xi_1 (1 + e^{\gamma \alpha_2} \gamma \epsilon \eta_2 \xi_2)\right) \\ e^{\gamma \alpha_2} \epsilon \left(\eta_2 + e^{\gamma \alpha_1} \eta_1 (1 + \gamma \epsilon \eta_2 \xi_1)\right) & 1 + e^{\gamma \alpha_1} \gamma \epsilon \eta_1 \xi_1 + e^{\gamma \alpha_2} \gamma \epsilon \left(\eta_2 + e^{\gamma \alpha_1} \eta_1 (1 + \gamma \epsilon \eta_2 \xi_1)\right) \xi_2 \end{pmatrix}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
  MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
 e^(τ0 ϵ γ) MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$\{\{e^{\gamma \alpha_2} (e^{\gamma \alpha_1} + e^{\gamma \alpha_1} \gamma \epsilon \eta_2 \xi_1), e^{\gamma \alpha_1} \gamma \xi_1 + e^{\gamma \alpha_2} \gamma (e^{\gamma \alpha_1} + e^{\gamma \alpha_1} \gamma \epsilon \eta_2 \xi_1) \xi_2\}, \{e^{\gamma \alpha_2} (e^{\gamma \alpha_1} \epsilon \eta_1 + \epsilon \eta_2 (1 + e^{\gamma \alpha_1} \gamma \epsilon \eta_1 \xi_1)), 1 + e^{\gamma \alpha_1} \gamma \epsilon \eta_1 \xi_1 + e^{\gamma \alpha_2} \gamma (e^{\gamma \alpha_1} \epsilon \eta_1 + \epsilon \eta_2 (1 + e^{\gamma \alpha_1} \gamma \epsilon \eta_1 \xi_1)) \xi_2\}\} == \{\{e^{\alpha 0 \gamma + \gamma \epsilon \tau 0}, e^{\alpha 0 \gamma + \gamma \epsilon \tau 0} \gamma \xi 0\}, \{e^{\alpha 0 \gamma + \gamma \epsilon \tau 0} \epsilon \eta 0, e^{\gamma \epsilon \tau 0} (1 + e^{\alpha 0 \gamma} \gamma \epsilon \eta 0 \xi 0)\}\}$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]][[1]]
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$\text{Log}[e^{\gamma(\alpha 0+\epsilon\,\tau 0)}] - \text{Log}[e^{\gamma\alpha_2}(e^{\gamma\,\text{Subscript}[\ll 2\gg]} + e^{\text{Times}[\ll 2\gg]}\gamma\,\epsilon\,\eta_2\,\xi_1)] == 0.$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\tau 0 \to \frac{-\text{Log}\left[e^{\alpha 0\,\gamma}\right] + \text{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2} + e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_2\,\xi_1\right]}{\gamma\,\epsilon},\right.$$

$$\eta 0 \to \left(e^{-\gamma\alpha_1}\left(\frac{1}{2} + \frac{1}{2}e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\xi_1 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_1\,\xi_2 + \frac{1}{2}e^{\gamma\alpha_2}\gamma\,\epsilon\,\eta_2\,\xi_2 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\,\eta_2\,\xi_1\,\xi_2 - \frac{1}{2}\sqrt{\left(\left(-1 - e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\xi_1 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_1\,\xi_2 - e^{\gamma\alpha_2}\gamma\,\epsilon\,\eta_2\,\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\,\eta_2\,\xi_1\,\xi_2\right)^2 + 4\,e^{-\alpha 0\,\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\left(-e^{\gamma\alpha_1}\eta_1\,\xi_1 - \eta_2\,\xi_1 - e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1^2 - e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\,\xi_2 - e^{\gamma\alpha_2}\eta_2\,\xi_2 - 2\,e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1\,\xi_2 - e^{\gamma\alpha_2}\gamma\,\epsilon\,\eta_2^2\,\xi_1\,\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\,\eta_2^2\,\xi_1^2\,\xi_2\right)\right)}\right)\right) \Big/$$

$$\left(\gamma\,\epsilon\left(\xi_1 + e^{\gamma\alpha_2}\xi_2 + e^{\gamma\alpha_2}\gamma\,\epsilon\,\eta_2\,\xi_1\,\xi_2\right)\right),\ \xi 0 \to \frac{1}{e^{\gamma\alpha_1}\eta_1 + \eta_2 + e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1}$$

$$e^{-\gamma\alpha_2}\left(\frac{1}{2\gamma\,\epsilon} + \frac{1}{2}e^{\gamma\alpha_1}\eta_1\,\xi_1 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\,\xi_2 + \frac{1}{2}e^{\gamma\alpha_2}\eta_2\,\xi_2 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1\,\xi_2 - \frac{1}{2\gamma\,\epsilon}\left(\sqrt{\left(\left(-1 - e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\xi_1 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_1\,\xi_2 - e^{\gamma\alpha_2}\gamma\,\epsilon\,\eta_2\,\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\,\eta_2\,\xi_1\,\xi_2\right)^2 + 4\,e^{-\alpha 0\,\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\left(-e^{\gamma\alpha_1}\eta_1\,\xi_1 - \eta_2\,\xi_1 - e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1^2 - e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\,\xi_2 - e^{\gamma\alpha_2}\eta_2\,\xi_2 - 2\,e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1\,\xi_2 - e^{\gamma\alpha_2}\gamma\,\epsilon\,\eta_2^2\,\xi_1\,\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\,\eta_2^2\,\xi_1^2\,\xi_2\right)\right)}\right)\right)\Bigg\}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
    MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
  T0 MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\left\{\left\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1} + e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_2\,\xi_1\right),\ e^{\gamma\alpha_1}\gamma\,\xi_1 + e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1} + e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_2\,\xi_1\right)\xi_2\right\},\right.$$
$$\left\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1}\epsilon\,\eta_1 + \epsilon\,\eta_2\left(1 + e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\xi_1\right)\right),\right.$$
$$\left.\left.1 + e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\xi_1 + e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1}\epsilon\,\eta_1 + \epsilon\,\eta_2\left(1 + e^{\gamma\alpha_1}\gamma\,\epsilon\,\eta_1\,\xi_1\right)\right)\xi_2\right\}\right\} ==$$
$$\left\{\left\{e^{\alpha 0\,\gamma}\,T0,\ e^{\alpha 0\,\gamma}\,T0\,\gamma\,\xi 0\right\},\ \left\{e^{\alpha 0\,\gamma}\,T0\,\epsilon\,\eta 0,\ T0\left(1 + e^{\alpha 0\,\gamma}\gamma\,\epsilon\,\eta 0\,\xi 0\right)\right\}\right\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {T0, η0, α0, ξ0}]][[1]]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{T0 \to \frac{1}{1 + \gamma\,\epsilon\,\eta_2\,\xi_1},\ \eta 0 \to \frac{\eta_1 + e^{-\gamma\alpha_1}\eta_2 + \gamma\,\epsilon\,\eta_1\,\eta_2\,\xi_1}{1 + \gamma\,\epsilon\,\eta_2\,\xi_1},\right.$$
$$\left.\alpha 0 \to \frac{\text{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2}\left(1 + \gamma\,\epsilon\,\eta_2\,\xi_1\right)^2\right]}{\gamma},\ \xi 0 \to \frac{e^{-\gamma\alpha_2}\xi_1 + \xi_2 + \gamma\,\epsilon\,\eta_2\,\xi_1\,\xi_2}{1 + \gamma\,\epsilon\,\eta_2\,\xi_1}\right\}$$

E

$\mathbb{E}[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a's, Q is a combination of $x_i\, y_j$, and P is a perturbation polynomial. It should be interpreted via $\mathbb{CO}[\mathbb{E}[\,...], \{x_1, a_1, y_1\}_i, ...]$ (with some default for direct interpretation), or likewise via $\mathbb{QO}[\mathbb{E}[\,...], \{x_1, a_1, y_1\}_i, ...]$. In themselves, $\mathbb{CO}$ and $\mathbb{QO}$ should have an interpretation in CU/QU by casting.