

Pensieve header: A unified verification notebook for the \$sl_2\$-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

Go;

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ $\mathcal{E}$ _] := Style[ $\mathcal{E}$ , Background  $\rightarrow$  Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is \hbar , and all “coupling constants” are proportional to it.

TD

```
$T $\hbar$ D = 3; $TeD = 2;  $\epsilon$  /:  $e^{d \cdot}$  /;  $d > $TeD := 0$ ;
(* $TeD can't be  $\infty$  at least because of Quesne. Can't be  $\leq$ 
  1 at least because of the explicit  $e^2$  in SD$g. *)
SetAttributes[{SS, SST}, HoldAll];
SS[ $\mathcal{E}$ _] := Block[{ $\hbar$ ,  $\epsilon$ }, (* Shielded Series *)
  Collect[Normal@Series[ $\mathcal{E}$ , { $\hbar$ , 0, $T $\hbar$ D}],  $\hbar$ , Together] ];
SST[ $\mathcal{E}$ _] := Block[{ $\hbar$ ,  $\epsilon$ },
  Collect[Normal@Series[ $\mathcal{E}$  /. {T $_i$   $\rightarrow e^{\hbar t_i/2}$ , T  $\rightarrow e^{\hbar t/2}$ }, { $\hbar$ , 0, $T $\hbar$ D}],  $\hbar$ , Together] ];
Simp[ $\mathcal{E}$ _, op_] := Collect[ $\mathcal{E}$ , _CU | _QU, op];
Simp[ $\mathcal{E}$ _] := Simp[ $\mathcal{E}$ , Collect[Normal@Series[#, { $\hbar$ , 0, $T $\hbar$ D}],  $\hbar$ , Expand] &];
SimpT[ $\mathcal{E}$ _] := Collect[ $\mathcal{E}$ , _CU | _QU,
  Collect[Normal@Series[#, {T $_i$   $\rightarrow e^{\hbar t_i/2}$ , T  $\rightarrow e^{\hbar t/2}$ }, { $\hbar$ , 0, $T $\hbar$ D}],  $\hbar$ , Expand] &];
```

Differential polynomials (DP):

Utils

```
DP $_{\alpha \rightarrow D_x, \beta \rightarrow D_y}$ [P_] [ $\lambda$ _] :=
  Total[CoefficientRules[P, { $\alpha$ ,  $\beta$ }] /. ({m_, n_}  $\rightarrow$  c_)  $\Rightarrow$  c D[ $\lambda$ , {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```

DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, CE, pow,
  gs = Generators /. {opts}, cs = Centrals /. {opts}},
  (#U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp; (* gen's pattern *)
  sr = Thread[gs → Range@Length@gs]; (* sorting rule *)
  cp = Alternatives @@ cs; (* cent's pattern *)
  CE[ε_] := Collect[ε, _U, (Expand[#] /. h^d_ /; d > $ThD → 0) &];
  Ui[ε_] := ε /. {t : cp → ti, u_U → Replace[u, x_ → xi, 1]};
  Ui[NCM[]] := U[];
  B[U@(x_)i_, U@(y_)i_] := B[U@xi, U@yi] = Ui@B[U@x, U@y];
  B[U@(x_)i_, U@(y_)j_] /; i != j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** U[] := x; U[] ** x_ := x;
  (a_. * x_U) ** (b_. * y_U) := If[ab === 0, 0, CE[ab (x ** y)]];
  U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
    U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
  U@{c_. * (L : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[L, {n}] ** U@{r}];
  U@{c_. * L : gp, r___} := CE[c U[L] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{} = U[];
  U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
  U@{L_, r___} := U@{Expand[L], r};
  U[ε_NonCommutativeMultiply] := U /@ ε;
  Ou[poly_, specs___] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, L_List → L_null, {1}];
    vs = Join@@(First /@ sp);
    us = Join@@(sp /. L_s_ → (L /. x_i_ → xs));
    CE[Total[
      CoefficientRules[poly, vs] /. (p_ → c_) → c U@(us^p)
    ]] /. x_null → x
  ];
  pow[ε_, 0] = U[]; pow[ε_, n_] := pow[ε, n - 1] ** ε;
  Su[ε_, ss__Rule] := CE@Total[
    CoefficientRules[ε, First /@ {ss}] /.
      (p_ → c_) → c NCM@@MapThread[pow, {Last /@ {ss}, p}]];
  Si[c_. * u_U] := CE[(c /. Si[U, Centrals]) DeleteCases[u, _i] **
    Ui[NCM@@Reverse@Cases[u, x_i_ → S@U@x]]];
]

```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ -> V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ -> img_) -> (m[U[g]] = img), {1}];
  m[U[]] = V[];
  m[U[g_i]] := V_i[m[U@g]];
  m[U[vs_]] := NCM@@(m/@U/@{vs});
  m[ε_] := Simp[ε /. oncs /. u_U -> m[u]];
```

Meta-Operations

QLImplementation

```
S_i_[ε_Plus] := Simp[S_i_/@ε];
```

Implementing $sl_2^{\vee\epsilon}$

CU

```
DeclareAlgebra[CU, Generators -> {y, a, x}, CentralS -> {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ε a_CU - t CU[];
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU);
S_i_[CU, CentralS] = {t_i -> -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a "random" triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp // HL
}] // Timing
{0.828125, {(28 t^2 γ^4 + 116 t γ^5 ε + 120 γ^6 ε^2) CU[y, y, y, x, x] + <<24>>, 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
    {z1, bas}, {z2, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Implementing $\mathcal{U}_{\gamma\epsilon; \hbar}$

With $q = e^{\hbar\gamma\epsilon}$, $A = e^{-\hbar\epsilon a}$, $T = e^{\hbar/2}$, and $[f, g]_q := fg - qgf$, our algebra is $\mathcal{U}_{\gamma\epsilon; \hbar} = \langle t, y, a, x \rangle / \mathcal{R}$, where $\mathcal{R} = \langle [t, *] = 0, [a, y] = -\gamma y, [x, y]_q = (1 - T^2 A^2) / \hbar, [x, a] = -\gamma x \rangle$.

QU

```
DeclareAlgebra[QU, Generators -> {y, a, x}, CentralS -> {t, T}];
q = SS[e^{\gamma\epsilon\hbar}]; (*T=SS[e^{\hbar t/2}];*)
B[aQU, yQU] = -\gamma yQU; B[xQU, aQU] = -\gamma QU@x;
B[xQU, yQU] = (q - 1) QU@{y, x} + OQU[SS[(1 - T^2 e^{-2\epsilon a\hbar}) / \hbar], {a}];
(S@yQU = OQU[SS[-T^2 e^{\hbar\epsilon a} y], {a, y}]; S@aQU = -aQU; S@xQU = OQU[SS[-e^{\hbar\epsilon a} x], {a, x}];)
Si_ [QU, CentralS] = {ti -> -ti, Ti -> Ti^{-1}};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} -> Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas} ] ]
{{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> \gamma QU[y], {QU[y], QU[x]} ->
  \frac{(-1 + T^2) QU[]}{\hbar} - 2 T^2 \epsilon QU[a] + 2 T^2 \epsilon^2 \hbar QU[a, a] + \left(-\gamma \epsilon \hbar - \frac{1}{2} \gamma^2 \epsilon^2 \hbar^2\right) QU[y, x]},
 {{QU[a], QU[y]} -> -\gamma QU[y], {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> \gamma QU[x]},
 {{QU[x], QU[y]} -> \frac{(1 - T^2) QU[]}{\hbar} + 2 T^2 \epsilon QU[a] - 2 T^2 \epsilon^2 \hbar QU[a, a] + \left(\gamma \epsilon \hbar + \frac{1}{2} \gamma^2 \epsilon^2 \hbar^2\right) QU[y, x],
 {QU[x], QU[a]} -> -\gamma QU[x], {QU[x], QU[x]} -> 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
  Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a "random" triple (~34 secs @ $\$T\hbar D=5$, $\$T\epsilon D=2$):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  HL[z1 ** (z2 ** z3) - rhs // Simp]
}] // Timing
{32.625, {<<1>>, 0}}
```

Verifying that S is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
  Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
    {z1, bas}, {z2, bas} ] ]
{{QU[y1], QU[y1]} → 0, {QU[y1], QU[a1]} → 0, {QU[y1], QU[x1]} → 0},
{QU[a1], QU[y1]} → 0, {QU[a1], QU[a1]} → 0, {QU[a1], QU[x1]} → 0},
{QU[x1], QU[y1]} → 0, {QU[x1], QU[a1]} → 0, {QU[x1], QU[x1]} → 0}}
```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs @ $T\hbar D=5, T\epsilon D=2$):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  Short[lhs = z1 ** (z2 ** z3)],
  Short[rhs = (QU@@z1) ** ((QU@@z2) ** (QU@@z3))],
  Expand[Limit[rhs /. {QU → CU, T → e^{\hbar t/2}}, \hbar → 0] - lhs] // HL
}] // Timing
{34., {2 (8 t^2 \gamma^4 + 16 t \gamma^5 \epsilon) CU[y, y, y, x, x] + <<107>> + CU[y, y, y, y, <<5>>, x, x, x, x],
(-8 T^2 \gamma^6 \epsilon^2 + 8 T^4 \gamma^6 \epsilon^2) QU[y, y, y, x, x] + <<489>> + (\gamma \epsilon \hbar + <<1>>) QU[<<1>>, 0]}}
```

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -xCU, a → -aCU, x → -yCU}, {t → -t, T → T-1});
DeclareMorphism[Qθ, QU → QU, {y → 0QU[SS[-T-1 e^{\hbar \epsilon^a} x], {a, x}],
  a → -aQU, x → 0QU[SS[-T-1 e^{\hbar \epsilon^a} y], {a, y}], {t → -t, T → T-1}}
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas} ] ]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas} ] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[z → Qθ[z] → HL[Qθ[Qθ[z]]], {z, bas} ] ]
{QU[y] → -\frac{QU[x]}{T} - \frac{\epsilon \hbar QU[a, x]}{T} - \frac{\epsilon^2 \hbar^2 QU[a, a, x]}{2 T} → QU[y], QU[a] → -QU[a] → QU[a],
QU[x] → \left(-\frac{1}{T} + \frac{\gamma \epsilon \hbar}{T} - \frac{\gamma^2 \epsilon^2 \hbar^2}{2 T}\right) QU[y] + \left(-\frac{\epsilon \hbar}{T} + \frac{\gamma \epsilon^2 \hbar^2}{T}\right) QU[y, a] - \frac{\epsilon^2 \hbar^2 QU[y, a, a]}{2 T} → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$AD\$f = \frac{\gamma}{\hbar} e^{\hbar \left(\frac{t}{2} - (a+\gamma)\epsilon \right)} \left(\left(\text{Cosh} \left[\hbar \left(a\epsilon + \frac{\gamma\epsilon}{2} - \frac{t}{2} \right) \right] - \text{Cosh} \left[\hbar \sqrt{\left(\frac{t-\gamma\epsilon}{2} \right)^2 + \epsilon\omega} \right] \right) / \right. \\ \left. \left(\text{Sinh} \left[\frac{\gamma\epsilon\hbar}{2} \right] (a^2\epsilon + a\gamma\epsilon - at - \omega) \right) \right);$$

Scaling behaviour of AD\$f:

```
HL@Simplify[AD$f == ((AD$f /. γ → 1) /. {ε → γε, a → γ-1a, ω → γ-1ω})]
```

True

```
HL@FullSimplify[
  AD$f == ((AD$f /. γ → 1) /. {ħ → γ2ħ, ε → ε/γ, a → a/γ, t → γ-2t, ω → γ-3ω})]
```

True

ADeq

$$AD\$ω = \gamma CU[y, x] + \epsilon CU[a, a] - (t - \gamma\epsilon) CU[a];$$

ADeq

```
DeclareMorphism[AD, QU → CU,
  {a → aCU, x → CU@x, y → SCU[SS[AD$f], a → aCU, ω → AD$ω] ** yCU}]
```

Verifying that the asymmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$\text{SD}\$g = \sqrt{\left(\left(\text{Cosh}\left[\frac{\hbar}{2} \sqrt{t^2 + \gamma^2 \epsilon^2 + 4 \epsilon \varpi}\right] - \text{Cosh}\left[\frac{\hbar}{2} (t - (2a + \gamma) \epsilon)\right] \right) / \left(\text{Sinh}\left[\frac{\gamma \epsilon \hbar}{2}\right] (t(2a + \gamma) - 2a(a + \gamma) \epsilon + 2\varpi) \hbar / (2\gamma) \right) \right)}$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

$$\{\text{SD}\$P = \frac{\text{Cosh}\left[\hbar \left(\frac{\epsilon - t}{2} + \epsilon a\right)\right] - \text{Cosh}\left[\hbar \sqrt{\frac{t^2 + \epsilon^2}{4} + \epsilon \varpi}\right]}{\hbar \text{Sinh}\left[\frac{-\epsilon \hbar}{2}\right] (\varpi - \epsilon a^2 + (t - \epsilon) a + t/2)}\},$$

```
Simplify[SD$P == (SD$P /. {a -> -a - 1, t -> -t})] // HL,
PowerExpand@Simplify[(SD$P /. {h -> \gamma^2 h, \epsilon -> \epsilon / \gamma, a -> a / \gamma, t -> \gamma^{-2} t, \varpi -> \gamma^{-3} \varpi}) ==
SD$g (SD$g /. {a -> -a - \gamma, t -> -t})] // HL,
SD$Q = Simplify[SD$P /. {a -> c - 1/2}],
Simplify[SD$Q == (SD$Q /. {c -> -c, t -> -t})] // HL,
Simplify[SD$g == FullSimplify[
\sqrt{SD$Q} /. c -> a + 1/2 /. {h -> \gamma^2 h, \epsilon -> \epsilon / \gamma, a -> a / \gamma, t -> \gamma^{-2} t, \varpi -> \gamma^{-3} \varpi}]] // HL
}
```

$$\left\{ - \left(\left(\left(\text{Cosh}\left[\left(a \epsilon + \frac{1}{2} (-t + \epsilon)\right) \hbar\right] - \text{Cosh}\left[\sqrt{\frac{1}{4} (t^2 + \epsilon^2) + \epsilon \varpi} \hbar\right] \text{Csch}\left[\frac{\epsilon \hbar}{2}\right] \right) / \left(\left(\frac{t}{2} + a(t - \epsilon) - a^2 \epsilon + \varpi\right) \hbar \right) \right), \text{True, True}, \right. \\ \left. - \left(\left(4 \left(\text{Cosh}\left[\frac{1}{2} (t - 2c \epsilon) \hbar\right] - \text{Cosh}\left[\frac{1}{2} \sqrt{t^2 + \epsilon^2 + 4 \epsilon \varpi} \hbar\right] \right) \text{Csch}\left[\frac{\epsilon \hbar}{2}\right] \right) / \left((4c t + \epsilon - 4c^2 \epsilon + 4\varpi) \hbar \right) \right), \text{True, True} \right\}$$

SDeq

$$\text{SD}\$f = \text{FullSimplify}\left[e^{\hbar (t/2 - \epsilon a)} (\text{SD}\$g /. \{a \rightarrow -a, t \rightarrow -t\})\right];$$

SDeq

$$\text{SD}\$\varpi = \gamma \text{CU}[y, x] + \epsilon \text{CU}[a, a] - (t - \gamma \epsilon) \text{CU}[a] - t \gamma \text{CU}[] / 2;$$

SDeq

```
DeclareMorphism[SD, QU -> CU, {a -> a_CU,
x -> S_CU[SS[SD$f], a -> a_CU, \varpi -> SD$\varpi] ** X_CU,
y -> S_CU[SS[SD$g], a -> a_CU, \varpi -> SD$\varpi] ** Y_CU
}]
```

Verifying the θ -symmetry:

```
Table[HL@SimpT[C\theta[SD[z]] == SD[Q\theta[z]]], {z, QU /@ {y, a, x}}]
{True, True, True}
```

Verifying that the symmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} -> HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> 0, {QU[y], QU[x]} -> 0},
 {{QU[a], QU[y]} -> 0, {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> 0},
 {{QU[x], QU[y]} -> 0, {QU[x], QU[a]} -> 0, {QU[x], QU[x]} -> 0}}
```

R in QU.

Quesne's formula:

Quesne

```
e_{q-,n-}[x_] := e ^ ( Sum_{k=1}^n ( (1-q)^k x^k / (k (1-q^k)) ); e_{q-}[x_] := e_{q,TEd}[x]
```

```
Table[Together@SeriesCoefficient[e_{rho,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{ 1, 1, \frac{1}{1+\rho}, \frac{1}{(1+\rho)(1+\rho+\rho^2)}, \frac{1}{(1+\rho)^2(1+\rho^2)(1+\rho+\rho^2)}, \frac{1}{(1+\rho)^2(1+\rho^2)(1+\rho+\rho^2)(1+\rho+\rho^2+\rho^4)} \right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, rho] SeriesCoefficient[e_{rho,5}[x], {x, 0, n}]], {n, 0, 5}]
```

```
{1, 1, 1, 1, 1, 1}
```

R

```
QU[R_{i,j-}] := O_{QU}[SS[e^{h b_1 a_2} e_q[h y_1 x_2] /. b_1 -> gamma^{-1} (epsilon a_1 - t_i)], {y_1, a_1}_i, {a_2, x_2}_j];
QU[R_{i,j-}^{-1}] := S_j @ QU[R_{i,j}];
```

```
QU[R_{3,4}] // Short
```

$$QU[] + \frac{\epsilon \hbar \ll 1 \gg}{\gamma} + \ll 21 \gg + \frac{\hbar^3 \ll 1 \gg \ll 1 \gg}{2 \gamma^2} - \frac{\hbar^3 QU[a_4, a_4, a_4] t_3^3}{6 \gamma^3}$$

Verifying R2 (~2 secs @ \$TħD=4, \$TεD=2):

```
QU[R_{1,2} ** R_{1,2}^{-1}] // Simp // HL // Timing
{0.546875, QU[]}
```

Verifying R3 (~156 secs @ \$TħD=4, \$TεD=2):

```
{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}], HL@SimpT[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]] // Timing
```

$$\left\{ 11.3125, \left\{ QU[] + \ll 456 \gg + QU[y_1, y_1, y_1, x_3, x_3, x_3] \left(\frac{\hbar^3}{6} - \frac{1}{2} \hbar^3 T_2^2 + \frac{1}{2} \hbar^3 T_2^4 - \frac{1}{6} \hbar^3 T_2^6 \right), 0 \right\} \right\}$$

The representation ρ

rho

```

rho@yCU = rho@yQU =  $\begin{pmatrix} \theta & \theta \\ \epsilon & \theta \end{pmatrix}$ ; rho@aCU = rho@aQU =  $\begin{pmatrix} \gamma & \theta \\ \theta & \theta \end{pmatrix}$ ;
rho@xCU =  $\begin{pmatrix} \theta & \gamma \\ \theta & \theta \end{pmatrix}$ ; rho@xQU = SS@ $\begin{pmatrix} \theta & (1 - e^{-\gamma \epsilon \hbar}) / (\epsilon \hbar) \\ \theta & \theta \end{pmatrix}$ ;
rho[e^delta_] := MatrixExp[rho[delta]];
rho[delta_] :=
(delta /. {t -> gamma epsilon, T -> e^{hbar gamma epsilon/2}) /. (U : CU | QU)[u___] => Fold[Dot,  $\begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}$ , rho/@U/@{u}]

```

Verifying that ρ represents CU and QU:

```

Table[rho[z1 ** z2] == rho[z1].rho[z2] // SS // HL,
{U, {CU, QU}}, {z1, U/@{y, a, x}}, {z2, U/@{y, a, x}}]
{{{True, True, True}, {True, True, True}, {True, True, True}},
{{True, True, True}, {True, True, True}, {True, True, True}}}

```

The Classical Logos CA

Lemma 3C. To degree k ,

$\mathcal{O}_{CU}(e^{\eta y + \xi x + \delta y x} \mid x y) = \mathcal{O}_{CU}(v e^{v(-t \xi \eta + \eta y + \xi x + \delta y x)} \mathcal{C}\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta) \mid y a x)$, with $v = (1 + t \delta)^{-1}$ and where $\mathcal{C}\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta)$ is a fixed polynomial of degree at most $4 k$ in $y, \sqrt{a}, x, \eta, \xi$, with scalar coefficients.

Comment. Even better, $\log(\mathcal{C}\Lambda_k)$ is of degree at most $2 k + 2$ in said variables.

```

eqn = rho[e^{xi x cu}].rho[e^{eta y cu}] == rho[e^{d y cu}].rho[e^{c (t CU[] - 2 epsilon a cu)}].rho[e^{b x cu}]
{{1 + gamma epsilon eta xi, gamma xi}, {epsilon eta, 1}} == {{e^{-c gamma epsilon}, b e^{-c gamma epsilon gamma}, {d e^{-c gamma epsilon} epsilon, e^{c gamma epsilon} + b d e^{-c gamma epsilon} gamma epsilon}}

```

```

sol = Solve[Thread[Flatten/@eqn], {d, b, c}][[1]] /. C[1] -> 0

```

$$\left\{ d \rightarrow \frac{\eta}{1 + \gamma \epsilon \eta \xi}, b \rightarrow \frac{\xi}{1 + \gamma \epsilon \eta \xi}, c \rightarrow \frac{\text{Log}\left[\frac{1}{1 + \gamma \epsilon \eta \xi}\right]}{\gamma \epsilon} \right\}$$

Proof of Lemma 3C. We know that $\mathcal{O}_{CU}(e^{\xi x + \eta y} \mid x y) = \mathcal{O}_{CU}(e^{ct + ay - 2 \epsilon ca + bx} \mid y a x)$, with

$\left\{ d \rightarrow \frac{\eta}{1 + \gamma \epsilon \eta \xi}, b \rightarrow \frac{\xi}{1 + \gamma \epsilon \eta \xi}, c \rightarrow \frac{\text{Log}[1 + \gamma \epsilon \eta \xi]}{-\gamma \epsilon} \right\}$. Expanding in ϵ we get

$$\mathcal{O}_{CU}(e^{\xi x + \eta y} \mid x y) = \mathcal{O}_{CU}(\lambda_\epsilon(\xi, \eta) e^{\eta y + \xi x - \eta \xi t} \mid y a x) = \mathcal{O}_{CU}(\lambda_\epsilon(\partial_x, \partial_y) e^{\eta y + \xi x - \eta \xi t} \mid y a x) \text{ and so}$$

$$\mathcal{O}_{CU}(e^{\eta y + \xi x + \delta y x} \mid x y) = \mathcal{O}(\lambda_\epsilon(\partial_x, \partial_y) e^{\delta \partial_\eta \partial_\xi} e^{\eta y + \xi x - \eta \xi t} \mid y a x) = \mathcal{O}(\lambda_\epsilon(\partial_x, \partial_y) v e^{v(-t \xi \eta + \eta y + \xi x + \delta y x)} \mid y a x).$$

Logos

```

SS $\epsilon$ [ $\delta$ _] := Block[{ $\epsilon$ }, Collect[Normal@Series[ $\delta$ , { $\epsilon$ , 0, $TeD}],  $\epsilon$ , Together]];
(* Shielded  $\epsilon$ -Series *)
C $\Delta$ [t1_, y1_, a1_, x1_,  $\xi$ 1_,  $\eta$ 1_,  $\delta$ _] := Module[
  {eqn, d, b, c, sol,  $\lambda$ , q, v,  $\xi$ ,  $\eta$ },
  eqn =  $\rho$ [e $^{\xi x_{cu}}$ ]. $\rho$ [e $^{\eta y_{cu}}$ ] ==  $\rho$ [e $^{d y_{cu}}$ ]. $\rho$ [e $^{c (t_{cu} - 2 \epsilon a_{cu})}$ ]. $\rho$ [e $^{b x_{cu}}$ ];
  sol = Solve[Thread[Flatten/@eqn], {d, b, c}] [[1]] /. C[1]  $\rightarrow$  0;
   $\lambda$  = Simplify[e $^{-\eta y - \xi x + \eta \xi t}$  SS $\epsilon$ [e $^{c t + d y - 2 \epsilon c a + b x}$  /. sol]];
  q = e $^{v (-t \xi \eta + \eta y + \xi x + \delta y x)}$ ;
  Collect[v q $^{-1}$  DP $_{\xi \rightarrow D_x, \eta \rightarrow D_y}$ [ $\lambda$ ][q] /. v  $\rightarrow$  (1 + t  $\delta$ ) $^{-1}$ ,  $\epsilon$ , Simplify] /.
  {t  $\rightarrow$  t1, y  $\rightarrow$  y1, a  $\rightarrow$  a1, x  $\rightarrow$  x1,  $\xi$   $\rightarrow$   $\xi$ 1,  $\eta$   $\rightarrow$   $\eta$ 1}
];

```

$CA[t, y, a, x, \xi, \eta, \delta]$

$$\begin{aligned}
& \frac{1}{1+t\delta} + \frac{1}{24(1+t\delta)^9} \\
& \epsilon^2 \left(48 a^2 (1+t\delta)^4 \left(2\delta^2 (1+t\delta)^2 + 4\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) - \right. \\
& \quad 24 a \gamma (1+t\delta)^4 \left(2\delta^2 (1+t\delta)^2 + 4\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) - \\
& \quad 48 a y \gamma (1+t\delta)^3 (x\delta+\eta) \\
& \quad \left(6\delta^2 (1+t\delta)^2 + 6\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) + 24 y \gamma^2 (1+t\delta)^3 \\
& \quad (x\delta+\eta) \left(6\delta^2 (1+t\delta)^2 + 6\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) - 48 a x \gamma \\
& \quad (1+t\delta)^3 (y\delta+\xi) \left(6\delta^2 (1+t\delta)^2 + 6\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) + \\
& \quad 24 x \gamma^2 (1+t\delta)^3 (y\delta+\xi) \left(6\delta^2 (1+t\delta)^2 + 6\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + \right. \\
& \quad \quad \left. (x\delta+\eta)^2 (y\delta+\xi)^2 \right) + 12 y^2 \gamma^2 (1+t\delta)^2 (x\delta+\eta)^2 \\
& \quad \left(12\delta^2 (1+t\delta)^2 + 8\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) + 12 x^2 \gamma^2 \\
& \quad (1+t\delta)^2 (y\delta+\xi)^2 \left(12\delta^2 (1+t\delta)^2 + 8\delta (1+t\delta) (x\delta+\eta) (y\delta+\xi) + (x\delta+\eta)^2 (y\delta+\xi)^2 \right) + \\
& \quad 24 a t \gamma (1+t\delta)^2 \left(6\delta^3 (1+t\delta)^3 + 18\delta^2 (1+t\delta)^2 (x\delta+\eta) (y\delta+\xi) + \right. \\
& \quad \quad \left. 9\delta (1+t\delta) (x\delta+\eta)^2 (y\delta+\xi)^2 + (x\delta+\eta)^3 (y\delta+\xi)^3 \right) - \\
& \quad 8 t (\gamma+t\gamma\delta)^2 \left(6\delta^3 (1+t\delta)^3 + 18\delta^2 (1+t\delta)^2 (x\delta+\eta) (y\delta+\xi) + \right. \\
& \quad \quad \left. 9\delta (1+t\delta) (x\delta+\eta)^2 (y\delta+\xi)^2 + (x\delta+\eta)^3 (y\delta+\xi)^3 \right) + \\
& \quad 24 x y (\gamma+t\gamma\delta)^2 \left(6\delta^3 (1+t\delta)^3 + 18\delta^2 (1+t\delta)^2 (x\delta+\eta) (y\delta+\xi) + \right. \\
& \quad \quad \left. 9\delta (1+t\delta) (x\delta+\eta)^2 (y\delta+\xi)^2 + (x\delta+\eta)^3 (y\delta+\xi)^3 \right) - \\
& \quad 12 t y \gamma^2 (1+t\delta) (x\delta+\eta) \left(24\delta^3 (1+t\delta)^3 + 36\delta^2 (1+t\delta)^2 (x\delta+\eta) (y\delta+\xi) + \right. \\
& \quad \quad \left. 12\delta (1+t\delta) (x\delta+\eta)^2 (y\delta+\xi)^2 + (x\delta+\eta)^3 (y\delta+\xi)^3 \right) - \\
& \quad 12 t x \gamma^2 (1+t\delta) (y\delta+\xi) \left(24\delta^3 (1+t\delta)^3 + 36\delta^2 (1+t\delta)^2 (x\delta+\eta) (y\delta+\xi) + \right. \\
& \quad \quad \left. 12\delta (1+t\delta) (x\delta+\eta)^2 (y\delta+\xi)^2 + (x\delta+\eta)^3 (y\delta+\xi)^3 \right) + \\
& \quad \left. 3 t^2 \gamma^2 \left(24\delta^4 (1+t\delta)^4 + 96\delta^3 (1+t\delta)^3 (x\delta+\eta) (y\delta+\xi) + 72\delta^2 (1+t\delta)^2 \right. \right. \\
& \quad \quad \left. \left. (x\delta+\eta)^2 (y\delta+\xi)^2 + 16\delta (1+t\delta) (x\delta+\eta)^3 (y\delta+\xi)^3 + (x\delta+\eta)^4 (y\delta+\xi)^4 \right) \right) + \\
& \frac{1}{2(1+t\delta)^5} \epsilon \left(4 a (1+t\delta)^2 \left((t+xy) \delta^2 + \eta \xi + \delta (1+y\eta+x\xi) \right) + \right. \\
& \quad \gamma \left(2 t^3 \delta^4 + 4 t^2 \delta^2 (\delta - x y \delta^2 + \eta \xi) - 2 (y \eta (\delta (2+y \eta) + \eta \xi) + x^2 \delta (2 y^2 \delta^2 + 3 y \delta \xi + \xi^2) + \right. \\
& \quad \quad \left. x (3 y^2 \delta^2 \eta + 4 y \delta (\delta + \eta \xi) + \xi (2 \delta + \eta \xi))) - t (3 x^2 y^2 \delta^4 - 4 \delta \eta \xi - \eta^2 \xi^2 + \right. \\
& \quad \quad \left. 4 x y \delta^3 (3 + y \eta + x \xi) + \delta^2 (-2 + y^2 \eta^2 + 4 x \xi + x^2 \xi^2 + 4 y (\eta + x \eta \xi))) \right) \left. \right)
\end{aligned}$$

```
{Short[1hs = Ocu[SS[e^h (xi x + eta y + delta xy)], {x, y}], 5], HL[1hs ==
  Ocu[SS[e^h v (xi x + eta y + delta xy - t h xi eta) CA[t, y, a, x, h xi, h eta, h delta] /. v -> (1 + h t delta)^-1], {y, a, x}]]]
{ (1 - t delta h + t^2 delta^2 h^2 + t gamma delta^2 in h^2 - t eta xi h^2 -
  t^3 delta^3 h^3 - 3 t^2 gamma delta^3 in h^3 - 2 t gamma^2 delta^3 e^2 h^3 + 2 t^2 delta eta xi h^3 + 2 t gamma delta eta xi h^3) CU[] +
  (2 delta eta h - 4 t delta^2 eta h^2 - 2 gamma delta^2 e^2 h^2 + 2 eta xi h^2 + 6 t^2 delta^3 in h^3 + 12 t gamma delta^3 e^2 h^3 -
  8 t delta eta xi h^3 - 4 gamma delta e^2 eta xi h^3) CU[a] +
  (xi h - 2 t delta xi h^2 - 2 gamma delta eta xi h^2 + 3 t^2 delta^2 xi h^3 + 9 t gamma delta^2 eta xi h^3 + 6 gamma^2 delta^2 e^2 xi h^3 - t eta xi^2 h^3 - gamma eta xi^2 h^3)
  CU[x] + <<24>> + 1/2 delta^2 eta h^3 CU[y, y, y, x, x] + 1/6 delta^3 h^3 CU[y, y, y, x, x, x], True }
```

CO and Swaps

Swaps from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
SetAttributes[CO, Orderless];
CU@CO[specs___, E[L_, Q_, P_]] := Ocu[SS[e^{L+Q} P], specs]

CU@CO[E[h t1 a2, h t1^-1 (e^{t1} - 1) y1 x2, 1 + e x1 y2], {y1, x1}1, {x2, a2, y2}2] // Short
CU[] + <<27>> +
  CU[y1, x1] ( -gamma eta h^2 t2 + e^{t1} gamma eta h^2 t2 + <<1>>/t1 - <<1>>/<<1>> + 1/2 gamma^2 eta h^3 t1 t2 - 1/2 e^{t1} gamma^2 eta h^3 t1 t2 )

HL[rho[e^{xi CUex}].rho[e^{alpha CUea}] == rho[e^{alpha CUea}].rho[e^{-gamma alpha xi CUex}]]
True
```

SW

```
SW_{xi, aj}[CO[{Lh___, xi_, aj_, rh___}s_, more___, E[L_, Q_, P_]]] :=
  CO[{Lh, aj, xi, rh}s, more,
  With[{q = e^{-gamma alpha xi xi + alpha aj},
  E[L, e^{-gamma alpha xi xi + (Q /. xi -> theta)}, e^{-q} DP_{xi -> D_e, aj -> D_alpha}[P][e^q]] /. {alpha -> partial_{aj} L, xi -> partial_{xi} Q}]]]

co = CO[E[h t1 a2, h t1^-1 (e^{t1} - 1) y1 x2, 1 + e x1 y2], {y1, x1}1, {x2, a2, y2}2]
CO[{y1, x1}1, {x2, a2, y2}2, E[h a2 t1, (-1 + e^{t1}) h x2 y1 / t1, 1 + e x1 y2]]

SW_{x2, a2}[co]
CO[{y1, x1}1, {a2, x2, y2}2, E[h a2 t1, (e^{-gamma h t1} (-1 + e^{t1}) h x2 y1) / t1, 1 + e x1 y2]]

With[{co = CO[{y1, x1}1, {x2, a2, y2}2, E[h t1 a2, h t1^-1 (e^{t1} - 1) y1 x2, 1 + e x1 y2]}],
  HL[CU[co] == CU[co // SW_{x2, a2}]]]
True
```

```
With[{c0 = CO[{y1, a1, x1}1, {x2, a2, y2}2,
  E[h (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2), h (g11 x1 y1 + g12 x1 y2 + g21 x2 y1 + g22 x2 y2),
  1 + e (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2) ] ]},
{CU[c0] // Short, HL[CU[c0] == CU[c0 // SWx2,a2] ]}
]
{CU[a1, a1, a1, a1] (1/6 e h^3 l1 l11^3 t1^3 + 1/2 e h^3 l1 l11^2 l21 t1^2 t2 + 1/2 e h^3 l1 l21^2 l21 t1^2 t2 + 1/2 e h^3 l1 l21^3 t2^3) +
  <<180>> + <<1>>, True}
```

SW

```
SWx_i,y_j->k_ [CO[{Lh___, x_i_, y_j_, rh___}s_, more___, E[L_, Q_, P_]]] :=
CO[{Lh, y_k, a_k, x_k, rh}S, more,
With[{q = v (xi x_k + eta y_k + delta x_k y_k - t_k xi eta)},
  E[L, q + (Q /. x_i | y_j -> theta), e^-q DP_{x_i->D_xi, y_j->D_yj}[P][CΔ[t_k, y_k, a_k, x_k, xi, eta, delta] e^q]] /.
  v -> (1 + t_k delta)^-1 /. {xi -> (partial_{x_i} Q /. y_j -> theta), eta -> (partial_{y_j} Q /. x_i -> theta), delta -> partial_{x_i, y_j} Q}] ]
```

```
With[{c0 = CO[{x1, y1}1, {x2, a2, y2}2,
  E[h (l12 t1 a2 + l22 t2 a2), h (g11 x1 y1 + g12 x1 y2 + g21 x2 y1 + g22 x2 y2),
  1 + e (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2) ] ]},
{CU[c0] // Short, HL[CU[c0] == CU[c0 // SWx1,y1->1] ]}
]
{12 e^2 h^3 CU[y1, a1, a1, x1] g11^3 + 16/3 e^2 <<3>> g11^3 +
  <<158>> + CU[] (<<301>> + e h^3 p11 t1 t2^3 g22^3 + 4 e h^3 p22 t2^4 g22^3), True}
```

The Quantum Logos QΛ

Goal 1: In QU, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$.

First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum.

```
adx[ε_] := Simp[QU@x ** ε - ε ** QU@x];
G = Simp[NestList[adx, QU@y, $TeD + 1].Table[ξ^k/k!, {k, 0, $TeD + 1}]]
(ξ - T^2 ξ) QU[] / h + 2 T^2 e ξ QU[a] + (1/2 γ e ξ^2 - 3/2 T^2 γ e ξ^2 + (1/4 γ^2 e^2 ξ^2 - 5/4 T^2 γ^2 e^2 ξ^2) h) QU[x] +
QU[y] - 2 T^2 e^2 ξ h QU[a, a] + 3 T^2 γ e^2 ξ^2 h QU[a, x] + (1/6 γ^2 e^2 ξ^3 - 7/6 T^2 γ^2 e^2 ξ^3) h QU[x, x] +
(γ e ξ h + 1/2 γ^2 e^2 ξ h^2) QU[y, x] + 1/2 γ^2 e^2 ξ^2 h^2 QU[y, x, x]
```

$G /. e \rightarrow \theta$

```
(ξ - T^2 ξ) QU[] / h + QU[y]
```

Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta=0) = 1$. We set it up and solve:

$$F = \text{Sum}[f_{1,i,j,k}[\eta] \epsilon^1 \text{QU}@ \{y^i, a^j, x^k\}, \\ \{1, 0, \text{\$TeD}\}, \{i, 0, 1\}, \{j, 0, 1\}, \{k, 0, \text{Min}[1, 21 - i - j]\}]$$

$$\begin{aligned} & \text{QU}[] f_{0,0,0,0}[\eta] + \epsilon \text{QU}[] f_{1,0,0,0}[\eta] + \epsilon \text{QU}[x] f_{1,0,0,1}[\eta] + \epsilon \text{QU}[a] f_{1,0,1,0}[\eta] + \\ & \epsilon \text{QU}[a, x] f_{1,0,1,1}[\eta] + \epsilon \text{QU}[y] f_{1,1,0,0}[\eta] + \epsilon \text{QU}[y, x] f_{1,1,0,1}[\eta] + \\ & \epsilon \text{QU}[y, a] f_{1,1,1,0}[\eta] + \epsilon^2 \text{QU}[] f_{2,0,0,0}[\eta] + \epsilon^2 \text{QU}[x] f_{2,0,0,1}[\eta] + \\ & \epsilon^2 \text{QU}[x, x] f_{2,0,0,2}[\eta] + \epsilon^2 \text{QU}[a] f_{2,0,1,0}[\eta] + \epsilon^2 \text{QU}[a, x] f_{2,0,1,1}[\eta] + \\ & \epsilon^2 \text{QU}[a, x, x] f_{2,0,1,2}[\eta] + \epsilon^2 \text{QU}[a, a] f_{2,0,2,0}[\eta] + \epsilon^2 \text{QU}[a, a, x] f_{2,0,2,1}[\eta] + \\ & \epsilon^2 \text{QU}[a, a, x, x] f_{2,0,2,2}[\eta] + \epsilon^2 \text{QU}[y] f_{2,1,0,0}[\eta] + \epsilon^2 \text{QU}[y, x] f_{2,1,0,1}[\eta] + \\ & \epsilon^2 \text{QU}[y, x, x] f_{2,1,0,2}[\eta] + \epsilon^2 \text{QU}[y, a] f_{2,1,1,0}[\eta] + \epsilon^2 \text{QU}[y, a, x] f_{2,1,1,1}[\eta] + \\ & \epsilon^2 \text{QU}[y, a, x, x] f_{2,1,1,2}[\eta] + \epsilon^2 \text{QU}[y, a, a] f_{2,1,2,0}[\eta] + \epsilon^2 \text{QU}[y, a, a, x] f_{2,1,2,1}[\eta] + \\ & \epsilon^2 \text{QU}[y, y] f_{2,2,0,0}[\eta] + \epsilon^2 \text{QU}[y, y, x] f_{2,2,0,1}[\eta] + \epsilon^2 \text{QU}[y, y, x, x] f_{2,2,0,2}[\eta] + \\ & \epsilon^2 \text{QU}[y, y, a] f_{2,2,1,0}[\eta] + \epsilon^2 \text{QU}[y, y, a, x] f_{2,2,1,1}[\eta] + \epsilon^2 \text{QU}[y, y, a, a] f_{2,2,2,0}[\eta] \end{aligned}$$

$$\text{unowns} = \text{Cases}[F, f_{___}[\eta], \infty]$$

$$\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta], f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], \\ f_{2,0,0,0}[\eta], f_{2,0,0,1}[\eta], f_{2,0,0,2}[\eta], f_{2,0,1,0}[\eta], f_{2,0,1,1}[\eta], f_{2,0,1,2}[\eta], f_{2,0,2,0}[\eta], f_{2,0,2,1}[\eta], \\ f_{2,0,2,2}[\eta], f_{2,1,0,0}[\eta], f_{2,1,0,1}[\eta], f_{2,1,0,2}[\eta], f_{2,1,1,0}[\eta], f_{2,1,1,1}[\eta], f_{2,1,1,2}[\eta], f_{2,1,2,0}[\eta], \\ f_{2,1,2,1}[\eta], f_{2,2,0,0}[\eta], f_{2,2,0,1}[\eta], f_{2,2,0,2}[\eta], f_{2,2,1,0}[\eta], f_{2,2,1,1}[\eta], f_{2,2,2,0}[\eta]\}$$

$$\text{bas} = \text{Union}@\text{Table}[\epsilon^1 \text{Cases}[\text{Coefficient}[F, \epsilon, 1], _ \text{QU}, \infty], \{1, 0, \text{\$TeD}\}]$$

$$\{\text{QU}[], \epsilon \text{QU}[], \epsilon^2 \text{QU}[], \epsilon \text{QU}[a], \epsilon^2 \text{QU}[a], \epsilon \text{QU}[x], \epsilon^2 \text{QU}[x], \epsilon \text{QU}[y], \epsilon^2 \text{QU}[y], \\ \epsilon^2 \text{QU}[a, a], \epsilon \text{QU}[a, x], \epsilon^2 \text{QU}[a, x], \epsilon^2 \text{QU}[x, x], \epsilon \text{QU}[y, a], \epsilon^2 \text{QU}[y, a], \epsilon \text{QU}[y, x], \\ \epsilon^2 \text{QU}[y, x], \epsilon^2 \text{QU}[y, y], \epsilon^2 \text{QU}[a, a, x], \epsilon^2 \text{QU}[a, x, x], \epsilon^2 \text{QU}[y, a, a], \epsilon^2 \text{QU}[y, a, x], \\ \epsilon^2 \text{QU}[y, x, x], \epsilon^2 \text{QU}[y, y, a], \epsilon^2 \text{QU}[y, y, x], \epsilon^2 \text{QU}[a, a, x, x], \epsilon^2 \text{QU}[y, a, a, x], \\ \epsilon^2 \text{QU}[y, a, x, x], \epsilon^2 \text{QU}[y, y, a, a], \epsilon^2 \text{QU}[y, y, a, x], \epsilon^2 \text{QU}[y, y, x, x]\}$$

$$\text{Short}[\text{eqns} = \text{Flatten}[\{(\text{Coefficient}[F - \text{QU}[], \#] /. \eta \rightarrow 0) == 0, \\ \text{Expand}[\text{Coefficient}[\text{Simp}[F ** G - \text{QU}[y] ** F - \partial_\eta F], \#] == 0] \& /@ \text{bas}], 8]$$

$$\begin{aligned} & \{-1 + f_{0,0,0,0}[0] + \epsilon f_{1,0,0,0}[0] + \epsilon^2 f_{2,0,0,0}[0] = 0, \\ & \frac{\xi f_{0,0,0,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{0,0,0,0}[\eta]}{\hbar} + \frac{\epsilon \xi f_{1,0,0,0}[\eta]}{\hbar} - \frac{T^2 \epsilon \xi f_{1,0,0,0}[\eta]}{\hbar} + \frac{\epsilon f_{1,0,0,1}[\eta]}{\hbar} - \\ & \frac{T^2 \epsilon f_{1,0,0,1}[\eta]}{\hbar} + \frac{\epsilon^2 \xi f_{2,0,0,0}[\eta]}{\hbar} - \frac{T^2 \epsilon^2 \xi f_{2,0,0,0}[\eta]}{\hbar} + \frac{\epsilon^2 f_{2,0,0,1}[\eta]}{\hbar} - \\ & \frac{T^2 \epsilon^2 f_{2,0,0,1}[\eta]}{\hbar} - f_{0,0,0,0}'[\eta] - \epsilon f_{1,0,0,0}'[\eta] - \epsilon^2 f_{2,0,0,0}'[\eta] = 0, f_{1,0,0,0}[0] = 0, \\ & \frac{\xi f_{1,0,0,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{1,0,0,0}[\eta]}{\hbar} + \frac{f_{1,0,0,1}[\eta]}{\hbar} - \frac{T^2 f_{1,0,0,1}[\eta]}{\hbar} - f_{1,0,0,0}'[\eta] = 0, \\ & f_{2,0,0,0}[0] = 0, \ll 53 \gg, f_{2,2,1,1}[0] = 0, \\ & \gamma \xi \hbar f_{1,1,1,0}[\eta] - 2 \gamma f_{2,1,2,1}[\eta] + \frac{\xi f_{2,2,1,1}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,1,1}[\eta]}{\hbar} - f_{2,2,1,1}'[\eta] = 0, \\ & f_{2,2,0,2}[0] = 0, \gamma \xi \hbar f_{1,1,0,1}[\eta] - \gamma f_{2,1,1,2}[\eta] + \frac{\xi f_{2,2,0,2}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,0,2}[\eta]}{\hbar} - f_{2,2,0,2}'[\eta] = 0 \} \end{aligned}$$

Short[{sol} = DSolve[eqns, unowns, η], 8]

$$\left\{ \left\{ \begin{aligned} f_{0,0,0,0}[\eta] &\rightarrow e^{-\frac{\eta(-\xi+T^2\xi)}{h}}, f_{1,0,0,0}[\eta] \rightarrow \frac{e^{-\frac{\eta(-\xi+T^2\xi)}{h}}(-1+T^2)(-1+3T^2)\gamma\eta^2\xi^2}{4\hbar}, \\ f_{1,0,0,1}[\eta] &\rightarrow -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}(-1+3T^2)\gamma\eta\xi^2, f_{1,0,1,0}[\eta] \rightarrow 2e^{-\frac{\eta(-\xi+T^2\xi)}{h}}T^2\eta\xi, \\ f_{1,0,1,1}[\eta] &\rightarrow 0, f_{1,1,0,0}[\eta] \rightarrow -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}(-1+3T^2)\gamma\eta^2\xi, f_{1,1,0,1}[\eta] \rightarrow e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma\eta\xi\hbar, \\ f_{1,1,1,0}[\eta] &\rightarrow 0, f_{2,0,0,0}[\eta] \rightarrow \frac{1}{288\hbar^2}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}(-1+T^2)\gamma^2\eta^2\xi^2(-9\eta^2\xi^2+63T^2\eta^2\xi^2- \\ &135T^4\eta^2\xi^2+81T^6\eta^2\xi^2-40\eta\xi\hbar+272T^2\eta\xi\hbar-328T^4\eta\xi\hbar-36\hbar^2+180T^2\hbar^2), \\ f_{2,0,0,1}[\eta] &\rightarrow -\frac{1}{24\hbar}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta\xi^2(-3\eta^2\xi^2+21T^2\eta^2\xi^2-45T^4\eta^2\xi^2+ \\ &27T^6\eta^2\xi^2-10\eta\xi\hbar+68T^2\eta\xi\hbar-82T^4\eta\xi\hbar-6\hbar^2+30T^2\hbar^2), \\ f_{2,0,0,2}[\eta] &\rightarrow \frac{1}{24}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta\xi^3(3\eta\xi-18T^2\eta\xi+27T^4\eta\xi+4\hbar-28T^2\hbar), \\ f_{2,0,1,0}[\eta] &\rightarrow \frac{1}{2\hbar}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}T^2\gamma\eta^2\xi^2(\eta\xi-4T^2\eta\xi+3T^4\eta\xi+4\hbar-6T^2\hbar), \\ f_{2,0,1,1}[\eta] &\rightarrow -e^{-\frac{\eta(-\xi+T^2\xi)}{h}}T^2\gamma\eta\xi^2(-\eta\xi+3T^2\eta\xi-3\hbar), f_{2,0,1,2}[\eta] \rightarrow 0, \\ f_{2,0,2,0}[\eta] &\rightarrow 2e^{-\frac{\eta(-\xi+T^2\xi)}{h}}T^2\eta\xi(T^2\eta\xi-\hbar), f_{2,0,2,1}[\eta] \rightarrow 0, f_{2,0,2,2}[\eta] \rightarrow 0, \\ f_{2,1,0,0}[\eta] &\rightarrow -\frac{1}{24\hbar}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta^2\xi(-3\eta^2\xi^2+21T^2\eta^2\xi^2- \\ &45T^4\eta^2\xi^2+27T^6\eta^2\xi^2-10\eta\xi\hbar+68T^2\eta\xi\hbar-82T^4\eta\xi\hbar-6\hbar^2+30T^2\hbar^2), \\ f_{2,1,0,1}[\eta] &\rightarrow \frac{1}{4}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta\xi(2\eta^2\xi^2-10T^2\eta^2\xi^2+12T^4\eta^2\xi^2+5\eta\xi\hbar-21T^2\eta\xi\hbar+2\hbar^2), \\ f_{2,1,0,2}[\eta] &\rightarrow -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta\xi^2(-\eta\xi+3T^2\eta\xi-\hbar)\hbar, \\ f_{2,1,1,0}[\eta] &\rightarrow -e^{-\frac{\eta(-\xi+T^2\xi)}{h}}T^2\gamma\eta^2\xi(-\eta\xi+3T^2\eta\xi-3\hbar), \\ f_{2,1,1,1}[\eta] &\rightarrow 2e^{-\frac{\eta(-\xi+T^2\xi)}{h}}T^2\gamma\eta^2\xi^2\hbar, f_{2,1,1,2}[\eta] \rightarrow 0, f_{2,1,2,0}[\eta] \rightarrow 0, f_{2,1,2,1}[\eta] \rightarrow 0, \\ f_{2,2,0,0}[\eta] &\rightarrow \frac{1}{24}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta^3\xi(3\eta\xi-18T^2\eta\xi+27T^4\eta\xi+4\hbar-28T^2\hbar), \\ f_{2,2,0,1}[\eta] &\rightarrow -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta^2\xi(-\eta\xi+3T^2\eta\xi-\hbar)\hbar, \\ f_{2,2,0,2}[\eta] &\rightarrow \frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{h}}\gamma^2\eta^2\xi^2\hbar^2, f_{2,2,1,0}[\eta] \rightarrow 0, f_{2,2,1,1}[\eta] \rightarrow 0, f_{2,2,2,0}[\eta] \rightarrow 0 \end{aligned} \right\} \right\}$$

Union@Cases[sol, e-, ∞]

$$\left\{ e^{-\frac{\eta(-\xi+T^2\xi)}{h}} \right\}$$

FF = Collect[F /. sol /. {e- → 1, QU → Times}, ε, Simplify]

$$1 + \frac{1}{4\hbar} \epsilon \eta \xi \left(8 a T^2 \hbar + (-1 + 3 T^2) \gamma \eta \left((-1 + T^2) \xi - 2 y \hbar \right) + 2 x \gamma \hbar \left(\xi - 3 T^2 \xi + 2 y \hbar \right) \right) +$$

$$\frac{1}{288 \hbar^2} \epsilon^2 \eta \xi \left(576 a^2 T^2 \left(T^2 \eta \xi - \hbar \right) \hbar^2 + 144 a T^2 \gamma \hbar \left(6 x \xi \hbar^2 + \right.$$

$$\left. (-1 + 3 T^2) \eta^2 \xi \left((-1 + T^2) \xi - 2 y \hbar \right) + 2 \eta \hbar \left((x - 3 T^2 x) \xi^2 + 3 y \hbar + \xi \left(2 - 3 T^2 + 2 x y \hbar \right) \right) \right) +$$

$$\gamma^2 \left(9 \left(1 - 3 T^2 \right)^2 \eta^3 \xi \left(\xi - T^2 \xi + 2 y \hbar \right)^2 + 24 x \hbar^3 \left(2 \left(1 - 7 T^2 \right) x \xi^2 + 6 y \hbar + \xi \left(3 - 15 T^2 + 6 x y \hbar \right) \right) + \right.$$

$$12 \eta \hbar^2 \left(3 \left(1 - 3 T^2 \right)^2 x^2 \xi^3 + 6 y \hbar \left(1 - 5 T^2 + 2 x y \hbar \right) + \right.$$

$$3 \xi \left(1 + 5 T^4 + 10 x y \hbar + 4 x^2 y^2 \hbar^2 - 6 T^2 \left(1 + 7 x y \hbar \right) \right) +$$

$$2 x \xi^2 \left(5 + 41 T^4 + 6 x y \hbar - 2 T^2 \left(17 + 9 x y \hbar \right) \right) \left. - 4 \eta^2 \hbar \left(9 \left(1 - 3 T^2 \right)^2 \left(-1 + T^2 \right) x \xi^3 + \right.$$

$$12 \left(-1 + 7 T^2 \right) y^2 \hbar^2 + 6 y \xi \hbar \left(-5 - 41 T^4 - 6 x y \hbar + 2 T^2 \left(17 + 9 x y \hbar \right) \right) + \right.$$

$$\left. 2 \xi^2 \left(-5 + 41 T^6 - 18 x y \hbar - 3 T^4 \left(25 + 36 x y \hbar \right) + T^2 \left(39 + 9 \theta x y \hbar \right) \right) \right)$$

{Short[lhs = SimpT@OQu[SS[e^{ħ(ξx+ηy)}], {x, y}], 5],

HL[lhs = SimpT@OQu[SS[e^{ħ(ξx+ηy+(1-T²)ξη)}](FF /. {ξ → ħξ, η → ħη})], {y, a, x}]]}

$$\left\{ \left(1 - t \eta \xi \hbar^2 - \frac{1}{2} t^2 \eta \xi \hbar^3 \right) \text{QU}[\] + \left(2 \epsilon \eta \xi \hbar^2 + 2 t \epsilon \eta \xi \hbar^3 \right) \text{QU}[a] + \right.$$

$$\left(\xi \hbar + (-t \eta \xi^2 - \gamma \epsilon \eta \xi^2) \hbar^3 \right) \text{QU}[x] + \left(\eta \hbar + (-t \eta^2 \xi - \gamma \epsilon \eta^2 \xi) \hbar^3 \right) \text{QU}[y] -$$

$$2 \epsilon^2 \eta \xi \hbar^3 \text{QU}[a, a] + 2 \epsilon \eta \xi^2 \hbar^3 \text{QU}[a, x] + \frac{1}{2} \xi^2 \hbar^2 \text{QU}[x, x] + 2 \epsilon \eta^2 \xi \hbar^3 \text{QU}[y, a] +$$

$$\left(\eta \xi \hbar^2 + \gamma \epsilon \eta \xi \hbar^3 \right) \text{QU}[y, x] + \frac{1}{2} \eta^2 \hbar^2 \text{QU}[y, y] + \frac{1}{6} \xi^3 \hbar^3 \text{QU}[x, x, x] +$$

$$\frac{1}{2} \eta \xi^2 \hbar^3 \text{QU}[y, x, x] + \frac{1}{2} \eta^2 \xi \hbar^3 \text{QU}[y, y, x] + \frac{1}{6} \eta^3 \hbar^3 \text{QU}[y, y, y], \text{True} \}$$

Logos

```
QA[T_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
  {adx, G, F, f, unowns, bas, eqns, sol, λ, q, v, ξ, η, t},
  adx[ε_] := Simp[xQu ** ε - ε ** xQu];
  G = Simp[NestList[adx, yQu, $TeD + 1].Table[ξk/k!, {k, 0, $TeD + 1}]];
  F = Sum[f1,i,j,k[η] ε1QU@{yi, aj, xk},
    {1, 0, $TeD}, {i, 0, 1}, {j, 0, 1}, {k, 0, Min[1, 21 - i - j]}];
  unowns = Cases[F, f__[η], ∞];
  bas = Union@@Table[ε1Cases[Coefficient[F, ε, 1], _QU, ∞], {1, 0, $TeD}];
  eqns = Flatten[{(Coefficient[F - QU[], #] /. η → 0) == 0,
    Expand[Coefficient[Simp[F ** G - yQu ** F - ∂ηF], #]] == 0} & /@ bas];
  {sol} = DSolve[eqns, unowns, η];
  λ = Collect[F /. sol /. {e- → 1, QU → Times}, ε, Simplify];
  q = ev(-tξη+ηy+ξx+δyx);
  Collect[v q-1DPξ→Dx, η→Dy[λ][q] /. v → (1 + t δ)-1 /. t → (T2 - 1)/ħ, ε, Simplify] /.
    {y → y1, a → a1, x → x1, ξ → ξ1, η → η1}
];
```

QA[T, y, a, x, ξ, η, δ]

$$\frac{\hbar}{(-1 + T^2) \delta + \hbar} + \frac{1}{288 \left((-1 + T^2) \delta + \hbar \right)^9}$$

$$\begin{aligned} & 72 \delta^2 (x \delta + \eta)^2 (y \delta + \xi)^2 \hbar^2 ((-1 + T^2) \delta + \hbar)^2 + \\ & 96 \delta^3 (x \delta + \eta) (y \delta + \xi) \hbar ((-1 + T^2) \delta + \hbar)^3 + 24 \delta^4 ((-1 + T^2) \delta + \hbar)^4 \Big) + \\ & \frac{1}{4 ((-1 + T^2) \delta + \hbar)^5} \in \hbar^2 \left(8 a T^2 ((-1 + T^2) \delta + \hbar)^2 (\eta \xi \hbar + \delta (1 + y \eta + x \xi) \hbar + \delta^2 (-1 + T^2 + x y \hbar)) + \right. \\ & \gamma (\eta \xi \hbar^2 ((-1 + 3 T^2) \eta ((-1 + T^2) \xi - 2 y \hbar) + 2 x \hbar (\xi - 3 T^2 \xi + 2 y \hbar)) + \\ & (-1 + T^2) \delta^4 (-2 + 6 T^6 - x^2 y^2 \hbar^2 - 2 T^4 (7 + 4 x y \hbar) + T^2 (10 + 8 x y \hbar - 5 x^2 y^2 \hbar^2)) - \\ & 4 \delta^3 \hbar (1 - 3 T^6 + x^2 y^2 \hbar^2 + T^4 (7 + 2 x y (3 + y \eta) \hbar + 2 x^2 y \xi \hbar) + \\ & T^2 (-5 - 2 x y (3 + y \eta) \hbar + x^2 y \hbar (-2 \xi + y \hbar))) + \\ & 2 \delta \hbar^2 ((1 - 3 T^2) y^2 \eta^2 \hbar + 2 \eta (\xi + 3 T^4 \xi - 4 T^2 \xi (1 + x y \hbar) + y \hbar (1 - 3 T^2 + x y \hbar)) + \\ & x \hbar ((x - 3 T^2 x) \xi^2 + 2 y \hbar + \xi (2 - 6 T^2 + 2 x y \hbar))) - \\ & \left. \delta^2 \hbar ((1 - 4 T^2 + 3 T^4) y^2 \eta^2 \hbar + \hbar (-2 + 3 T^4 (-2 + 4 x \xi + x^2 \xi^2) + 4 x (\xi + y \hbar) + \right. \\ & \left. x^2 (\xi^2 + 2 y \xi \hbar - 4 y^2 \hbar^2) - 2 T^2 (-4 + x (8 \xi - 6 y \hbar) + x^2 \xi (2 \xi - 5 y \hbar))) + 2 \eta \right. \\ & \left. (-2 (-1 + T^2) \xi (1 + 3 T^4 - 2 T^2 (2 + x y \hbar)) + y \hbar (2 + 6 T^4 + x y \hbar + T^2 (-8 + 5 x y \hbar))) \right) \Big) \end{aligned}$$

{Short[lhs = SimpT@OQu[SS[e^ħ(ξx+ηy+δxy)], {x, y}], 5],

rhs = SimpT@OQu[SS[
 $e^{\hbar v (\xi x + \eta y + \delta x y - (T^2 - 1) \xi \eta)}$ QΛ[T, y, a, x, ħ ξ, ħ η, ħ δ] /. v → (1 + (T² - 1) δ)⁻¹], {y, a, x}];
 HL[Simplify[lhs == rhs]]]

$$\begin{aligned} & \left\{ \left(1 - t \delta \hbar + \left(-\frac{t^2 \delta}{2} + t^2 \delta^2 + t \gamma \delta^2 \epsilon - t \eta \xi \right) \hbar^2 + \left(-\frac{t^3 \delta}{6} + t^3 \delta^2 - t^3 \delta^3 + 2 t^2 \gamma \delta^2 \epsilon - \right. \right. \right. \\ & \left. \left. \left. 3 t^2 \gamma \delta^3 \epsilon + t \gamma^2 \delta^2 \epsilon^2 - 2 t \gamma^2 \delta^3 \epsilon^2 - \frac{1}{2} t^2 \eta \xi + 2 t^2 \delta \eta \xi + 2 t \gamma \delta \epsilon \eta \xi \right) \hbar^3 \right) \right. \\ & \text{QU}[] + (2 \delta \epsilon \hbar + (2 t \delta \epsilon - 4 t \delta^2 \epsilon - 2 \gamma \delta^2 \epsilon^2 + 2 \epsilon \eta \xi) \hbar^2 + \\ & (t^2 \delta \epsilon - 6 t^2 \delta^2 \epsilon + 6 t^2 \delta^3 \epsilon - 8 t \gamma \delta^2 \epsilon^2 + 12 t \gamma \delta^3 \epsilon^2 + 2 t \epsilon \eta \xi - 8 t \delta \epsilon \eta \xi - 4 \gamma \delta \epsilon^2 \eta \xi) \hbar^3) \\ & \left. \text{QU}[a] + \ll 25 \gg + \frac{1}{2} \delta^2 \eta \hbar^3 \text{QU}[y, y, y, x, x] + \frac{1}{6} \delta^3 \hbar^3 \text{QU}[y, y, y, x, x, x], \text{True} \right\} \end{aligned}$$

Stitching Direct

MatrixExp[η₁ ρ[CU@y]].MatrixExp[α₁ ρ[CU@a]].MatrixExp[ξ₁ ρ[CU@x]].MatrixExp[η₂ ρ[CU@y]].
 MatrixExp[α₂ ρ[CU@a]].MatrixExp[ξ₂ ρ[CU@x]] // Simplify // MatrixForm

$$\begin{pmatrix} e^{\gamma (\alpha_1 + \alpha_2)} (1 + \gamma \in \eta_2 \xi_1) & e^{\gamma \alpha_1} \gamma (e^{\gamma \alpha_2} \xi_2 + \xi_1 (1 + e^{\gamma \alpha_2} \gamma \in \eta_2 \xi_2)) \\ e^{\gamma \alpha_2} \in (\eta_2 + e^{\gamma \alpha_1} \eta_1 (1 + \gamma \in \eta_2 \xi_1)) & 1 + e^{\gamma \alpha_1} \gamma \in \eta_1 \xi_1 + e^{\gamma \alpha_2} \gamma \in (\eta_2 + e^{\gamma \alpha_1} \eta_1 (1 + \gamma \in \eta_2 \xi_1)) \xi_2 \end{pmatrix}$$

eqn = MatrixExp[η₁ ρ[CU@y]].MatrixExp[α₁ ρ[CU@a]].MatrixExp[ξ₁ ρ[CU@x]].
 MatrixExp[η₂ ρ[CU@y]].MatrixExp[α₂ ρ[CU@a]].MatrixExp[ξ₂ ρ[CU@x]] ==
 e^{τθ ∈ γ} MatrixExp[ηθ ρ[CU@y]].MatrixExp[αθ ρ[CU@a]].MatrixExp[ξθ ρ[CU@x]]

$$\begin{aligned} & \{ \{ e^{\gamma \alpha_2} (e^{\gamma \alpha_1} + e^{\gamma \alpha_1} \gamma \in \eta_2 \xi_1), e^{\gamma \alpha_1} \gamma \xi_1 + e^{\gamma \alpha_2} \gamma (e^{\gamma \alpha_1} + e^{\gamma \alpha_1} \gamma \in \eta_2 \xi_1) \xi_2 \}, \\ & \{ e^{\gamma \alpha_2} (e^{\gamma \alpha_1} \in \eta_1 + \in \eta_2 (1 + e^{\gamma \alpha_1} \gamma \in \eta_1 \xi_1)), \\ & 1 + e^{\gamma \alpha_1} \gamma \in \eta_1 \xi_1 + e^{\gamma \alpha_2} \gamma (e^{\gamma \alpha_1} \in \eta_1 + \in \eta_2 (1 + e^{\gamma \alpha_1} \gamma \in \eta_1 \xi_1)) \xi_2 \} \} == \\ & \{ \{ e^{\alpha \theta \gamma + \gamma \in \tau \theta}, e^{\alpha \theta \gamma + \gamma \in \tau \theta} \gamma \xi \theta \}, \{ e^{\alpha \theta \gamma + \gamma \in \tau \theta} \in \eta \theta, e^{\gamma \in \tau \theta} (1 + e^{\alpha \theta \gamma} \gamma \in \eta \theta \xi \theta) \} \} \end{aligned}$$

sol = Block[{ ϵ }, Solve[Thread[Flatten /@ eqn], { $\tau\theta$, $\eta\theta$, $\alpha\theta$, $\xi\theta$ }]][1]

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $\text{Log}[e^{\gamma(\alpha\theta + \epsilon\tau\theta)}] - \text{Log}[e^{\gamma\alpha_2} (e^{\gamma \text{Subscript}[\epsilon, 2]} + e^{\gamma \text{Times}[\epsilon, 2]} \gamma \in \eta_2 \xi_1)] = 0$.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

$$\begin{aligned} \tau\theta &\rightarrow \frac{1}{\gamma \in} \left(-\text{Log}[e^{\alpha\theta \gamma}] + \text{Log}[e^{\gamma\alpha_1 + \gamma\alpha_2} + e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_2 \xi_1] \right), \\ \eta\theta &\rightarrow \left(e^{-\gamma\alpha_1} \left(\frac{1}{2} + \frac{1}{2} e^{\gamma\alpha_1} \gamma \in \eta_1 \xi_1 + \frac{1}{2} e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_1 \xi_2 + \frac{1}{2} e^{\gamma\alpha_2} \gamma \in \eta_2 \xi_2 + \frac{1}{2} e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma^2 \epsilon^2 \eta_1 \eta_2 \xi_1 \xi_2 - \right. \right. \\ &\quad \left. \frac{1}{2} \sqrt{\left((-1 - e^{\gamma\alpha_1} \gamma \in \eta_1 \xi_1 - e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_1 \xi_2 - e^{\gamma\alpha_2} \gamma \in \eta_2 \xi_2 - e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma^2 \epsilon^2 \eta_1 \eta_2 \xi_1 \xi_2)^2 + \right. \right. \\ &\quad \left. \left. 4 e^{-\alpha\theta \gamma + \gamma\alpha_1 + \gamma\alpha_2} \gamma \in (-e^{\gamma\alpha_1} \eta_1 \xi_1 - \eta_2 \xi_1 - e^{\gamma\alpha_1} \gamma \in \eta_1 \eta_2 \xi_1^2 - e^{\gamma\alpha_1 + \gamma\alpha_2} \eta_1 \xi_2 - e^{\gamma\alpha_2} \eta_2 \xi_2 - \right. \right. \\ &\quad \left. \left. 2 e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_1 \eta_2 \xi_1 \xi_2 - e^{\gamma\alpha_2} \gamma \in \eta_2^2 \xi_1 \xi_2 - e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma^2 \epsilon^2 \eta_1 \eta_2^2 \xi_1^2 \xi_2) \right) \right) \Big/ \\ &\quad \left(\gamma \in (\xi_1 + e^{\gamma\alpha_2} \xi_2 + e^{\gamma\alpha_2} \gamma \in \eta_2 \xi_1 \xi_2) \right), \xi\theta \rightarrow \left(e^{-\gamma\alpha_2} \left(\frac{1}{2\gamma \in} + \frac{1}{2} e^{\gamma\alpha_1} \eta_1 \xi_1 + \right. \right. \\ &\quad \left. \frac{1}{2} e^{\gamma\alpha_1 + \gamma\alpha_2} \eta_1 \xi_2 + \frac{1}{2} e^{\gamma\alpha_2} \eta_2 \xi_2 + \frac{1}{2} e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_1 \eta_2 \xi_1 \xi_2 - \right. \\ &\quad \left. \frac{1}{2\gamma \in} \sqrt{\left((-1 - e^{\gamma\alpha_1} \gamma \in \eta_1 \xi_1 - e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_1 \xi_2 - e^{\gamma\alpha_2} \gamma \in \eta_2 \xi_2 - e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma^2 \epsilon^2 \eta_1 \eta_2 \xi_1 \xi_2)^2 + \right. \right. \\ &\quad \left. \left. 4 e^{-\alpha\theta \gamma + \gamma\alpha_1 + \gamma\alpha_2} \gamma \in (-e^{\gamma\alpha_1} \eta_1 \xi_1 - \eta_2 \xi_1 - e^{\gamma\alpha_1} \gamma \in \eta_1 \eta_2 \xi_1^2 - e^{\gamma\alpha_1 + \gamma\alpha_2} \eta_1 \right. \right. \\ &\quad \left. \left. \xi_2 - e^{\gamma\alpha_2} \eta_2 \xi_2 - 2 e^{\gamma\alpha_1 + \gamma\alpha_2} \gamma \in \eta_1 \eta_2 \xi_1 \xi_2 - e^{\gamma\alpha_2} \gamma \in \eta_2^2 \xi_1 \xi_2 - e^{\gamma\alpha_1 + \gamma\alpha_2} \right. \right. \\ &\quad \left. \left. \gamma^2 \epsilon^2 \eta_1 \eta_2^2 \xi_1^2 \xi_2) \right) \right) \Big/ (e^{\gamma\alpha_1} \eta_1 + \eta_2 + e^{\gamma\alpha_1} \gamma \in \eta_1 \eta_2 \xi_1) \} \end{aligned}$$

eqn = MatrixExp[$\eta_1 \rho$ [CU@y]].MatrixExp[$\alpha_1 \rho$ [CU@a]].MatrixExp[$\xi_1 \rho$ [CU@x]].MatrixExp[$\eta_2 \rho$ [CU@y]].MatrixExp[$\alpha_2 \rho$ [CU@a]].MatrixExp[$\xi_2 \rho$ [CU@x]] == T0 MatrixExp[$\eta\theta \rho$ [CU@y]].MatrixExp[$\alpha\theta \rho$ [CU@a]].MatrixExp[$\xi\theta \rho$ [CU@x]]

$$\begin{aligned} &\left\{ \left\{ e^{\gamma\alpha_2} (e^{\gamma\alpha_1} + e^{\gamma\alpha_1} \gamma \in \eta_2 \xi_1), e^{\gamma\alpha_1} \gamma \xi_1 + e^{\gamma\alpha_2} \gamma (e^{\gamma\alpha_1} + e^{\gamma\alpha_1} \gamma \in \eta_2 \xi_1) \xi_2 \right\}, \right. \\ &\quad \left\{ e^{\gamma\alpha_2} (e^{\gamma\alpha_1} \in \eta_1 + \in \eta_2 (1 + e^{\gamma\alpha_1} \gamma \in \eta_1 \xi_1)), \right. \\ &\quad \left. 1 + e^{\gamma\alpha_1} \gamma \in \eta_1 \xi_1 + e^{\gamma\alpha_2} \gamma (e^{\gamma\alpha_1} \in \eta_1 + \in \eta_2 (1 + e^{\gamma\alpha_1} \gamma \in \eta_1 \xi_1)) \xi_2 \right\} \} = \\ &\left\{ \{e^{\alpha\theta \gamma} \tau\theta, e^{\alpha\theta \gamma} \tau\theta \gamma \xi\theta\}, \{e^{\alpha\theta \gamma} \tau\theta \in \eta\theta, \tau\theta (1 + e^{\alpha\theta \gamma} \gamma \in \eta\theta \xi\theta)\} \right\} \end{aligned}$$

sol = Block[{ ϵ }, Solve[Thread[Flatten /@ eqn], { $T\theta$, $\eta\theta$, $\alpha\theta$, $\xi\theta$ }]][1]

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\begin{aligned} T\theta &\rightarrow \frac{1}{1 + \gamma \in \eta_2 \xi_1}, \eta\theta \rightarrow \frac{\eta_1 + e^{-\gamma\alpha_1} \eta_2 + \gamma \in \eta_1 \eta_2 \xi_1}{1 + \gamma \in \eta_2 \xi_1}, \\ \alpha\theta &\rightarrow \frac{\text{Log}[e^{\gamma\alpha_1 + \gamma\alpha_2} (1 + \gamma \in \eta_2 \xi_1)^2]}{\gamma}, \xi\theta \rightarrow \frac{e^{-\gamma\alpha_2} \xi_1 + \xi_2 + \gamma \in \eta_2 \xi_1 \xi_2}{1 + \gamma \in \eta_2 \xi_1} \} \end{aligned}$$

E

$E[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a 's, Q is a combination of $x_i y_j$, and P is a perturbation polynomial. It should be interpreted via $\text{CO}[E[...], \{x_1, a_1, y_1\}_j, ...]$ (with some default for direct interpretation), or likewise via $\text{QO}[E[...], \{x_1, a_1, y_1\}_j, ...]$. In themselves, CO and QO should have an interpretation in CU/QU by casting.