Pensieve header: A unified verification testing suite for the sl_2-portfolio project, Uxi version. Continues pensieve://Projects/SL2Portfolio/nb/Verification.pdf.

Also continues pensieve://Projects/PPSA/nb/Verification.pdf and pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
In[]:= wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
<< "SL2PortfolioProgram.m"
```

```
In[]:= $p = 2; $k = 1; $U = QU;
```

```
In[]:= HL[ℰ_] := Style[ℰ, Background → Yellow];
```

DocileQ

In[]:= DQ /@ $\{\epsilon^2\, x\, y\, a_2,\ \epsilon^2\, x^2\, y^3\}$

Out[]= {True, False}

Initialization / Utilities

$HL\left[DP_{x\to D_\xi, y\to D_\eta}\left[x^2 y^3\right]\left[e^{\delta\,\xi\,\eta}\right] == 6\, e^{\delta\,\eta\,\xi}\,\delta^3\,\xi + 6\, e^{\delta\,\eta\,\xi}\,\delta^4\,\eta\,\xi^2 + e^{\delta\,\eta\,\xi}\,\delta^5\,\eta^2\,\xi^3\right]$

True

$SP_{\{\xi\to x\}}\left[\left(\xi^2+\xi+3\right)\left(x^5\, e^x + 7\,x\right) + 99\,a\right]$

$7 + 99\,a + 21\,x + 20\,e^x\,x^3 + 15\,e^x\,x^4 + 5\,e^x\,x^5$

$SP_{\{\xi\to x, \eta\to y\}}\left[\left(\xi^2+\xi+3+2\,\xi\,\eta\right)\left(x^5\, e^x + 7\,x\right) + 99\,a + e^{\delta\,x\,y}\,\xi\,\eta\right]$

$7 + 99\,a + 21\,x + 20\,e^x\,x^3 + 15\,e^x\,x^4 + 5\,e^x\,x^5 + e^{x\,y\,\delta}\,\delta + e^{x\,y\,\delta}\,x\,y\,\delta^2$

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}]]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

{1.32813,
 {($28 t^2 \gamma^4 + 116 t \gamma^5 \epsilon$) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

```
Series[(1 - T e^(-2 ϵ a ħ)) / ħ, {a, 0, 3}]
```

$$\frac{1-T}{\hbar} + 2 T \epsilon a - 2 (T \epsilon^2 \hbar) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

In[∘]:= `HL /@ DQ /@ Series[{(1 - T e^(-2 ϵ a ħ)) / ħ, e^(ħ ϵ a)}, {ϵ, 0, 5}]`

Out[∘]= {True, True}

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
 {QU[y], QU[x]} → $\frac{(-1+T)\,QU[]}{\hbar} - 2 T \epsilon\, QU[a] - \gamma \epsilon \hbar\, QU[y, x]$},
 {{QU[a], QU[y]} → $-\gamma$ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} → $\frac{(1-T)\,QU[]}{\hbar} + 2 T \epsilon\, QU[a] + \gamma \epsilon \hbar\, QU[y, x]$,
 {QU[x], QU[a]} → $-\gamma$ QU[x], {QU[x], QU[x]} → 0}}

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

{3.78125, {($\frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \epsilon - 280 T <<1>> \epsilon + 198 T^2 \gamma^5 \epsilon}{\hbar}$) QU[y, y, y, x, x] +
 <<18>> + ($1 + 8 \gamma \epsilon \hbar$) QU[y, <<11>>, x], 0}}

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. T2t ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

{10.125, {28 t² γ⁴ CU[y, y, y, x, x] +
  116 t γ⁵ ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
 $2\left(\frac{\gamma^4}{\hbar^2}-\frac{2\,T\,\gamma^4}{\hbar^2}+\frac{T^2\,\gamma^4}{\hbar^2}+\frac{\gamma^5\,\epsilon}{\hbar}-\frac{2\,T\,\gamma^5\,\epsilon}{\hbar}+\frac{T^2\,\gamma^5\,\epsilon}{\hbar}\right)$ QU[y, y, y, x, x] +
  <<209>> + (1 + 8 γ ∈ ℏ) QU[y, y, y, <<7>>, x, x, x], 0}}

Verifying σ, m, S, and Δ.

Verifying $\sigma_{i\to j,k\to l}$:

In[∘]:= `CU@x₁ + CU@x₂ // σ₁→₃,₂→₄`

Out[∘]= $CU[x_3] + CU[x_4]$

Verifying relabeling using m:

In[∘]:= `t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃`

Out[∘]= $CU[a_2, x_2, y_3]\, t_3^2 + CU[x_2, y_3, a_3]\, t_3^2$

Verifying the meta-associativity of m:

In[∘]:=
```
Module[{z, u},
 Table[u = CU[z⟦1⟧₁, z⟦2⟧₂, z⟦3⟧₃]; HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
```

Out[∘]= {{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
 {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
 {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
 {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
 {True, True}, {True, True}, {True, True}, {True, True}, {True, True}}

Verifying the involutivity of S on CU on products of triples:

In[∘]:=
```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

Out[∘]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying that S is an anti-homomorphism on CU/QU:

In[∘]:=
```
With[{bas = U /@ {y₁, a₁, x₁}},
 Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} , {U, {CU, QU}}] ]
```

Out[∘]= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}

Verifying the co-associativity of Δ:

```
In[∘]:= Block[{bas = U /@ {y_1, a_1, x_1}},
          Table[(z1 ** z2 ** z3 // Δ_{1→1,2} // Δ_{2→2,3}) - (z1 ** z2 ** z3 // Δ_{1→1,3} // Δ_{1→1,2}) // Simp // HL,
           {z1, bas}, {z2, bas}, {z3, bas} , {U, {CU, QU}}] ]

Out[∘]= {{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
         {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
         {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying S-Δ compatibility:

```
In[∘]:= Block[{bas = U /@ {y_1, a_1, x_1}},
          Table[z1 ** z2 ** z3 // Δ_{1→1,2} // S_i // m_{1,2→1} // Simp // HL,
           {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]

Out[∘]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
         {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying S-Δ compatibility for opposite m, only for CU:

```
In[∘]:= Block[{bas = CU /@ {y_1, a_1, x_1}},
          Table[z1 ** z2 ** z3 // Δ_{1→1,2} // S_i // m_{2,1→1} // Simp // HL,
           {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ]]

Out[∘]= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying m-Δ compatibility:

```
In[∘]:= Block[{bas1 = U /@ {y_1, a_1, x_1}, bas2 = U /@ {y_2, a_2, x_2}},
          Table[(z1 ** z2 ** z3 ** z4 // m_{1,2→1} // Δ_{1→1,2}) -
              (z1 ** z2 ** z3 ** z4 // Δ_{1→3,4} // Δ_{2→5,6} // m_{3,5→1} // m_{4,6→2}) // Simp // HL,
           {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2} ] ]

Out[∘]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
          {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
         {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
          {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Implementing θ

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}]]
```

{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}]]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}]]
```

$$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\,\hbar\,QU[a,x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$$
$$\left. QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\,\epsilon\,\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\,\hbar\,QU[y,a]}{\sqrt{T}} \to QU[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Docility of A𝔻$f:

In[∘]:=
```
HL@DQ@Block[{$p = 4}, Collect[SS@A𝔻$f /. ω → a1, ϵ]]
```

Out[∘]= True

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

In[∘]:=
```
{S𝔻$P = (Cosh[ℏ ((ϵ-t)/2 + ϵ a)] - Cosh[ℏ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) /
        (ℏ Sinh[(-ϵ ℏ)/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[S𝔻$P == (S𝔻$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(S𝔻$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
     S𝔻$g (S𝔻$g /. {a → -a - γ, t → -t})] // HL,
 S𝔻$Q = Simplify[S𝔻$P /. {a → c - 1/2}],
 Simplify[S𝔻$Q == (S𝔻$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[S𝔻$g == FullSimplify[
     Sqrt[S𝔻$Q] /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL,
 HL@DQ@Block[{$p = 4}, Collect[SS@S𝔻$g /. ω → a₁, ϵ]],
 HL@DQ@Block[{$p = 4}, Collect[SS@S𝔻$f /. ω → a₁, ϵ]]
}
```

Out[∘]=
$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\right.$$
$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left((4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi)\,\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\ \mathrm{True},\ \mathrm{True}\right\}$$

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[S𝔻[z]] == S𝔻[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ϵ^k_· /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}}]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[𝕖^(ξ U@x)].ρ[𝕖^(α U@a)] == ρ[𝕖^(α U@a)].ρ[𝕖^(e^(-γα) ξ U@x)]], {U, {CU, QU}}]
```

```
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

```
Table[
 {Λ_{U,1}[{α, β}, {u, u}],
  lhs = U@Œ_U[{u1, u2}, ℏ (α u1 + β u2), 1], HL[lhs == U@Λ_{U,1}[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

$$\{\{\{Œ_{CU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2], \mathrm{CU}[\,] + (\alpha\hbar+\beta\hbar)\,\mathrm{CU}[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)\mathrm{CU}[y, y], \mathrm{True}\},$$

$$\{Œ_{CU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2], \mathrm{CU}[\,] + (\alpha\hbar+\beta\hbar)\,\mathrm{CU}[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)\mathrm{CU}[a, a], \mathrm{True}\}, \{Œ_{CU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$$

$$\mathrm{CU}[\,] + (\alpha\hbar+\beta\hbar)\,\mathrm{CU}[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)\mathrm{CU}[x, x], \mathrm{True}\}\},$$

$$\{\{Œ_{QU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2], \mathrm{QU}[\,] + (\alpha\hbar+\beta\hbar)\,\mathrm{QU}[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)\mathrm{QU}[y, y],$$

$$\mathrm{True}\}, \{Œ_{QU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2], \mathrm{QU}[\,] + (\alpha\hbar+\beta\hbar)\,\mathrm{QU}[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)\mathrm{QU}[a, a], \mathrm{True}\}, \{Œ_{QU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$$

$$\mathrm{QU}[\,] + (\alpha\hbar+\beta\hbar)\,\mathrm{QU}[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)\mathrm{QU}[x, x], \mathrm{True}\}\}\}$$

```
{Λ#,1[{ξ, α}, {x, a}], lhs = #@Œ#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ#,1[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$$\Big\{\Big\{\mathbb{E}_{CU}\big[\{a, x\}, a\,\alpha + e^{-\alpha\gamma}\,x\,\xi, 1 + O[\epsilon]^2\big],\ CU[\,] + \alpha\,\hbar\,CU[a] + \left(\xi\,\hbar - \alpha\,\gamma\,\xi\,\hbar^2\right) CU[x] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\,\xi\,\hbar^2\,CU[a, x] + \frac{1}{2}\xi^2\hbar^2\,CU[x, x],\ \text{True}\Big\},\ \Big\{\mathbb{E}_{QU}\big[\{a, x\}, a\,\alpha + e^{-\alpha\gamma}\,x\,\xi, 1 + O[\epsilon]^2\big],\ QU[\,] + \alpha\,\hbar\,QU[a] + \left(\xi\,\hbar - \alpha\,\gamma\,\xi\,\hbar^2\right) QU[x] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\,\xi\,\hbar^2\,QU[a, x] + \frac{1}{2}\xi^2\hbar^2\,QU[x, x],\ \text{True}\Big\}\Big\}$$

```
{Λ#,2[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#,2[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$$\Big\{\Big\{\mathbb{E}_{CU}\big[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1 + O[\epsilon]^3\big],\ CU[\,] + \alpha\,\hbar\,CU[a] + \left(\eta\,\hbar - \alpha\,\gamma\,\eta\,\hbar^2\right) CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\,\eta\,\hbar^2\,CU[y, a] + \frac{1}{2}\eta^2\hbar^2\,CU[y, y],\ \text{True}\Big\},\ \Big\{\mathbb{E}_{QU}\big[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1 + O[\epsilon]^3\big],\ QU[\,] + \alpha\,\hbar\,QU[a] + \left(\eta\,\hbar - \alpha\,\gamma\,\eta\,\hbar^2\right) QU[y] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\,\eta\,\hbar^2\,QU[y, a] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y],\ \text{True}\Big\}\Big\}$$

In[•]:=
```
Timing@ΛQU,2[{ξ, η}, {x, y}]
```

Out[•]=

$$\Big\{1.64063,\ \mathbb{E}_{QU}\Big[\{y, a, x\},\ y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar},\ 1 + \frac{1}{4\hbar}\eta\,\xi\left(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\right)\epsilon +$$
$$\Big(-a\,T\,y\,\gamma\,\eta^2\,\xi\left(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar\right) - a\,T\,x\,\gamma\,\eta\,\xi^2\left(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar\right) + 2\,a^2\,T\,\eta\,\xi\,(T\,\eta\,\xi - \hbar) +$$
$$2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar - \frac{1}{2}x\,y^2\gamma^2\eta^2\,\xi\left(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar\right)\hbar - \frac{1}{2}x^2\,y\,\gamma^2\,\eta\,\xi^2\left(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar\right)\hbar +$$
$$\frac{1}{2}x^2\,y^2\gamma^2\eta^2\xi^2\hbar^2 + \frac{1}{24}y^2\gamma^2\eta^3\,\xi\left(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar\right) + \frac{1}{24}x^2\gamma^2\,\eta\,\xi^3$$
$$\left(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar\right) + \frac{1}{2\hbar}a\,T\,\gamma\,\eta^2\,\xi^2\left(\eta\,\xi - 4\,T\,\eta\,\xi + 3\,T^2\,\eta\,\xi + 4\,\hbar - 6\,T\,\hbar\right) +$$
$$\frac{1}{4}x\,y\,\gamma^2\,\eta\,\xi\left(2\,\eta^2\xi^2 - 10\,T\,\eta^2\xi^2 + 12\,T^2\eta^2\xi^2 + 5\,\eta\,\xi\,\hbar - 21\,T\,\eta\,\xi\,\hbar + 2\,\hbar^2\right) -$$
$$\frac{1}{24\hbar}y\,\gamma^2\,\eta^2\,\xi\left(-3\,\eta^2\xi^2 + 21\,T\,\eta^2\xi^2 - 45\,T^2\eta^2\xi^2 + 27\,T^3\eta^2\xi^2 - 10\,\eta\,\xi\,\hbar +\right.$$
$$\left.68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar - 6\,\hbar^2 + 30\,T\,\hbar^2\right) - \frac{1}{24\hbar}x\,\gamma^2\,\eta\,\xi^2\left(-3\,\eta^2\xi^2 + 21\,T\,\eta^2\xi^2 -\right.$$
$$\left.45\,T^2\eta^2\xi^2 + 27\,T^3\eta^2\xi^2 - 10\,\eta\,\xi\,\hbar + 68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar - 6\,\hbar^2 + 30\,T\,\hbar^2\right) +$$
$$\frac{1}{288\,\hbar^2}(-1 + T)\,\gamma^2\,\eta^2\,\xi^2\left(-9\,\eta^2\xi^2 + 63\,T\,\eta^2\xi^2 - 135\,T^2\eta^2\xi^2 + 81\,T^3\eta^2\xi^2 -\right.$$
$$\left.40\,\eta\,\xi\,\hbar + 272\,T\,\eta\,\xi\,\hbar - 328\,T^2\,\eta\,\xi\,\hbar - 36\,\hbar^2 + 180\,T\,\hbar^2\right)\Big)\epsilon^2 + O[\epsilon]^3\Big]\Big\}$$

```
{Λ_{CU,1}[{ξ, η}, {x, y}], lhs = CU@Œ_{CU}[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@Λ_{CU,1}[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{Œ_{CU}\left[\{y, a, x\}, y\,\eta + x\,\xi - t\,\eta\,\xi, 1 + \frac{1}{2}\eta\,\xi\left(4\,a - 2\,y\,\gamma\,\eta - 2\,x\,\gamma\,\xi + t\,\gamma\,\eta\,\xi\right)\epsilon + O[\epsilon]^2\right],\right.$$
$$\left(1 - t\,\eta\,\xi\,\hbar^2\right) CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] +$$
$$\left.\frac{1}{2}\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\eta^2\,\hbar^2\,CU[y, y], \text{True}\right\}$$

In[∘]:=
```
{Λ_{QU,1}[{ξ, η}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η}, {x, y}]]}
```

Out[∘]=

$$\left\{Œ_{QU}\left[\{y, a, x\}, y\,\eta + x\,\xi + \frac{(1 - T)\,\eta\,\xi}{\hbar}, 1 + \frac{1}{4\,\hbar}\right.\right.$$
$$\eta\,\xi\left(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\right)\epsilon +$$
$$\left.O[\epsilon]^2\right], \left(1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\right) QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] +$$
$$\left.\eta\,\hbar\,QU[y] + \frac{1}{2}\xi^2\,\hbar^2\,QU[x, x] + \eta\,\xi\,\hbar^2\,QU[y, x] + \frac{1}{2}\eta^2\,\hbar^2\,QU[y, y], \text{True}\right\}$$

```
{tt = Last[Λ_{CU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand
```

$$\left\{1 + \left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon +\right.$$
$$\left(2\,a^2\,\eta^2\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 2\,a\,y\,\gamma\,\eta^3\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + \frac{1}{2}y^2\,\gamma^2\,\eta^4\,\xi^2 - 2\,a\,x\,\gamma\,\eta^2\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 + a\,t\,\gamma\,\eta^3\,\xi^3 -\right.$$
$$\left.\frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3 + x\,y\,\gamma^2\,\eta^3\,\xi^3 - \frac{1}{2}t\,y\,\gamma^2\,\eta^4\,\xi^3 + \frac{1}{2}x^2\,\gamma^2\,\eta^2\,\xi^4 - \frac{1}{2}t\,x\,\gamma^2\,\eta^3\,\xi^4 + \frac{1}{8}t^2\,\gamma^2\,\eta^4\,\xi^4\right)\epsilon^2 + O[\epsilon]^3,$$
$$\left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(-a\,\gamma\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + x\,\gamma^2\,\eta^2\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3\right)\epsilon^2 +$$
$$\left.O[\epsilon]^3, 6\right\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{1 + \Big(2\,a\,T\,\eta\,\xi + \frac{1}{2}\,y\,\gamma\,\eta^2\,\xi - \frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi +$$
$$\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2 - \frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2 + \frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar} - \frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar} + \frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar} + x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon +$$
$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2 + 2\,a\,T\,\gamma\,\eta^2\,\xi^2 - 3\,a\,T^2\,\gamma\,\eta^2\,\xi^2 + \frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2 - \frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2 + \frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2 +$$
$$a\,T\,y\,\gamma\,\eta^3\,\xi^2 - 3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2 + \frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2 - \frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2 + \frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2 + \frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2 -$$
$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2 + \frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2 + a\,T\,x\,\gamma\,\eta^2\,\xi^3 - 3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3 + \frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3 - \frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3 +$$
$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3 + \frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3 - \frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3 + 3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4 -$$
$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4 + \frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4 + \frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2} - \frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2} + \frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2} - \frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2} +$$
$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2} + \frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar} - \frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar} + \frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar} + \frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar} - \frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar} +$$
$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar} - \frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar} + \frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar} - \frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar} + \frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar} - \frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar} +$$
$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar} - \frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar} + \frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar} - \frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar} - 2\,a^2\,T\,\eta\,\xi\,\hbar + 3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar +$$
$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar - \frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar + \frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar - \frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar + 3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar + \frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar -$$
$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar + 2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar + \frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar - \frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar + \frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar -$$
$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar + \frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar - \frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar + \frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar - \frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar +$$
$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2 + \frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2 + \frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2 + \frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2 + O[\epsilon]^3,$$
$$\Big(2\,a\,T\,\eta\,\xi + \frac{1}{2}\,y\,\gamma\,\eta^2\,\xi - \frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi + \frac{1}{2}\,x\,\gamma\,\eta\,\xi^2 - \frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2 + \frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar} -$$
$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar} + \frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar} + x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon +$$
$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2 - 3\,a\,T^2\,\gamma\,\eta^2\,\xi^2 + \frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2 - \frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2 + \frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2 + \frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2 -$$
$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2 + \frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2 + \frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3 - \frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3 + \frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3 +$$
$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar} - \frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar} + \frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar} - \frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar} - 2\,a^2\,T\,\eta\,\xi\,\hbar + 3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar +$$
$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar - \frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar + \frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar - \frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar + 3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar +$$
$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar - \frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar + \frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar - \frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar + \frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar -$$
$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar + \frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2 + \frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2 + \frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6\Big\}$$

```
Block[{$p = 4, $k = 1},
 {Λ_{CU,$k}[ℏ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@Œ_{CU}[{x, y}, ℏ (ξ x + η y + δ x y), 1_{$k}], 5],
   HL@Simp[lhs - CU@Λ_{CU,$k}[ℏ {ξ, η, δ}, {x, y}]]}
]
```

$\{Œ_{CU}[\{y, a, x\}, \frac{x y \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar},$

$\frac{1}{1 + t \delta \hbar} + ((4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 -$
$4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 +$
$4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 -$
$6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 +$
$8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 +$
$4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 +$
$4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 +$
$4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4) \epsilon) /$
$(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5) + O[\epsilon]^2],$

$(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 +$
$2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 -$
$9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4) CU[] +$
$(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 +$
$18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4) CU[a] +$
$\ll 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 CU[y, y, y, y, x, x, x] +$
$\frac{1}{24} \delta^4 \hbar^4$
$CU[y, y, y, y, x, x, x, x], 0\}$

```
{Λ_{QU,2}[{ξ, η, δ}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η, δ}, {x, y}]]}
```

$\{Œ_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$
$\frac{\hbar}{-\delta + T \delta + \hbar} + ((-8 a T \delta^4 \hbar^2 + 24 a T^2 \delta^4 \hbar^2 - 24 a T^3 \delta^4 \hbar^2 + 8 a T^4 \delta^4 \hbar^2 + \cdots 149 \cdots +$
$4 x^2 y^2 \gamma \delta^2 \hbar^6 + 4 x y^2 \gamma \delta \eta \hbar^6 + 4 x^2 y \gamma \delta \xi \hbar^6 + 4 x y \gamma \eta \xi \hbar^6) \epsilon) /$
$(-4 \delta^5 + 20 T \delta^5 - 40 T^2 \delta^5 + 40 T^3 \delta^5 - 20 T^4 \delta^5 + 4 T^5 \delta^5 + \cdots 12 \cdots + 40 T^3 \delta^3 \hbar^2 +$
$40 \delta^2 \hbar^3 - 80 T \delta^2 \hbar^3 + 40 T^2 \delta^2 \hbar^3 - 20 \delta \hbar^4 + 20 T \delta \hbar^4 + 4 \hbar^5) +$
$\frac{(\cdots 1 \cdots) \cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3], \cdots 1 \cdots, \text{True}\}$

large output | show less | show more | show all | set size limit...

```
{tt = ComposeSeries[(1 + t δ) Last[Λ_{CU,2}[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$$\Big\{\Big(2 a \delta + 6 a t \delta^2 + 2 a x y \delta^2 + t \gamma \delta^2 - 4 x y \gamma \delta^2 + 6 a t^2 \delta^3 + 4 a t x y \delta^3 + 2 t^2 \gamma \delta^3 - 6 t x y \gamma \delta^3 -$$
$$2 x^2 y^2 \gamma \delta^3 + 2 a t^3 \delta^4 + 2 a t^2 x y \delta^4 + t^3 \gamma \delta^4 - 2 t^2 x y \gamma \delta^4 - \frac{3}{2} t x^2 y^2 \gamma \delta^4 + 2 a y \delta \eta -$$
$$2 y \gamma \delta \eta + 4 a t y \delta^2 \eta - 2 t y \gamma \delta^2 \eta - 3 x y^2 \gamma \delta^2 \eta + 2 a t^2 y \delta^3 \eta - 2 t x y^2 \gamma \delta^3 \eta -$$
$$y^2 \gamma \delta \eta^2 - \frac{1}{2} t y^2 \gamma \delta^2 \eta^2 + 2 a x \delta \xi - 2 x \gamma \delta \xi + 4 a t x \delta^2 \xi - 2 t x \gamma \delta^2 \xi - 3 x^2 y \gamma \delta^2 \xi +$$
$$2 a t^2 x \delta^3 \xi - 2 t x^2 y \gamma \delta^3 \xi + 2 a \eta \xi + 4 a t \delta \eta \xi + 2 t \gamma \delta \eta \xi - 4 x y \gamma \delta \eta \xi + 2 a t^2 \delta^2 \eta \xi +$$
$$2 t^2 \gamma \delta^2 \eta \xi - 2 t x y \gamma \delta^2 \eta \xi - y \gamma \eta^2 \xi - x^2 \gamma \delta \xi^2 - \frac{1}{2} t x^2 \gamma \delta^2 \xi^2 - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2\Big) \epsilon +$$
$$\Big(2 a^2 \delta^2 - 2 a \gamma \delta^2 + 12 a^2 t \delta^3 + 4 a^2 x y \delta^3 - 8 a t \gamma \delta^3 - 20 a x y \gamma \delta^3 - 2 t \gamma^2 \delta^3 + 18 x y \gamma^2 \delta^3 +$$
$$30 a^2 t^2 \delta^4 + 20 a^2 t x y \delta^4 - 10 a t^2 \gamma \delta^4 - 88 a t x y \gamma \delta^4 - 13 a x^2 y^2 \gamma \delta^4 - \frac{15}{2} t^2 \gamma^2 \delta^4 +$$
$$64 t x y \gamma^2 \delta^4 + 34 x^2 y^2 \gamma^2 \delta^4 + 40 a^2 t^3 \delta^5 + 40 a^2 t^2 x y \delta^5 - 152 a t^2 x y \gamma \delta^5 - 48 a t x^2 y^2 \gamma \delta^5 -$$
$$10 t^3 \gamma^2 \delta^5 + 86 t^2 x y \gamma^2 \delta^5 + 107 t x^2 y^2 \gamma^2 \delta^5 + 11 x^3 y^3 \gamma^2 \delta^5 + 30 a^2 t^4 \delta^6 + 40 a^2 t^3 x y \delta^6 +$$
$$10 a t^4 \gamma \delta^6 - 128 a t^3 x y \gamma \delta^6 - 66 a t^2 x^2 y^2 \gamma \delta^6 - 5 t^4 \gamma^2 \delta^6 + 54 t^3 x y \gamma^2 \delta^6 + \frac{247}{2} t^2 x^2 y^2 \gamma^2 \delta^6 +$$
$$\frac{80}{3} t x^3 y^3 \gamma^2 \delta^6 + 12 a^2 t^5 \delta^7 + 20 a^2 t^4 x y \delta^7 + 8 a t^5 \gamma \delta^7 - 52 a t^4 x y \gamma \delta^7 - 40 a t^3 x^2 y^2 \gamma \delta^7 +$$
$$16 t^4 x y \gamma^2 \delta^7 + 62 t^3 x^2 y^2 \gamma^2 \delta^7 + \frac{64}{3} t^2 x^3 y^3 \gamma^2 \delta^7 + 2 a^2 t^6 \delta^8 + 4 a^2 t^5 x y \delta^8 + 2 a t^6 \gamma \delta^8 -$$
$$8 a t^5 x y \gamma \delta^8 - 9 a t^4 x^2 y^2 \gamma \delta^8 + \frac{1}{2} t^6 \gamma^2 \delta^8 + 2 t^5 x y \gamma^2 \delta^8 + \frac{23}{2} t^4 x^2 y^2 \gamma^2 \delta^8 + \frac{17}{3} t^3 x^3 y^3 \gamma^2 \delta^8 +$$
$$4 a^2 y \delta^2 \eta - 12 a y \gamma \delta^2 \eta + 6 y \gamma^2 \delta^2 \eta + 20 a^2 t y \delta^3 \eta - 48 a t y \gamma \delta^3 \eta - 20 a x y^2 \gamma \delta^3 \eta +$$
$$14 t y \gamma^2 \delta^3 \eta + 40 x y^2 \gamma^2 \delta^3 \eta + 40 a^2 t^2 y \delta^4 \eta - 72 a t^2 y \gamma \delta^4 \eta - 72 a t x y^2 \gamma \delta^4 \eta + 6 t^2 y \gamma^2 \delta^4 \eta +$$
$$115 t x y^2 \gamma^2 \delta^4 \eta + 23 x^2 y^3 \gamma^2 \delta^4 \eta + 40 a^2 t^3 y \delta^5 \eta - 48 a t^3 y \gamma \delta^5 \eta - 96 a t^2 x y^2 \gamma \delta^5 \eta -$$
$$6 t^3 y \gamma^2 \delta^5 \eta + 118 t^2 x y^2 \gamma^2 \delta^5 \eta + 53 t x^2 y^3 \gamma^2 \delta^5 \eta + 20 a^2 t^4 y \delta^6 \eta - 12 a t^4 y \gamma \delta^6 \eta -$$
$$56 a t^3 x y^2 \gamma \delta^6 \eta - 4 t^4 y \gamma^2 \delta^6 \eta + 51 t^3 x y^2 \gamma^2 \delta^6 \eta + 40 t^2 x^2 y^3 \gamma^2 \delta^6 \eta + 4 a^2 t^5 y \delta^7 \eta -$$
$$12 a t^4 x y^2 \gamma \delta^7 \eta + 8 t^4 x y^2 \gamma^2 \delta^7 \eta + 10 t^3 x^2 y^3 \gamma^2 \delta^7 \eta - 7 a y^2 \gamma \delta^2 \eta^2 + 10 y^2 \gamma^2 \delta^2 \eta^2 -$$
$$24 a t y^2 \gamma \delta^3 \eta^2 + 24 t y^2 \gamma^2 \delta^3 \eta^2 + 15 x y^3 \gamma^2 \delta^3 \eta^2 - 30 a t^2 y^2 \gamma \delta^4 \eta^2 + \frac{37}{2} t^2 y^2 \gamma^2 \delta^4 \eta^2 +$$
$$32 t x y^3 \gamma^2 \delta^4 \eta^2 - 16 a t^3 y^2 \gamma \delta^5 \eta^2 + 5 t^3 y^2 \gamma^2 \delta^5 \eta^2 + 22 t^2 x y^3 \gamma^2 \delta^5 \eta^2 - 3 a t^4 y^2 \gamma \delta^6 \eta^2 +$$
$$\frac{1}{2} t^4 y^2 \gamma^2 \delta^6 \eta^2 + 5 t^3 x y^3 \gamma^2 \delta^6 \eta^2 + 3 y^3 \gamma^2 \delta^2 \eta^3 + \frac{17}{3} t y^3 \gamma^2 \delta^3 \eta^3 + \frac{10}{3} t^2 y^3 \gamma^2 \delta^4 \eta^3 +$$
$$\frac{2}{3} t^3 y^3 \gamma^2 \delta^5 \eta^3 + 4 a^2 x \delta^2 \xi - 12 a x \gamma \delta^2 \xi + 6 x \gamma^2 \delta^2 \xi + 20 a^2 t x \delta^3 \xi - 48 a t x \gamma \delta^3 \xi -$$
$$20 a x^2 y \gamma \delta^3 \xi + 14 t x \gamma^2 \delta^3 \xi + 40 x^2 y \gamma^2 \delta^3 \xi + 40 a^2 t^2 x \delta^4 \xi - 72 a t^2 x \gamma \delta^4 \xi - 72 a t x^2 y \gamma \delta^4 \xi +$$
$$6 t^2 x \gamma^2 \delta^4 \xi + 115 t x^2 y \gamma^2 \delta^4 \xi + 23 x^3 y^2 \gamma^2 \delta^4 \xi + 40 a^2 t^3 x \delta^5 \xi - 48 a t^3 x \gamma \delta^5 \xi -$$
$$96 a t^2 x^2 y \gamma \delta^5 \xi - 6 t^3 x \gamma^2 \delta^5 \xi + 118 t^2 x^2 y \gamma^2 \delta^5 \xi + 53 t x^3 y^2 \gamma^2 \delta^5 \xi + 20 a^2 t^4 x \delta^6 \xi -$$
$$12 a t^4 x \gamma \delta^6 \xi - 56 a t^3 x^2 y \gamma \delta^6 \xi - 4 t^4 x \gamma^2 \delta^6 \xi + 51 t^3 x^2 y \gamma^2 \delta^6 \xi + 40 t^2 x^3 y^2 \gamma^2 \delta^6 \xi +$$
$$4 a^2 t^5 x \delta^7 \xi - 12 a t^4 x^2 y \gamma \delta^7 \xi + 8 t^4 x^2 y \gamma^2 \delta^7 \xi + 10 t^3 x^3 y^2 \gamma^2 \delta^7 \xi + 4 a^2 \delta \eta \xi - 4 a \gamma \delta \eta \xi +$$
$$20 a^2 t \delta^2 \eta \xi - 8 a t \gamma \delta^2 \eta \xi - 28 a x y \gamma \delta^2 \eta \xi - 6 t \gamma^2 \delta^2 \eta \xi + 38 x y \gamma^2 \delta^2 \eta \xi + 40 a^2 t^2 \delta^3 \eta \xi +$$

$$8\,a\,t^2\,\gamma\,\delta^3\,\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\,\eta\,\xi - 14\,t^2\,\gamma^2\,\delta^3\,\eta\,\xi + 88\,t\,x\,y\,\gamma^2\,\delta^3\,\eta\,\xi + 44\,x^2\,y^2\,\gamma^2\,\delta^3\,\eta\,\xi +$$
$$40\,a^2\,t^3\,\delta^4\,\eta\,\xi + 32\,a\,t^3\,\gamma\,\delta^4\,\eta\,\xi - 120\,a\,t^2\,x\,y\,\gamma\,\delta^4\,\eta\,\xi - 6\,t^3\,\gamma^2\,\delta^4\,\eta\,\xi + 62\,t^2\,x\,y\,\gamma^2\,\delta^4\,\eta\,\xi +$$
$$93\,t\,x^2\,y^2\,\gamma^2\,\delta^4\,\eta\,\xi + 20\,a^2\,t^4\,\delta^5\,\eta\,\xi + 28\,a\,t^4\,\gamma\,\delta^5\,\eta\,\xi - 64\,a\,t^3\,x\,y\,\gamma\,\delta^5\,\eta\,\xi + 6\,t^4\,\gamma^2\,\delta^5\,\eta\,\xi +$$
$$12\,t^3\,x\,y\,\gamma^2\,\delta^5\,\eta\,\xi + 63\,t^2\,x^2\,y^2\,\gamma^2\,\delta^5\,\eta\,\xi + 4\,a^2\,t^5\,\delta^6\,\eta\,\xi + 8\,a\,t^5\,\gamma\,\delta^6\,\eta\,\xi - 12\,a\,t^4\,x\,y\,\gamma\,\delta^6\,\eta\,\xi +$$
$$4\,t^5\,\gamma^2\,\delta^6\,\eta\,\xi + 14\,t^3\,x^2\,y^2\,\gamma^2\,\delta^6\,\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\,\xi + 6\,y\,\gamma^2\,\delta\,\eta^2\,\xi - 24\,a\,t\,y\,\gamma\,\delta^2\,\eta^2\,\xi +$$
$$5\,t\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi + 25\,x\,y^2\,\gamma^2\,\delta^2\,\eta^2\,\xi - 24\,a\,t^2\,y\,\gamma\,\delta^3\,\eta^2\,\xi - 8\,t^2\,y\,\gamma^2\,\delta^3\,\eta^2\,\xi + 45\,t\,x\,y^2\,\gamma^2\,\delta^3\,\eta^2\,\xi -$$
$$8\,a\,t^3\,y\,\gamma\,\delta^4\,\eta^2\,\xi - 7\,t^3\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi + 24\,t^2\,x\,y^2\,\gamma^2\,\delta^4\,\eta^2\,\xi + 4\,t^3\,x\,y^2\,\gamma^2\,\delta^5\,\eta^2\,\xi + 4\,y^2\,\gamma^2\,\delta\,\eta^3\,\xi +$$
$$5\,t\,y^2\,\gamma^2\,\delta^2\,\eta^3\,\xi + t^2\,y^2\,\gamma^2\,\delta^3\,\eta^3\,\xi - 7\,a\,x^2\,\gamma\,\delta^2\,\xi^2 + 10\,x^2\,\gamma^2\,\delta^2\,\xi^2 - 24\,a\,t\,x^2\,\gamma\,\delta^3\,\xi^2 +$$
$$24\,t\,x^2\,\gamma^2\,\delta^3\,\xi^2 + 15\,x^3\,y\,\gamma^2\,\delta^3\,\xi^2 - 30\,a\,t^2\,x^2\,\gamma\,\delta^4\,\xi^2 + \frac{37}{2}\,t^2\,x^2\,\gamma^2\,\delta^4\,\xi^2 + 32\,t\,x^3\,y\,\gamma^2\,\delta^4\,\xi^2 -$$
$$16\,a\,t^3\,x^2\,\gamma\,\delta^5\,\xi^2 + 5\,t^3\,x^2\,\gamma^2\,\delta^5\,\xi^2 + 22\,t^2\,x^3\,y\,\gamma^2\,\delta^5\,\xi^2 - 3\,a\,t^4\,x^2\,\gamma\,\delta^6\,\xi^2 + \frac{1}{2}\,t^4\,x^2\,\gamma^2\,\delta^6\,\xi^2 +$$
$$5\,t^3\,x^3\,y\,\gamma^2\,\delta^6\,\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\,\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\,\eta\,\xi^2 + 5\,t\,x\,\gamma^2\,\delta^2\,\eta\,\xi^2 +$$
$$25\,x^2\,y\,\gamma^2\,\delta^2\,\eta\,\xi^2 - 24\,a\,t^2\,x\,\gamma\,\delta^3\,\eta\,\xi^2 - 8\,t^2\,x\,\gamma^2\,\delta^3\,\eta\,\xi^2 + 45\,t\,x^2\,y\,\gamma^2\,\delta^3\,\eta\,\xi^2 - 8\,a\,t^3\,x\,\gamma\,\delta^4\,\eta\,\xi^2 -$$
$$7\,t^3\,x\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 24\,t^2\,x^2\,y\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 4\,t^3\,x^2\,y\,\gamma^2\,\delta^5\,\eta\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 3\,t\,\gamma^2\,\delta\,\eta^2\,\xi^2 +$$
$$11\,x\,y\,\gamma^2\,\delta\,\eta^2\,\xi^2 + 6\,a\,t^2\,\gamma\,\delta^2\,\eta^2\,\xi^2 - \frac{5}{2}\,t^2\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 12\,t\,x\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 8\,a\,t^3\,\gamma\,\delta^3\,\eta^2\,\xi^2 +$$
$$4\,t^3\,\gamma^2\,\delta^3\,\eta^2\,\xi^2 + 3\,a\,t^4\,\gamma\,\delta^4\,\eta^2\,\xi^2 + \frac{7}{2}\,t^4\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 - t^3\,x\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 -$$
$$t\,y\,\gamma^2\,\delta\,\eta^3\,\xi^2 - 2\,t^2\,y\,\gamma^2\,\delta^2\,\eta^3\,\xi^2 + 3\,x^3\,\gamma^2\,\delta^2\,\xi^3 + \frac{17}{3}\,t\,x^3\,\gamma^2\,\delta^3\,\xi^3 + \frac{10}{3}\,t^2\,x^3\,\gamma^2\,\delta^4\,\xi^3 +$$
$$\frac{2}{3}\,t^3\,x^3\,\gamma^2\,\delta^5\,\xi^3 + 4\,x^2\,\gamma^2\,\delta\,\eta\,\xi^3 + 5\,t\,x^2\,\gamma^2\,\delta^2\,\eta\,\xi^3 + t^2\,x^2\,\gamma^2\,\delta^3\,\eta\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 - t\,x\,\gamma^2\,\delta\,\eta^2\,\xi^3 -$$
$$2\,t^2\,x\,\gamma^2\,\delta^2\,\eta^2\,\xi^3 - \frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{3}\,t^2\,\gamma^2\,\delta\,\eta^3\,\xi^3 + \frac{2}{3}\,t^3\,\gamma^2\,\delta^2\,\eta^3\,\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}$$

```
{tt = Last[ΛQU,2[{ξ, η, δ}, {x, y}]];
  Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{\mathrm{Log}\Big[\frac{\hbar}{-\delta+T\,\delta+\hbar}\Big] + \Big(\frac{2\,a\,T\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} - \frac{8\,a\,T^2\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} + \frac{12\,a\,T^3\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} -$$
$$\frac{8\,a\,T^4\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} + \cdots 267 \cdots + \frac{x^2\,y^2\,\gamma\,\delta^2\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x\,y\,\gamma\,\eta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}\Big)\,\epsilon +$$
$$\Big(-\frac{32\,a^2\,T^2\,\delta^{10}\,\hbar^2}{(\cdots 1 \cdots)^2} + \cdots 8307 \cdots + \cdots 1 \cdots + \frac{144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^{11}}{\cdots 1 \cdots}\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6\Big\}$$

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Reorderings with Rord

```
With[{c⊙ = ℭCU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (e^t1 - 1) y1 x2, 12 + ϵ x1 y2]},
 {Short[rhs = c⊙ // Rord_{x2,a2→3}, 3], HL[CU[c⊙] == CU[rhs]]}]
```

$$\Big\{\mathfrak{C}_{CU}\big[\{y_1, x_1\}_1,\ \{a_3, x_3, y_2\}_2,\ \frac{e^{-\gamma\,\hbar\,t_1}\,\big(e^{\gamma\,\hbar\,t_1}\,\hbar\,a_3\,t_1^2 - \hbar\,x_3\,y_1 + e^{t_1}\,\hbar\,x_3\,y_1\big)}{t_1},\ 1 + x_1\,y_2\,\epsilon + O[\epsilon]^3\big],\ \mathrm{True}\Big\}$$

```
With[{c⊕ = ŒCU[{y1, a1, a2}1, {x2, x1, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = c⊕ // Rord_{a1,a2→3} // Rord_{x2,x1→4}, 3], HL[CU[c⊕] == CU[rhs]]}]

{ŒCU[{y1, a3}1, {x4, y2}2,
  ℏ a3 l11 t1 + ℏ a3 l12 t1 + ℏ a3 l21 t2 + ℏ a3 l22 t2 + ℏ x4 y1 γ11 + ℏ x4 y2 γ12 + ℏ x4 y1 γ21 + ℏ x4 y2 γ22,
  1 + (a3 l1 + a3 l2 + p11 x4 y1 + p21 x4 y1 + p12 x4 y2 + p22 x4 y2) ∈ + O[∈]^3], True}

ℏ a3 l11 t1 + ℏ a3 l12 t1 + ℏ a3 l21 t2 + ℏ a3 l22 t2 +
  ℏ x4 y1 γ11 + ℏ x4 y2 γ12 + ℏ x4 y1 γ21 + ℏ x4 y2 γ22 // Simplify

ℏ (a3 (l11 t1 + l12 t1 + (l21 + l22) t2) + x4 (y1 (γ11 + γ21) + y2 (γ12 + γ22)))

With[{c⊕ = ŒCU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = c⊕ // Rord_{x2,a2→3}, 3], HL[CU[c⊕] == CU[rhs]]}]

{ŒCU[{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
  1 + e^(-γ ℏ l12 t1 - γ ℏ l22 t2) (e^(γ ℏ l12 t1 + γ ℏ l22 t2) a1 l1 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) a3 l2 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) p11 x1 y1 + p21 x3 y1 +
      e^(<<1>> + <<1>>) p12 x1 y2 + p22 x3 y2 - γ ℏ l2 x3 y1 γ21 - γ ℏ l2 x3 y2 γ22) ∈ + O[∈]^3], True}

With[{q⊕ = ŒQU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊕ // Rord_{x2,a2→3}, 3], HL[QU[q⊕] == QU[rhs]]}]

{ŒQU[{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
  1 + e^(-γ ℏ l12 t1 - γ ℏ l22 t2) (e^(γ ℏ l12 t1 + γ ℏ l22 t2) a1 l1 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) a3 l2 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) p11 x1 y1 + p21 x3 y1 +
      e^(<<1>> + <<1>>) p12 x1 y2 + p22 x3 y2 - γ ℏ l2 x3 y1 γ21 - γ ℏ l2 x3 y2 γ22) ∈ + O[∈]^3], True}

With[{q⊕ = ŒQU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊕ // Rord_{a2,y2→3}, 3], HL[QU[q⊕] == QU[rhs]]}]

{<<1>>, True}

Timing@With[{q⊕ = ŒQU[{x1, y1}1, {x2, a2, y2}2,
     ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     θ2 + ∈ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
   {Short[rhs = q⊕ // Rord_{x1,y1→3}, 5]}]

{116.156, {ŒQU[{y3, a3, x3}1, <<1>>2, <<1>> / (1 - <<1>> + <<1>>),
   ((ℏ a2 l2 + p11 - p11 T1 + ℏ p22 x2 y2 + ℏ p12 x3 y2 + <<46>> + 2 ℏ p12 T1 x2 y2 γ11 γ21 -
        ℏ p12 T1^2 x2 y2 γ11 γ21 + ℏ p11 x2 y2 γ12 γ21 - 2 ℏ p11 T1 x2 y2 γ12 γ21 + ℏ p11 T1^2 x2 y2 γ12 γ21) ∈) /
    (ℏ - 3 ℏ γ11 + 3 ℏ T1 γ11 + 3 ℏ γ11^2 - 6 ℏ T1 γ11^2 + 3 ℏ T1^2 γ11^2 - ℏ γ11^3 + 3 ℏ T1 γ11^3 - 3 ℏ T1^2 γ11^3 + ℏ T1^3 γ11^3) +
   ((8 a3 p11 T1 + <<1>> + <<2726>> + 3 γ <<6>> γ21^3) <<1>>) /
    (4 - 28 γ11 + <<48>> + 4 T1^7 γ11^7) + O[∈]^3]}}
```

```
Timing@With[{q⊙ = ℭQU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊙ // Ord_{x1,y1→3}, 5], HL@SimpT[QU[q⊙] == QU[rhs]]}]
```

{388.922, {ℭQU[{y3, a3, x3}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 − γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 − 4 p11 T1 + 4 ℏ p22 x2 y2 + <<339>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 − 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) / (4 ℏ − 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 − 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 − 40 ℏ γ11^3 + <<13>> + 20 ℏ T1 γ11^5 − 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 − 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) + ((576 a3 p11 T1 + <<8073>> + <<1>>) <<1>>) / (<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}

```
Timing@With[{q⊙ = ℭQU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊙ // Ord_{x1,y1→1}, 5], HL@SimpT[QU[q⊙] == QU[rhs]]}]
```

{336.781, {ℭQU[{y1, a1, x1}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 − γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 − 4 p11 T1 + 4 ℏ p11 x1 y1 + 4 ℏ p21 x2 y1 + <<338>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 − 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) / (4 ℏ − 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 − 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 − 40 ℏ γ11^3 + <<10>> + 20 ℏ T1^4 γ11^4 − 4 ℏ γ11^5 + 20 ℏ T1 γ11^5 − 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 − 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) + ((576 a1 p11 T1 + <<8073>> + <<1>>) <<1>>) / (<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}

Canonical ordering with Cord

```
Cord@ℭCU[{x1, y1}1, 0, 01 + x1 y1]
```

ℭCU[{y1, a1, x1}1, 0, (−t1 + x1 y1) + 2 a1 ϵ + O[ϵ]^2]

```
Block[{$p = 4, $k = 0, c⊙ = ℭCU[{y1, a1, x1, x2, a2, y2}1,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     10 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c⊙], 8], HL@Simp[CU[c⊙] − CU[Cord[c⊙]]]}]
```

{4.53125, {ℭCU[{y1, a1, x1}1, (e^{γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2} ℏ a1 l11 t1 + e^{γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2} ℏ a1 l12 t1 + <<12>> + e^{γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2} ℏ^2 a1 l22 t1 t2 γ22 + ℏ x1 y1 γ22) / (e^{γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2} + e^{γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2} ℏ t1 γ12 + e^{γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2} ℏ t1 γ22), e^{γ ℏ l12 t1+γ ℏ l22 t2} / (e^{γ ℏ l12 t1+γ ℏ l22 t2} + ℏ t1 γ12 + ℏ t1 γ22) + O[ϵ]^1], 0}}

```
Block[{$p = 4, $k = 1, cω = ŒCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    11 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[cω], 8], HL@Simp[CU[cω] - CU[Cord[cω]]]}]
```

$$\Big\{81.2656,\ \Big\{\text{Œ}_{CU}\big[\{y_1,a_1,x_1\}_1,\ \big(e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}\hbar a_1 l_{11}t_1+\ll 14\gg+\hbar x_1 y_1\gamma_{22}\big)\Big/$$
$$\big(e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}+e^{\gamma\hbar l_{11}t_1+\ll 2\gg+\gamma\hbar\ll 1\gg t_2}\hbar t_1\gamma_{12}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar t_1\gamma_{22}\big),$$
$$\frac{e^{\gamma\hbar l_{12}t_1+\gamma\hbar l_{22}t_2}}{e^{\gamma\hbar l_{12}t_1+\gamma\hbar l_{22}t_2}+\hbar t_1\gamma_{12}+\hbar t_1\gamma_{22}}+\Big(\big(2e^{2\gamma\hbar l_{11}t_1+6\gamma\hbar l_{12}t_1+2\gamma\hbar l_{21}t_2+6\gamma\hbar l_{22}t_2}a_1 l_1+\ll 419\gg\big)\epsilon\Big)\Big/$$
$$\big(2e^{2\gamma\hbar l_{11}t_1+6\gamma\hbar l_{12}t_1+2\gamma\hbar l_{21}t_2+6\gamma\hbar l_{22}t_2}+10e^{2\gamma\hbar l_{11}t_1+5\gamma\hbar l_{12}t_1+2\gamma\hbar l_{21}t_2+5\gamma\hbar l_{22}t_2}\hbar t_1\gamma_{12}+\ll 18\gg+2e^{2\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+2\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar^5 t_1^5\gamma_{22}^5\big)+O[\epsilon]^2\big],\ 0\Big\}\Big\}$$

```
With[{qω = ŒQU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    10 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Cord[
  qω]]
```

$$\text{Œ}_{QU}\big[\{y_1,a_1,x_1\}_1,$$
$$\big(e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}\hbar a_1 l_{11}t_1+e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}\hbar a_1 l_{12}t_1+$$
$$e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}\hbar a_1 l_{21}t_2+e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}\hbar a_1 l_{22}t_2+$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar x_1y_1\gamma_{11}-e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{11}t_1\gamma_{12}-$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{12}t_1\gamma_{12}-e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{21}t_2\gamma_{12}-$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{22}t_2\gamma_{12}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{11}t_1T_1\gamma_{12}+$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{12}t_1T_1\gamma_{12}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{21}t_2T_1\gamma_{12}+$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{22}t_2T_1\gamma_{12}+\hbar x_1y_1\gamma_{12}+$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar x_1y_1\gamma_{21}-e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{11}t_1\gamma_{22}-$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{12}t_1\gamma_{22}-e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{21}t_2\gamma_{22}-$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{22}t_2\gamma_{22}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{11}t_1T_1\gamma_{22}+$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{12}t_1T_1\gamma_{22}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{21}t_2T_1\gamma_{22}+$$
$$e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\hbar a_1 l_{22}t_2T_1\gamma_{22}+\hbar x_1y_1\gamma_{22}\big)\Big/$$
$$\big(e^{\gamma\hbar l_{11}t_1+2\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+2\gamma\hbar l_{22}t_2}-e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\gamma_{12}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}T_1\gamma_{12}-e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}\gamma_{22}+e^{\gamma\hbar l_{11}t_1+\gamma\hbar l_{12}t_1+\gamma\hbar l_{21}t_2+\gamma\hbar l_{22}t_2}T_1\gamma_{22}\big),$$
$$\frac{e^{\gamma\hbar l_{12}t_1+\gamma\hbar l_{22}t_2}}{e^{\gamma\hbar l_{12}t_1+\gamma\hbar l_{22}t_2}-\gamma_{12}+T_1\gamma_{12}-\gamma_{22}+T_1\gamma_{22}}+O[\epsilon]^1\big]$$

Stitching Œ's.

```
cω = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 12];
{cω // m3→4, HL@Simp[CU[m3→4[cω]] - m3→4[CU[cω]]]}
```

$$\big\{\text{Œ}_{CU}\big[\{y_1,a_1,x_1\}_1,\{y_2,a_2,x_2\}_2,\{y_4,a_4,x_4\}_4,$$
$$\hbar\,(a_1l_{11}t_1+a_2l_{12}t_1+a_4l_{13}t_1+a_1l_{21}t_2+a_2l_{22}t_2+a_4l_{23}t_2+a_1l_{31}t_4+a_2l_{32}t_4+a_4l_{33}t_4+x_1y_1\gamma_{11}+x_2y_1\gamma_{12}+x_4y_1\gamma_{13}+x_1y_2\gamma_{21}+$$
$$x_2y_2\gamma_{22}+x_4y_2\gamma_{23}+x_1y_4\gamma_{31}+x_2y_4\gamma_{32}+x_4y_4\gamma_{33}),\ 1+O[\epsilon]^3\big],\ 0\big\}$$

Verifying that *m* commutes with evaluation, in CU:

```
co = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{co // m2→3, HL@Simp[CU[m2→3[co]] - m2→3[CU[co]]]}
```

```
{513.453, {ŒCU[{y1, a1, x1}1, {y3, a3, x3}3, (⋯1⋯)/(e^(⋯1⋯) ⋯1⋯ ⋯1⋯),
  1/(1+ℏ t3 γ32) + (((4 e^(2 γ ℏ l12 t1+⋯4⋯+2 γ ℏ l33 t3) ℏ^2 a3 x1 y1 γ12 γ31 - 2 ⋯7⋯ γ31 + ⋯154⋯) ∈) /
   (2 e^(2 γ ℏ l12 t1+2 γ ℏ l13 t1+⋯3⋯+2 γ ℏ l33 t3) + 10 e^(⋯1⋯) ℏ t3 γ32 + ⋯2⋯ +
    ⋯1⋯ + 2 e^(2 γ ℏ l12 t1+⋯4⋯+⋯1⋯) ℏ^5 t3^5 γ32^5) + ((⋯1⋯) ∈^2)/(⋯1⋯) + O[∈]^3], 0}}
```

large output | show less | show more | show all | set size limit...

Verifying that *m* commutes with evaluation, in QU:

```
qo = ŒQU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 12];
Timing@{qo // m2→3, HL@SimpT[QU[m2→3[qo]] - m2→3[QU[qo]]]}
```

```
{7831.47, {ŒQU[{y1, a1, x1}1, {y3, a3, x3}3, (⋯1⋯)/(⋯1⋯), 1/(1-γ32+T3 γ32) +
  (((8 e^(2 γ ℏ l12 t1+⋯4⋯+2 γ ℏ l33 t3) ℏ^2 a3 T3 x1 y1 γ12 γ31 + 4 ⋯8⋯ γ31 + ⋯371⋯) ∈) /
   (4 e^(2 γ ℏ l12 t1+2 γ ℏ l13 t1+⋯3⋯+2 γ ℏ l33 t3) - 20 e^(⋯1⋯) γ32 + ⋯26⋯ +
    4 e^(2 γ ℏ l12 t1+⋯4⋯+⋯1⋯) T3^5 γ32^5) + ((⋯1⋯) ∈^2)/(⋯79⋯+⋯1⋯) + O[∈]^3], 0}}
```

large output | show less | show more | show all | set size limit...

Verifying meta-associativity in CU:

```
co = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 10];
Timing@HL[(lhs = co // m1,2→1 // m1,3→1) ≡ (rhs = co // m2,3→2 // m1,2→1)]
```

```
{41.9219, True}
```

```
co = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 11];
Timing@HL[(lhs = co // m1,2→1 // m1,3→1) ≡ (rhs = co // m2,3→2 // m1,2→1)]
```

```
{30119.8, True}
```

```
cⓞ = ŒCU[{y1, a1, x1}1, {y2, a2, x2}2, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 2}, {j, 2}], 11];
Short[Simplify /@ (cexample = cⓞ // m1→2), 12]
Short[Simplify /@ (qexample = (qⓞ = cⓞ /. CU → QU) // m1→2), 12]
```

$$\mathbb{E}_{CU}\Big[\{y_2,a_2,x_2\}_2,\ \hbar a_2 (l_{11}+l_{12}+l_{21}+l_{22})\, t_2 + \frac{1}{1+\hbar t_2\gamma_{21}}\, e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar x_2 y_2 \Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+\hbar t_2\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar t_2\gamma_{22}\big)\Big),$$

$$\frac{1}{1+\hbar t_2\gamma_{21}} + \frac{1}{2(1+\hbar t_2\gamma_{21})^5}\, e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar\Big(4a_2(1+\hbar t_2\gamma_{21})^2 \big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2 + x_2y_2)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22})\big)\big) - \gamma\hbar\Big(-2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2\gamma_{21}^2(1+\hbar t_2\gamma_{21})^2 + 4 \ll 5\gg (\ll 1 \gg) + \hbar\ll 4\gg \big(3\hbar t_2\gamma_{21}^2 + 2e^{\gamma\hbar(l_{12}+l_{\ll 2\gg})t_2}\gamma_{22} + \gamma_{21}(4+e^{\gamma\ll 3\gg}\hbar t_2\gamma_{22}) + e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}(2+\hbar t_2(\gamma_{21}-e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}))\big)\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

$$\mathbb{E}_{QU}\Big[\{y_2,a_2,x_2\}_2,\ \hbar a_2 (l_{11}+l_{12}+l_{21}+l_{22})\, t_2 + \frac{1}{1+(-1+T_2)\gamma_{21}}\, e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar x_2 y_2 \Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+(-1+T_2)\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-1+T_2)\gamma_{22}\big)\Big),$$

$$\frac{1}{1+(-1+T_2)\gamma_{21}} + \frac{1}{4(1+(-1+T_2)\gamma_{21})^5}\, e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar\Big(8a_2T_2(1+(-1+T_2)\gamma_{21})^2\big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}T_2 + \hbar x_2y_2)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2y_2(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22})\big)\big) + \gamma\Big(2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(1-4T_2+3T_2^2)\gamma_{21}^2(1+(-1+T_2)\gamma_{21})^2 + 4e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2y_2\gamma_{21}(1+(-1+T_2)\gamma_{21})(\ll 1\gg) - \ll 1\gg\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

R in QU.

```
Table[Series[eqℏ,k[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$$e^x$$

$$e^x - \frac{1}{4}(e^x x^2\gamma\hbar)\epsilon + \frac{1}{32}e^x x^4\gamma^2\hbar^2\epsilon^2 - \frac{1}{384}(e^x x^2(-8+x^4)\gamma^3\hbar^3)\epsilon^3 + \frac{e^x x^4(-32+x^4)\gamma^4\hbar^4\epsilon^4}{6144} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}(e^x x^2\gamma\hbar)\epsilon + \frac{1}{288}e^x x^3(32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{(e^x x^2(-24+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{e^x x^3(-4608-864x+1024x^3+576x^4+27x^5)\gamma^4\hbar^4\epsilon^4}{165\,888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}(e^x x^2\gamma\hbar)\epsilon + \frac{1}{288}e^x x^3(32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{(e^x x^2(-24+72x^2+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{e^x x^3(-4608-864x+3616x^3+576x^4+27x^5)\gamma^4\hbar^4\epsilon^4}{165\,888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}(e^x x^2\gamma\hbar)\epsilon + \frac{1}{288}e^x x^3(32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{(e^x x^2(-24+72x^2+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{1}{4\,147\,200}e^x x^3(-115\,200-21\,600x+165\,888x^2+90\,400x^3+14\,400x^4+675x^5)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$$

$$e^x - \frac{1}{4}(e^x x^2\gamma\hbar)\epsilon + \frac{1}{288}e^x x^3(32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{(e^x x^2(-24+72x^2+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{1}{4\,147\,200}e^x x^3(-115\,200-21\,600x+165\,888x^2+90\,400x^3+14\,400x^4+675x^5)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$$

```
Table[Together@SeriesCoefficient[𝕖q,5[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, 1\Big/\left((1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[𝕖q,5[x], {x, 0, n}]], {n, 0, 5}]
```

$\{1, 1, 1, 1, 1, 1\}$

```
QU[R3,4] // Short
```

$$QU[\,] + \frac{\epsilon\hbar\, QU[a_3, a_4]}{\gamma} + \hbar\, QU[y_3, x_4] + \frac{\epsilon \ll 1\gg \ll 1\gg}{\gamma} + \frac{1}{2}\ll 1\gg \ll 1\gg - \frac{\ll 1\gg}{\gamma} - \frac{\epsilon\hbar^2 \ll 1\gg t_3}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4]\, t_3^2}{2\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R1,2 ** R^-1 1,2] // Simp // HL // Timing
```

$\{0.078125, QU[\,]\}$

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$$\left\{0.203125, \left\{QU[\,] + \frac{\epsilon\hbar\, QU[a_1, a_2]}{\gamma} + \frac{\epsilon\hbar\, QU[a_1, a_3]}{\gamma} + \ll 73\gg + 2\epsilon\hbar^2 QU[y_1, a_2, x_3]\, T_2 + QU[y_1, x_3]\,(\hbar - \hbar T_2), 0\right\}\right\}$$

R in Œ$_{QU}$.

```
{ŒQU,1[R1,2], ŒQU,2[R1,2]}
```

$$\left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2\right)\epsilon + O[\epsilon]^2\right],\right.$$
$$\left. Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2\right)\epsilon + \frac{1}{288\gamma^2}\left(144\hbar^2 a_1^2 a_2^2 - 72\gamma^2\hbar^4 a_1 a_2 x_2^2 y_1^2 + 32\gamma^4\hbar^5 x_2^3 y_1^3 + 9\gamma^4\hbar^6 x_2^4 y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$$

The morphism Œ$_{U,k}$.

```
ŒQU,1[R1,2 R3,4 R5,6]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar a_2 t_1}{\gamma} - \frac{\hbar a_4 t_3}{\gamma} - \frac{\hbar a_6 t_5}{\gamma} + \hbar x_2 y_1 + \hbar x_4 y_3 + \hbar x_6 y_5,\right.$$
$$\left. 1 + \left(\frac{\hbar a_1 a_2}{\gamma} + \frac{\hbar a_3 a_4}{\gamma} + \frac{\hbar a_5 a_6}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2 - \frac{1}{4}\gamma\hbar^3 x_4^2 y_3^2 - \frac{1}{4}\gamma\hbar^3 x_6^2 y_5^2\right)\epsilon + O[\epsilon]^2\right]$$

$\text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{1,3\to 1}\ //\ m_{2,5\to 2}\ //\ m_{4,6\to 4}]$

$$\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\Big(-\hbar\, a_2\, t_1 - \hbar\, a_4\, t_1 - \hbar\, a_4\, t_2 + \gamma\,\hbar\, x_2\, y_1 + \gamma\,\hbar\, x_4\, y_1 + e^{\hbar\, t_2}\gamma\,\hbar\, x_4\, y_1 - \gamma\,\hbar\, T_2\, x_4\, y_1 + e^{\hbar\, t_1}\gamma\,\hbar\, x_4\, y_2\Big),$$
$$1 + \frac{1}{4\gamma}\Big(4\hbar\, a_1\, a_2 + 4\hbar\, a_1\, a_4 + 4\hbar\, a_2\, a_4 - 4\gamma\,\hbar^2\, a_4\, x_2\, y_1 - 8\, e^{\hbar\, t_2}\gamma\,\hbar^2\, a_2\, x_4\, y_1 + 8\gamma\,\hbar^2\, a_2\, T_2\, x_4\, y_1 - \gamma^2\hbar^3 x_2^2\, y_1^2 + 4\, e^{\hbar\, t_2}\gamma^2\hbar^3 x_2\, x_4\, y_1^2 - 4\gamma^2\hbar^3\, T_2\, x_2\, x_4\, y_1^2 - \gamma^2\hbar^3 x_4^2\, y_1^2 - e^{2\hbar\, t_2}\gamma^2\hbar^3 x_4^2\, y_1^2 + \gamma^2\hbar^3\, T_2^2\, x_4^2\, y_1^2 - 4\, e^{\hbar\, t_1}\gamma\,\hbar^2\, a_1\, x_4\, y_2 + 4\, e^{\hbar\, t_1}\gamma^2\hbar^3\, x_2\, x_4\, y_1\, y_2 + 4\, e^{\hbar\, t_1+\hbar\, t_2}\gamma^2\hbar^3\, x_4^2\, y_1\, y_2 - 4\, e^{\hbar\, t_1}\gamma^2\hbar^3\, T_2\, x_4^2\, y_1\, y_2 - e^{2\hbar\, t_1}\gamma^2\hbar^3\, x_4^2\, y_2^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$\text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{3,1\to 1}\ //\ m_{5,2\to 2}\ //\ m_{6,4\to 4}]$

$$\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\Big(-\hbar\, a_2\, t_1 - \hbar\, a_4\, t_1 - \hbar\, a_4\, t_2 + \gamma\,\hbar\, x_2\, y_1 + \gamma\,\hbar\, x_4\, y_1 + e^{\hbar\, t_1}\gamma\,\hbar\, x_4\, y_2\Big),$$
$$1 + \frac{1}{4\gamma}\Big(4\hbar\, a_1\, a_2 + 4\hbar\, a_1\, a_4 + 4\hbar\, a_2\, a_4 - 4\gamma\,\hbar^2\, a_4\, x_2\, y_1 - \gamma^2\hbar^3 x_2^2\, y_1^2 - \gamma^2\hbar^3 x_4^2\, y_1^2 - 4\, e^{\hbar\, t_1}\gamma\,\hbar^2\, a_1\, x_4\, y_2 + 4\, e^{\hbar\, t_1}\gamma^2\hbar^3\, x_2\, x_4\, y_1\, y_2 - e^{2\hbar\, t_1}\gamma^2\hbar^3\, x_4^2\, y_2^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$\text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{1,3\to 1}\ //\ m_{2,5\to 2}\ //\ m_{4,6\to 4}] \equiv \text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{3,1\to 1}\ //\ m_{5,2\to 2}\ //\ m_{6,4\to 4}]$

$$\hbar\left(e^{\hbar\, t_2} - T_2\right) x_4\, y_1 == 0\ \&\&\ \epsilon\,\hbar\left(e^{\hbar\, t_2} - T_2\right) x_4\, y_1 \left(8\, a_2 + \gamma\,\hbar\left(-4\, x_2\, y_1 + x_4\left(\left(e^{\hbar\, t_2} + T_2\right) y_1 - 4\, e^{\hbar\, t_1}\, y_2\right)\right)\right) == 0$$

Exponentials as needed.

In[∘]:=
```
Block[{$p = 2, $k = 2}, TableForm[StringSplit[
   "y|a|x|Cθ@y_CU|Cθ@a_CU|Cθ@x_CU|Qθ@y_QU|Qθ@a_QU|Qθ@x_QU|A𝔻@y_QU|A𝔻@a_QU|A𝔻@x_QU|S𝔻@y_QU|S𝔻@a_QU|S𝔻
     @x_QU|S@y_CU|S@a_CU|S@x_CU|S@y_QU|S@a_QU|S@x_QU|Δ@y_CU|Δ@a_CU|Δ@x_CU|Δ@y_QU|Δ@a_QU|Δ@x_QU",
   "|"] /. s_String ↦
   {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[∘]//TableForm=

y	y
a	a
x	x
$C\theta@y_{CU}$	$-x$
$C\theta@a_{CU}$	$-a$
$C\theta@x_{CU}$	$-y$
$Q\theta@y_{QU}$	$-\frac{x}{\sqrt{T}}-\frac{a\,x\,\epsilon\,\hbar}{\sqrt{T}}-\frac{a^2\,x\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$Q\theta@a_{QU}$	$-a$
$Q\theta@x_{QU}$	$-\frac{y}{\sqrt{T}}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{\sqrt{T}}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$A\mathbb{D}@y_{QU}$	$\frac{2}{3}a^2\,y\,\epsilon^2\,\hbar^2+\frac{1}{6}y\left(6+3\,t\,\hbar+t^2\,\hbar^2\right)+\frac{1}{12}y\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-4\,a\left(3+2\,t\,\hbar\right)\right)$
$A\mathbb{D}@a_{QU}$	a
$A\mathbb{D}@x_{QU}$	x
$S\mathbb{D}@y_{QU}$	$y+\frac{1}{48}t^2\,y\,\hbar^2+\frac{1}{24}y\left(-2\,a\,t+x\,y\,\gamma\right)\epsilon\,\hbar^2+\frac{1}{12}a^2\,y\,\epsilon^2\,\hbar^2$
$S\mathbb{D}@a_{QU}$	a
$S\mathbb{D}@x_{QU}$	$\frac{7}{12}a^2\,x\,\epsilon^2\,\hbar^2+x\left(1+\frac{t\,\hbar}{2}+\frac{7\,t^2\,\hbar^2}{48}\right)+\frac{1}{24}x\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-2\,a\left(12+7\,t\,\hbar\right)\right)$
$S@y_{CU}$	$-y$
$S@a_{CU}$	$-a$
$S@x_{CU}$	$-x$
$S@y_{QU}$	$-\frac{y}{T}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{T}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\,T}$
$S@a_{QU}$	$-a$
$S@x_{QU}$	$-x-a\,x\,\epsilon\,\hbar-\frac{1}{2}a^2\,x\,\epsilon^2\,\hbar^2$
$\Delta@y_{CU}$	y_1+y_2
$\Delta@a_{CU}$	a_1+a_2
$\Delta@x_{CU}$	x_1+x_2
$\Delta@y_{QU}$	$y_1+T_1\,y_2-\epsilon\,\hbar\,a_1\,T_1\,y_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,T_1\,y_2$
$\Delta@a_{QU}$	a_1+a_2
$\Delta@x_{QU}$	$x_1+x_2-\epsilon\,\hbar\,a_1\,x_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_2$

In[]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S1[QU[y1]] /. QU → Times,
   exps = Exp_{QU1,$k}[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S1@𝕆_QU[{y1}1, SS[e^{ħ η y1}]] - QU@(exps /. η → ħ η)]
  }]
```

Out[]= $\left\{35.8281, \left\{a_1 \left(-\frac{\epsilon \hbar}{T_1}+\frac{\gamma \epsilon^2 \hbar^2}{T_1}\right) y_1+\left(-\frac{1}{T_1}+\frac{\gamma \epsilon \hbar}{T_1}-\frac{\gamma^2 \epsilon^2 \hbar^2}{2 T_1}\right) y_1-\frac{\epsilon^2 \hbar^2 a_1^2 y_1}{2 T_1},\right.\right.$

$\mathbb{E}_{QU}\left[\{y_1, a_1, x_1\}_1, -\frac{\eta y_1}{T_1}, 1+\frac{\left(2 \gamma \eta \hbar T_1 y_1-2 \eta \hbar a_1 T_1 y_1-\gamma \eta^2 \hbar y_1^2\right) \epsilon}{2 T_1^2}+\right.$

$\left(-\frac{\gamma^2 \eta \hbar^2 y_1}{2 T_1}+\frac{\gamma \eta \hbar^2 a_1 y_1}{T_1}-\frac{\eta \hbar^2 a_1^2 y_1}{2 T_1}+\frac{7 \gamma^2 \eta^2 \hbar^2 y_1^2}{4 T_1^2}-\frac{2 \gamma \eta^2 \hbar^2 a_1 y_1^2}{T_1^2}+\right.$

$\left.\left.\left.\frac{\eta^2 \hbar^2 a_1^2 y_1^2}{2 T_1^2}-\frac{\gamma^2 \eta^3 \hbar^2 y_1^3}{T_1^3}+\frac{\gamma \eta^3 \hbar^2 a_1 y_1^3}{2 T_1^3}+\frac{\gamma^2 \eta^4 \hbar^2 y_1^4}{8 T_1^4}\right) \epsilon^2+O[\epsilon]^3\right], 0\right\}\right\}$

In[]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S1[QU[a1]] /. QU → Times,
   exps = Exp_{QU1,$k}[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S1@𝕆_QU[{a1}1, SS[e^{ħ α a1}]] - QU@(exps /. α → ħ α)]
  }]
```

Out[]= $\left\{33.5938, \left\{-a_1, \mathbb{E}_{QU}\left[\{y_1, a_1, x_1\}_1, -\alpha a_1, 1+O[\epsilon]^3\right], 0\right\}\right\}$

In[]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S1[QU[x1]] /. QU → Times,
   exps = Exp_{QU1,$k}[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S1@𝕆_QU[{x1}1, SS[e^{ħ ξ x1}]] - QU@(exps /. ξ → ħ ξ)]
  }]
```

Out[]= $\left\{34.0625, \left\{-x_1-\epsilon \hbar a_1 x_1-\frac{1}{2} \epsilon^2 \hbar^2 a_1^2 x_1,\right.\right.$

$\mathbb{E}_{QU}\left[\{y_1, a_1, x_1\}_1, -\xi x_1, 1+\left(-\xi \hbar a_1 x_1-\frac{1}{2} \gamma \xi^2 \hbar x_1^2\right) \epsilon+\left(-\frac{1}{2} \xi \hbar^2 a_1^2 x_1+\frac{1}{4} \gamma^2 \xi^2 \hbar^2 x_1^2-\right.\right.$

$\left.\left.\left.\gamma \xi^2 \hbar^2 a_1 x_1^2+\frac{1}{2} \xi^2 \hbar^2 a_1^2 x_1^2-\frac{1}{2} \gamma^2 \xi^3 \hbar^2 x_1^3+\frac{1}{2} \gamma \xi^3 \hbar^2 a_1 x_1^3+\frac{1}{8} \gamma^2 \xi^4 \hbar^2 x_1^4\right) \epsilon^2+O[\epsilon]^3\right], 0\right\}\right\}$

$S(\mathbb{e}^{\eta y}\,\mathbb{e}^{\alpha a}\,\mathbb{e}^{\xi x})$

In[∘]:=
```
Timing@Block[{$p = 3, $k = 1}, {
  (sexp = m_{3,2,1→1}[Exp_{QU_1,$k}[η, S_1[QU[y_1]] /. QU → Times] Exp_{QU_2,$k}[α, S_2[QU[a_2]] /. QU → Times]
      Exp_{QU_3,$k}[ξ, S_3[QU[x_3]] /. QU → Times]]) /. u_1 :> u,
  HL@SimpT[QU@(sexp /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ}) -
    S_1@𝕆_QU[{y_1, a_1, x_1}_1, SS[e^(ℏ (η y_1+α a_1+ξ x_1))]]]
 }]
```

Out[∘]=

$$\left\{15.2969,\left\{\mathbb{E}_{QU}\left[\{y_1,a_1,x_1\},\frac{1}{T\hbar}\left(e^{\alpha\gamma}\eta\xi-e^{\alpha\gamma}T\eta\xi-aT\alpha\hbar-e^{\alpha\gamma}y\eta\hbar-e^{\alpha\gamma}Tx\xi\hbar\right),\right.\right.\right.$$
$$1+\frac{1}{4T^2\hbar}\left(-3e^{2\alpha\gamma}\gamma\eta^2\xi^2+4e^{2\alpha\gamma}T\gamma\eta^2\xi^2-e^{2\alpha\gamma}T^2\gamma\eta^2\xi^2+8ae^{\alpha\gamma}T\eta\xi\hbar-4e^{\alpha\gamma}T\gamma\eta\xi\hbar+\right.$$
$$4e^{\alpha\gamma}T^2\gamma\eta\xi\hbar+6e^{2\alpha\gamma}y\gamma\eta^2\xi\hbar-2e^{2\alpha\gamma}Ty\gamma\eta^2\xi\hbar+6e^{2\alpha\gamma}Tx\gamma\eta\xi^2\hbar-$$
$$2e^{2\alpha\gamma}T^2x\gamma\eta\xi^2\hbar-4ae^{\alpha\gamma}Ty\eta\hbar^2+4e^{\alpha\gamma}Ty\gamma\eta\hbar^2-2e^{2\alpha\gamma}y^2\gamma\eta^2\hbar^2-$$
$$\left.\left.\left.4ae^{\alpha\gamma}T^2x\xi\hbar^2-4e^{2\alpha\gamma}Txy\gamma\eta\xi\hbar^2-2e^{2\alpha\gamma}T^2x^2\gamma\xi^2\hbar^2\right)\epsilon+O[\epsilon]^2\right],0\right\}\right\}$$

$\Delta_{1\to1,2}(\mathbb{e}^{\eta y_1}\,\mathbb{e}^{\alpha a_1}\,\mathbb{e}^{\xi x_1})$

```
Timing@Block[{$p = 4, $k = 2}, {
  sexp = m_{1,3,5→1}@m_{2,4,6→2}@Times[ (* Warning: wrong unless $p≥$k+1! *)
      Prepend[{y_2}_2]@Exp_{QU_1,$k}[η, Δ_{1→1,2}[QU[y_1]] /. QU → Times],
      Prepend[{a_4}_4]@Exp_{QU_3,$k}[α, Δ_{3→3,4}[QU[a_3]] /. QU → Times],
      Prepend[{x_6}_6]@Exp_{QU_5,$k}[ξ, Δ_{5→5,6}[QU[x_5]] /. QU → Times]
    ] /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ},
  HL@SimpT[QU@sexp - Δ_{1→1,2}@𝕆_QU[{y_1, a_1, x_1}_1, SS[e^(ℏ (η y_1+α a_1+ξ x_1))]]]
 }]
```

Out[∘]=

$$\left\{162.,\left\{\mathbb{E}_{QU}\left[\{y_2,a_2,x_2\}_2,\{y_1,a_1,x_1\}_1,\alpha\hbar a_1+\alpha\hbar a_2+\xi\hbar x_1+\xi\hbar x_2+\eta\hbar y_1+\eta\hbar T_1 y_2,\right.\right.\right.$$
$$1+\frac{1}{2}\left(-2\xi\hbar^2a_1x_2+\gamma\xi^2\hbar^3x_1x_2-2\eta\hbar^2a_1T_1y_2+\gamma\eta^2\hbar^3T_1y_1y_2\right)\epsilon+$$
$$\frac{1}{24}\left(12\xi\hbar^3a_1^2x_2+6\gamma^2\xi^2\hbar^4x_1x_2-12\gamma\xi^2\hbar^4a_1x_1x_2+4\gamma^2\xi^3\hbar^5x_1^2x_2+12\xi^2\hbar^4a_1^2x_2^2+\right.$$
$$4\gamma^2\xi^3\hbar^5x_1x_2^2-12\gamma\xi^3\hbar^5a_1x_1x_2^2+3\gamma^2\xi^4\hbar^6x_1^2x_2^2+12\eta\hbar^3a_1^2T_1y_2+$$
$$24\eta\xi\hbar^4a_1^2T_1x_2y_2-12\gamma\eta\xi^2\hbar^5a_1T_1x_1x_2y_2+6\gamma^2\eta^2\hbar^4T_1y_1y_2-12\gamma\eta^2\hbar^4a_1T_1y_1y_2-$$
$$12\gamma\eta^2\xi\hbar^5a_1T_1x_2y_1y_2+6\gamma^2\eta^2\xi^2\hbar^6T_1x_1x_2y_1y_2+4\gamma^2\eta^3\hbar^5T_1y_1^2y_2+12\eta^2\hbar^4a_1^2T_1^2y_2^2+$$
$$\left.\left.\left.4\gamma^2\eta^3\hbar^5T_1^2y_1y_2^2-12\gamma\eta^3\hbar^5a_1T_1^2y_1y_2^2+3\gamma^2\eta^4\hbar^6T_1^2y_1^2y_2^2\right)\epsilon^2+O[\epsilon]^3\right],0\right\}\right\}$$

Zip and Bind

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = P\mathbb{e}^{L+Q}$. Such zips regard the L variables as scalars.

```
In[∘]:= Timing@{E0 = 𝔼[0, Sum[a_{10 i+j} x_i ξ_j, {i, 3}, {j, 3}],
            1 + ϵ Sum[f_i[x_1, x_2, x_3] ξ_i, {i, 3}] + ϵ Sum[f_{10 i+j}[x_1, x_2, x_3] ξ_i ξ_j, {i, 3}, {j, 3}]],
          lhs = QZip_{ξ_1,ξ_2}@E0,
          HL[lhs == QZip_{ξ_1}@QZip_{ξ_2}@E0]}

Out[∘]= {38.6875, {𝔼[0, ⋯1⋯, 1 + ϵ (ξ_1 ⋯1⋯ + ⋯1⋯ + ⋯1⋯) +
            ϵ (ξ_1^2 f_11[x_1, x_2, x_3] + ⋯7⋯ + ξ_3^2 f_33[x_1, x_2, x_3])], ⋯1⋯, True}}
        large output | show less | show more | show all | set size limit...

In[∘]:= Timing@{
          Eh = 𝔼[0, h Sum[a_{10 i+j} x_i ξ_j, {i, 3}, {j, 3}],
            1 + ϵ Sum[f_i[x_1, x_2, x_3] ξ_i, {i, 3}] + ϵ Sum[f_{10 i+j}[x_1, x_2, x_3] ξ_i ξ_j, {i, 3}, {j, 3}]],
          lhs = Normal[Eh /. 𝔼[L_, Q_, P_] :> Series[P e^(L+Q), {h, 0, 2}]] // Zip_{ξ_1},
          HL@Simplify[lhs == Normal[QZip_{ξ_1}[Eh] /. 𝔼[L_, Q_, P_] :> Series[P e^(L+Q), {h, 0, 2}]]]}

Out[∘]= {18.4375, {𝔼[0, h ⋯1⋯, 1 + ϵ (ξ_1 ⋯1⋯ + ⋯1⋯ + ⋯1⋯) +
            ϵ (ξ_1^2 f_11[x_1, x_2, x_3] + ⋯7⋯ + ξ_3^2 f_33[x_1, x_2, x_3])], ⋯1⋯, True}}
        large output | show less | show more | show all | set size limit...
```

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single"P". Here the z's are t and α and the ζ's are τ and a.

```
In[∘]:= Bind_{2}[𝔼[0, ξ (x_1 + x_2), 1], 𝔼[0, ξ_2 (x_2 + x_3), 1]]
Out[∘]= 𝔼[0, ξ (x_1 + x_2 + x_3), 1]

In[∘]:= Bind_{2}[𝔼[0, (ξ_2 + ξ_3) x_2, 1], 𝔼[0, (ξ_1 + ξ_2) x, 1]]
Out[∘]= 𝔼[0, x (ξ_1 + ξ_2 + ξ_3), 1]

In[∘]:= Bind_{1,2}[𝔼[0, (ξ_2 + ξ_3) x_2 + ξ_1 x_1, 1], 𝔼[0, (ξ_1 + ξ_2) x, 1]]
Out[∘]= 𝔼[0, x (ξ_1 + ξ_2 + ξ_3), 1]
```

An xay → axy → ayx –> yax ≡ xay → xya → yxa → yax test:

```
In[∘]:= Bind[𝔼[α_1 a_1 + τ_1 t_1, e^(γ α_1) ξ_1 x_1 + η_1 y_1, 1], {1}, 𝔼[τ_1 t_1 + α_1 a_1, ξ_1 x_1 + η_1 y_1 + ξ_1 η_1 t_1, 1]]
Out[∘]= 𝔼[a_1 α_1 + t_1 τ_1, y_1 η_1 + e^(γ α_1) (x_1 + t_1 η_1) ξ_1, 1]

In[∘]:= Column@{Cord[Œ_CU[{x_1, a_1}_1, ξ_1 x_1 + α_1 a_1, 1 + 0_0]],
          Cord[Œ_CU[{x_1, y_1}_1, ξ_1 x_1 + η_1 y_1, 1 + 0_0]],
          Cord[Œ_CU[{a_1, y_1}_1, α_1 a_1 + η_1 y_1, 1 + 0_0]]}

        Œ_CU[{a_1, x_1}_1, e^(-γ α_1) (e^(γ α_1) a_1 α_1 + x_1 ξ_1), 1 + O[ϵ]^1]
Out[∘]= Œ_CU[{y_1, a_1, x_1}_1, y_1 η_1 + x_1 ξ_1 - t_1 η_1 ξ_1, 1 + O[ϵ]^1]
        Œ_CU[{y_1, a_1}_1, e^(-γ α_1) (e^(γ α_1) a_1 α_1 + y_1 η_1), 1 + O[ϵ]^1]
```

In[∘]:=
```
{rxa = 𝔼[τ1 t1 + α1 a1, e^(-γ α1) ξ1 x1 + η1 y1, 1];
 rxy = 𝔼[τ1 t1 + α1 a1, ξ1 x1 + η1 y1 - ξ1 η1 t1, 1];
 ray = 𝔼[τ1 t1 + α1 a1, e^(-γ α1) η1 y1 + ξ1 x1, 1];
 lhs = Expand /@ Bind[rxa, {1}, rxy, {1}, ray],
 HL[lhs == Expand /@ Bind[ray, {1}, rxy, {1}, rxa] ]}
```

Out[∘]= $\{\mathbb{E}[a_1\,\alpha_1 + t_1\,\tau_1,\ e^{-\gamma\alpha_1}\,y_1\,\eta_1 + e^{-\gamma\alpha_1}\,x_1\,\xi_1 - e^{-\gamma\alpha_1}\,t_1\,\eta_1\,\xi_1,\ 1],\ \text{True}\}$

In[∘]:=
```
Simplify /@ m_{i,j→k} @ Œ_CU[{y_i, a_i, x_i}_i, {y_j, a_j, x_j}_j, η_i y_i + α_i a_i + ξ_i x_i + η_j y_j + α_j a_j + ξ_j x_j, 1 + 0_1]
```

Out[∘]= $Œ_{CU}\Big[\{y_k, a_k, x_k\}_k,\ a_k\,(\alpha_i+\alpha_j) + y_k\,\left(\eta_i + e^{-\gamma\alpha_i}\,\eta_j\right) + e^{-\gamma\alpha_j}\,x_k\,\xi_i - t_k\,\eta_j\,\xi_i + x_k\,\xi_j,$
$1 + \frac{1}{2}\,\eta_j\,\xi_i\,\left(4\,a_k - 2\,e^{-\gamma\alpha_i}\,\gamma\,y_k\,\eta_j + \gamma\,\left(-2\,e^{-\gamma\alpha_j}\,x_k + t_k\,\eta_j\right)\,\xi_i\right)\,\epsilon + O[\epsilon]^2\Big]$

Tensorial Representations

Associativity of tm.

In[∘]:=
```
Table[Block[{$U = U, $k = kk},
   {lhs = Bind[tm_{1,2→2}, {2}, tm_{2,3→1}];
    {$U, $k} -> HL[lhs ≡ Bind[tm_{2,3→2}, {2}, tm_{1,2→1}]]}
 ], {U, {CU, QU}}, {kk, 0, 1}]
```

Out[∘]= {{{{CU, 0} → True}, {{CU, 1} → True}}, {{{QU, 0} → True}, {{QU, 1} → True}}}

In[∘]:=
```
Block[{$U = CU, $k = 2}, Timing@{lhs = Bind[tm_{1,2→2}, {2}, tm_{2,3→1}];
   HL[lhs ≡ Bind[tm_{2,3→2}, {2}, tm_{1,2→1}]]}]
```

Out[∘]= {65.75, {True}}

tS is an anti-homomorphism for tm.

In[∘]:=
```
HL[(tS_1 tS_2) ~ B_{1,2} ~ tm_{1,2→1} ≡ tm_{2,1→1} ~ B_1 ~ tS_1]
```

Out[∘]= True

Testing co-associativity.

In[∘]:=
```
HL[tΔ_{1→1,2} ~ B_2 ~ tΔ_{2→2,3} ≡ tΔ_{1→1,3} ~ B_1 ~ tΔ_{1→1,2}]
```

Out[∘]= True

Testing S is an anti-co-homomorphism

In[∘]:=
```
HL[tS_1 ~ B_1 ~ tΔ_{1→1,2} ≡ tΔ_{1→2,1} ~ B_{1,2} ~ (tS_1 tS_2)]
```

Out[∘]= True

Testing convolution inverse:

In[∘]:=
```
{HL[tΔ_{1→1,2} ~ B_1 ~ tS_1 ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙],
 HL[tΔ_{1→1,2} ~ B_2 ~ tS_2 ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙]}
```

Out[∘]= {True, True}

Testing R2

In[∘]:= $\mathrm{HL}\left[\left(\overline{\mathrm{tR}}_{1,2}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to1}\ \mathrm{tm}_{2,4\to2}\right) == \mathrm{t}\mathbb{1}\right]$

Out[∘]= True

Testing quasi-triangular axioms

In[∘]:= $\mathrm{HL}\left[\left(\mathrm{t}\Delta_{1\text{->}1,2}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to1}\ \mathrm{tm}_{2,4\to2}\right) \equiv \left(\mathrm{t}\Delta_{1\to2,1}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{3,1\to1}\ \mathrm{tm}_{4,2\to2}\right)\right]$

Out[∘]= True

In[∘]:= $\mathrm{HL}\left[\mathrm{tR}_{1,3} \sim \mathrm{B}_{1} \sim \mathrm{t}\Delta_{1\to1,2} \equiv \left(\mathrm{tR}_{1,4}\ \mathrm{tR}_{2,3}\right) \sim \mathrm{B}_{3,4} \sim \mathrm{tm}_{3,4\to3}\right]$

Out[∘]= True

Testing R3

In[∘]:= $\mathrm{HL}\left[\left(\mathrm{tR}_{2,3}\ \mathrm{tR}_{1,4}\ \mathrm{tR}_{5,6}\right) \sim \mathrm{B}_{\mathrm{Range@6}} \sim \left(\mathrm{tm}_{1,5\to1}\ \mathrm{tm}_{2,6\to2}\ \mathrm{tm}_{3,4\to3}\right) \equiv \left(\mathrm{tR}_{1,2}\ \mathrm{tR}_{5,3}\ \mathrm{tR}_{6,4}\right) \sim \mathrm{B}_{\mathrm{Range@6}} \sim \left(\mathrm{tm}_{1,5\to1}\ \mathrm{tm}_{2,6\to2}\ \mathrm{tm}_{3,4\to3}\right)\right]$

Out[∘]= True

tC is the counterclockwise spinner; $\overline{\mathrm{tC}}$ is its inverse:

In[∘]:= $\mathrm{Block}\left[\{\$k = 1\},\ \mathrm{HL}\left[\left(\mathrm{tC}_{1}\ \overline{\mathrm{tC}}_{2}\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \equiv \mathrm{t}\mathbb{1}\right]\right]$

Out[∘]= True

Cyclic R2 as on 180419 blackboard:

In[∘]:= $\mathrm{Block}\left[\{\$k = 2\},\ \mathrm{HL}\left[\left(\mathrm{tR}_{1,4}\ \overline{\mathrm{tR}}_{5,2}\ \overline{\mathrm{tC}}_{3}\right) \sim \mathrm{B}_{\{1,3,2,4\}} \sim \left(\mathrm{tm}_{1,3\to1}\ \mathrm{tm}_{2,4\to2}\right) \sim \mathrm{B}_{1,5} \sim \mathrm{tm}_{1,5\to1} \equiv \overline{\mathrm{tC}}_{1}\right]\right]$ // Timing

Out[∘]= {6.60938, True}

Swirl relation as on 180419 blackboard:

In[∘]:= $\mathrm{Block}\left[\{\$k = 1\},\ \mathrm{HL}\left[\mathrm{tR}_{1,2} \equiv \left(\mathrm{tC}_{1}\ \mathrm{tC}_{2}\ \mathrm{tR}_{3,4}\ \overline{\mathrm{tC}}_{5}\ \overline{\mathrm{tC}}_{6}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to1}\ \mathrm{tm}_{2,4\to2}\right) \sim \mathrm{B}_{1,2,5,6} \sim \left(\mathrm{tm}_{1,5\to1}\ \mathrm{tm}_{2,6\to2}\right)\right]\right]$ // Timing

Out[∘]= {5.15625, True}

The Four Kinks

In[∘]:= Block[{$k = 1, K1, K2, K3, K4},
Column@{ $K1 = \left(\mathrm{tR}_{1,3}\ \overline{\mathrm{tC}}_{2}\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \sim \mathrm{B}_{1,3} \sim \mathrm{tm}_{1,3\to1}$, $K2 = \left(\mathrm{tR}_{3,1}\ \mathrm{tC}_{2}\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \sim \mathrm{B}_{1,3} \sim \mathrm{tm}_{1,3\to1}$,
$K3 = \left(\overline{\mathrm{tR}}_{3,1}\ \overline{\mathrm{tC}}_{2}\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \sim \mathrm{B}_{1,3} \sim \mathrm{tm}_{1,3\to1}$, $K4 = \left(\overline{\mathrm{tR}}_{1,3}\ \mathrm{tC}_{2}\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \sim \mathrm{B}_{1,3} \sim \mathrm{tm}_{1,3\to1}$,
HL /@ { $K1 \equiv K2$, $K3 \equiv K4$, $\left(K1\ \left(K3 \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to2}\right)\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \equiv \mathrm{t}\mathbb{1}$ }}]

Out[∘]=

$$\mathbb{E}\left[-\frac{\hbar\, a_1\, t_1}{\gamma},\ \hbar\, x_1\, y_1,\ \frac{1}{\sqrt{T_1}} + \frac{\left(4\,\gamma\,\hbar\, a_1 + 4\,\hbar\, a_1^2 - \gamma^2\,\hbar^3\, x_1^2\, y_1^2\right)\epsilon}{4\,\gamma\,\sqrt{T_1}} + O[\epsilon]^2\right]$$

$$\mathbb{E}\left[-\frac{\hbar\, a_1\, t_1}{\gamma},\ \hbar\, x_1\, y_1,\ \frac{1}{\sqrt{T_1}} + \frac{\left(4\,\gamma\,\hbar\, a_1 + 4\,\hbar\, a_1^2 - \gamma^2\,\hbar^3\, x_1^2\, y_1^2\right)\epsilon}{4\,\gamma\,\sqrt{T_1}} + O[\epsilon]^2\right]$$

$$\mathbb{E}\left[\frac{\hbar\, a_1\, t_1}{\gamma},\ -\frac{\hbar\, x_1\, y_1}{T_1},\ \sqrt{T_1} - \frac{\left(\hbar\,\left(4\, a_1^2\, T_1^2 + 3\,\gamma^2\,\hbar^2\, x_1^2\, y_1^2 + 4\,\gamma\, a_1\, T_1\,\left(T_1 + 2\,\hbar\, x_1\, y_1\right)\right)\right)\epsilon}{4\,\left(\gamma\, T_1^{3/2}\right)} + O[\epsilon]^2\right]$$

$$\mathbb{E}\left[\frac{\hbar\, a_1\, t_1}{\gamma},\ -\frac{\hbar\, x_1\, y_1}{T_1},\ \sqrt{T_1} - \frac{\left(\hbar\,\left(4\, a_1^2\, T_1^2 + 3\,\gamma^2\,\hbar^2\, x_1^2\, y_1^2 + 4\,\gamma\, a_1\, T_1\,\left(T_1 + 2\,\hbar\, x_1\, y_1\right)\right)\right)\epsilon}{4\,\left(\gamma\, T_1^{3/2}\right)} + O[\epsilon]^2\right]$$

{True, True, True}

Trefoil as on 180419 blackboard:

In[∘]:=
```
Kink[QU, kk_] := Kink[QU, kk] = Block[{$k = kk}, (tR_{1,3} t̄C_2) ~B_{1,2}~tm_{1,2→1}~B_{1,3}~tm_{1,3→1}];
tKink_i_ := Kink[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i};
K̄ink[QU, kk_] := K̄ink[QU, kk] = Block[{$k = kk}, (t̄R_{1,3} tC_2) ~B_{1,2}~tm_{1,2→1}~B_{1,3}~tm_{1,3→1}];
t̄Kink_i_ := K̄ink[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i}
```

In[∘]:=
```
{tKink_1, t̄Kink_1}
```

Out[∘]=
$$\left\{\mathbb{E}\left[-\frac{\hbar a_1 t_1}{\gamma}, \hbar x_1 y_1, \frac{1}{\sqrt{T_1}}+\frac{\left(4\gamma\hbar a_1+4\hbar a_1^2-\gamma^2\hbar^3 x_1^2 y_1^2\right)\epsilon}{4\gamma\sqrt{T_1}}+O[\epsilon]^2\right], \mathbb{E}\left[\frac{\hbar a_1 t_1}{\gamma}, -\frac{\hbar x_1 y_1}{T_1}, \sqrt{T_1}-\frac{1}{4\left(\gamma T_1^{3/2}\right)}\left(\hbar\left(4 a_1^2 T_1^2+3\gamma^2\hbar^2 x_1^2 y_1^2+4\gamma a_1 T_1\left(T_1+2\hbar x_1 y_1\right)\right)\right)\epsilon+O[\epsilon]^2\right]\right\}$$

In[∘]:=
```
Timing[
 Z = tR_{1,5} tR_{6,2} tR_{3,7} t̄C_4 t̄Kink_8 t̄Kink_9 t̄Kink_10;
 Do[Echo@Z; Z = Z~B_{1,k}~tm_{1,k→1}, {k, 2, 10}];
 Z]
```

»
$$\mathbb{E}\left[-\frac{\hbar a_5 t_1}{\gamma}-\frac{\hbar a_7 t_3}{\gamma}-\frac{\hbar a_2 t_6}{\gamma}+\frac{\hbar a_8 t_8}{\gamma}+\frac{\hbar a_9 t_9}{\gamma}+\frac{\hbar a_{10} t_{10}}{\gamma},\right.$$
$$\hbar x_5 y_1+\hbar x_7 y_3+\hbar x_2 y_6-\frac{\hbar x_8 y_8}{T_8}-\frac{\hbar x_9 y_9}{T_9}-\frac{\hbar x_{10} y_{10}}{T_{10}}, \frac{\sqrt{T_8}\sqrt{T_9}\sqrt{T_{10}}}{\sqrt{T_4}}+$$
$$\left(\sqrt{T_{10}}\left(\sqrt{T_9}\left(\sqrt{T_8}\left(\frac{\hbar a_4}{\sqrt{T_4}}+\frac{\frac{\hbar a_1 a_5}{\gamma}-\frac{1}{4}\gamma\hbar^3 x_5^2 y_1^2}{\sqrt{T_4}}+\frac{\frac{\hbar a_3 a_7}{\gamma}-\frac{1}{4}\gamma\hbar^3 x_7^2 y_3^2}{\sqrt{T_4}}+\frac{\frac{\hbar a_2 a_6}{\gamma}-\frac{1}{4}\gamma\hbar^3 x_2^2 y_6^2}{\sqrt{T_4}}\right)-\right.\right.\right.$$
$$\left.\left(\hbar\left(4 a_8^2 T_8^2+3\gamma^2\hbar^2 x_8^2 y_8^2+4\gamma a_8 T_8\left(T_8+2\hbar x_8 y_8\right)\right)\right)\Big/\left(4\gamma\sqrt{T_4}\,T_8^{3/2}\right)\right)-$$
$$\left.\left(\hbar\sqrt{T_8}\left(4 a_9^2 T_9^2+3\gamma^2\hbar^2 x_9^2 y_9^2+4\gamma a_9 T_9\left(T_9+2\hbar x_9 y_9\right)\right)\right)\Big/\left(4\gamma\sqrt{T_4}\,T_9^{3/2}\right)\right)-$$
$$\left.\left.\frac{1}{4\gamma\sqrt{T_4}\,T_{10}^{3/2}}\hbar\sqrt{T_8}\sqrt{T_9}\left(4 a_{10}^2 T_{10}^2+3\gamma^2\hbar^2 x_{10}^2 y_{10}^2+4\gamma a_{10} T_{10}\left(T_{10}+2\hbar x_{10} y_{10}\right)\right)\right)\epsilon+O[\epsilon]^2\right]$$

»
$$\mathbb{E}\left[\frac{\hbar\left(-a_5 t_1-a_7 t_3-a_1 t_6+a_8 t_8+a_9 t_9+a_{10} t_{10}\right)}{\gamma},\right.$$
$$\hbar\left(x_5 y_1+x_7 y_3+x_1 y_6-\frac{x_8 y_8}{T_8}-\frac{x_9 y_9}{T_9}-\frac{x_{10} y_{10}}{T_{10}}\right), \frac{\sqrt{T_8}\sqrt{T_9}\sqrt{T_{10}}}{\sqrt{T_4}}-$$
$$\frac{1}{4\left(\gamma\sqrt{T_4}\,T_8^{3/2}\,T_9^{3/2}\,T_{10}^{3/2}\right)}\left(\hbar\left(-4\gamma a_4 T_8^2 T_9^2 T_{10}^2-4 a_1\left(a_5+a_6\right) T_8^2 T_9^2 T_{10}^2-4 a_3 a_7 T_8^2 T_9^2 T_{10}^2+4\gamma a_8 T_8^2 T_9^2 T_{10}^2+\right.\right.$$
$$4 a_8^2 T_8^2 T_9^2 T_{10}^2+4\gamma a_9 T_8^2 T_9^2 T_{10}^2+4 a_9^2 T_8^2 T_9^2 T_{10}^2+4\gamma a_{10} T_8^2 T_9^2 T_{10}^2+4 a_{10}^2 T_8^2 T_9^2 T_{10}^2+\gamma^2\hbar^2 T_8^2 T_9^2 T_{10}^2 x_5^2 y_1^2+$$
$$\gamma^2\hbar^2 T_8^2 T_9^2 T_{10}^2 x_7^2 y_3^2+\gamma^2\hbar^2 T_8^2 T_9^2 T_{10}^2 x_1^2 y_6^2+8\gamma\hbar a_8 T_8 T_9^2 T_{10}^2 x_8 y_8+3\gamma^2\hbar^2 T_9^2 T_{10}^2 x_8^2 y_8^2+$$
$$\left.\left.8\gamma\hbar a_9 T_8^2 T_9 T_{10}^2 x_9 y_9+3\gamma^2\hbar^2 T_8^2 T_{10}^2 x_9^2 y_9^2+8\gamma\hbar a_{10} T_8^2 T_9^2 T_{10} x_{10} y_{10}+3\gamma^2\hbar^2 T_8^2 T_9^2 x_{10}^2 y_{10}^2\right)\right)\epsilon+O[\epsilon]^2\Big]$$

»
$$\mathbb{E}\Big[\frac{\hbar\,(-a_5\,t_1-a_7\,t_1-a_1\,t_6+a_8\,t_8+a_9\,t_9+a_{10}\,t_{10})}{\gamma},$$
$$\hbar\left(x_5\,y_1+T_6\,x_7\,y_1+x_1\,y_6+x_7\,y_6-T_1\,x_7\,y_6-\frac{x_8\,y_8}{T_8}-\frac{x_9\,y_9}{T_9}-\frac{x_{10}\,y_{10}}{T_{10}}\right),$$
$$\frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_4}}-\frac{1}{4\left(\gamma\,\sqrt{T_4}\,T_8^{3/2}\,T_9^{3/2}\,T_{10}^{3/2}\right)}$$
$$\Big(\hbar\,\Big(-4\,\gamma\,a_4\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_8\,T_8^2\,T_9^2\,T_{10}^2+4\,a_8^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_9\,T_8^2\,T_9^2\,T_{10}^2+4\,a_9^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_{10}\,T_8^2\,T_9^2\,T_{10}^2+$$
$$4\,a_{10}^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,\hbar\,a_5\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+4\,\gamma\,\hbar\,a_6\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_5^2\,y_1^2+\gamma^2\,\hbar^2\,T_6^2\,T_8^2$$
$$T_9^2\,T_{10}^2\,x_7^2\,y_1^2+4\,\gamma\,\hbar\,a_7\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_6-4\,\gamma^2\,\hbar^2\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1\,y_6+4\,\gamma^2\,\hbar^2\,T_1\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1\,y_6+$$
$$\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_6^2+4\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_6^2+\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_6^2-\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_6^2-$$
$$4\,a_1\,T_8^2\,T_9^2\,T_{10}^2\,(a_5+a_6+a_7+2\,\gamma\,\hbar\,T_1\,x_7\,y_6)+8\,\gamma\,\hbar\,a_8\,T_8\,T_9^2\,T_{10}^2\,x_8\,y_8+3\,\gamma^2\,\hbar^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+$$
$$8\,\gamma\,\hbar\,a_9\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+3\,\gamma^2\,\hbar^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+8\,\gamma\,\hbar\,a_{10}\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+3\,\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

»
$$\mathbb{E}\Big[\frac{\hbar\,(-a_5\,t_1-a_7\,t_1-a_1\,t_6+a_8\,t_8+a_9\,t_9+a_{10}\,t_{10})}{\gamma},$$
$$\hbar\left((x_5+T_6\,x_7)\,y_1+x_1\,y_6-(-1+T_1)\,x_7\,y_6-\frac{x_8\,y_8}{T_8}-\frac{x_9\,y_9}{T_9}-\frac{x_{10}\,y_{10}}{T_{10}}\right),$$
$$\frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_1}}-\frac{1}{4\left(\gamma\,\sqrt{T_1}\,T_8^{3/2}\,T_9^{3/2}\,T_{10}^{3/2}\right)}$$
$$\Big(\hbar\,\Big(4\,a_8^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_9\,T_8^2\,T_9^2\,T_{10}^2+4\,a_9^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_{10}\,T_8^2\,T_9^2\,T_{10}^2+4\,a_{10}^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,\hbar\,a_5\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+$$
$$4\,\gamma\,\hbar\,a_6\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_5^2\,y_1^2+\gamma^2\,\hbar^2\,T_6^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2+4\,\gamma^2\,\hbar\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_6+$$
$$4\,\gamma\,\hbar\,a_7\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_6-4\,\gamma^2\,\hbar^2\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1\,y_6+4\,\gamma^2\,\hbar^2\,T_1\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1\,y_6+$$
$$\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_6^2+4\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_6^2+\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_6^2-\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_6^2-$$
$$4\,a_1\,T_8^2\,T_9^2\,T_{10}^2\,(\gamma+a_5+a_6+a_7+2\,\gamma\,\hbar\,T_1\,x_7\,y_6)+3\,\gamma^2\,\hbar^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+4\,\gamma\,a_8\,T_8\,T_9^2\,T_{10}^2\,(T_8+2\,\hbar\,x_8\,y_8)+$$
$$8\,\gamma\,\hbar\,a_9\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+3\,\gamma^2\,\hbar^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+8\,\gamma\,\hbar\,a_{10}\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+3\,\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[\frac{\hbar\,(-a_7\,t_1-a_1\,(t_1+t_6)+a_8\,t_8+a_9\,t_9+a_{10}\,t_{10})}{\gamma},$$
$$\hbar\left((x_1+T_6\,x_7)\,y_1+T_1\,x_1\,y_6-(-1+T_1)\,x_7\,y_6-\frac{x_8\,y_8}{T_8}-\frac{x_9\,y_9}{T_9}-\frac{x_{10}\,y_{10}}{T_{10}}\right),$$
$$\frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_1}}-\frac{1}{4\left(\gamma\,\sqrt{T_1}\,T_8^{3/2}\,T_9^{3/2}\,T_{10}^{3/2}\right)}$$
$$\Big(\hbar\,\Big(-4\,a_1^2\,T_8^2\,T_9^2\,T_{10}^2+4\,a_8^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_9\,T_8^2\,T_9^2\,T_{10}^2+4\,a_9^2\,T_8^2\,T_9^2\,T_{10}^2+4\,\gamma\,a_{10}\,T_8^2\,T_9^2\,T_{10}^2+4\,a_{10}^2\,T_8^2\,T_9^2\,T_{10}^2+$$
$$4\,\gamma\,\hbar\,a_6\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+\gamma^2\,\hbar^2\,T_6^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2+4\,\gamma^2\,\hbar\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_6+$$
$$4\,\gamma\,\hbar\,a_7\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_6-8\,\gamma^2\,\hbar^2\,T_1\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1\,y_6+4\,\gamma^2\,\hbar^2\,T_1\,T_6\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1\,y_6+$$
$$\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_6^2+4\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_6^2+\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_6^2-$$
$$\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_6^2+4\,a_1\,T_8^2\,T_9^2\,T_{10}^2\,(-a_6-a_7+\gamma\,(-1+\hbar\,T_6\,x_7\,y_1+\hbar\,T_1\,(x_1-2\,x_7)\,y_6))+$$
$$3\,\gamma^2\,\hbar^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+4\,\gamma\,a_8\,T_8\,T_9^2\,T_{10}^2\,(T_8+2\,\hbar\,x_8\,y_8)+8\,\gamma\,\hbar\,a_9\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+$$
$$3\,\gamma^2\,\hbar^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+8\,\gamma\,\hbar\,a_{10}\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+3\,\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[\frac{\hbar\,(-2\,a_1\,t_1-a_7\,t_1+a_8\,t_8+a_9\,t_9+a_{10}\,t_{10})}{\gamma},$$

$$\hbar\left((1+T_1)\,x_1\,y_1+x_7\,y_1-\frac{x_8\,y_8}{T_8}-\frac{x_9\,y_9}{T_9}-\frac{x_{10}\,y_{10}}{T_{10}}\right),\ \frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_1}\,\left(1-T_1+T_1^2\right)}+$$

$$\frac{1}{4\,\gamma\,\sqrt{T_1}\,\left(1-T_1+T_1^2\right)^3\,T_8^{3/2}\,T_9^{3/2}\,T_{10}^{3/2}}\,\hbar\,\Big(-4\,\gamma\,a_9\,T_8^2\,T_9^2\,T_{10}^2-4\,a_9^2\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_{10}\,T_8^2\,T_9^2\,T_{10}^2-4\,a_{10}^2\,T_8^2\,T_9^2\,T_{10}^2-$$

$$4\,\gamma^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_7\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_9\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,a_9^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_{10}\,T_1\,T_8^2\,T_9^2\,T_{10}^2+$$

$$8\,a_{10}^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_7\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,\gamma\,a_9\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,a_9^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-$$

$$12\,\gamma\,a_{10}\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,a_{10}^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,\gamma^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2-8\,\gamma\,a_7\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+$$

$$8\,\gamma\,a_9\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,a_9^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_{10}\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,a_{10}^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2+$$

$$4\,\gamma\,a_7\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_9\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,a_9^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_{10}\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,a_{10}^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2+$$

$$8\,a_1^2\,\left(1-T_1+T_1^2\right)^2\,T_8^2\,T_9^2\,T_{10}^2-4\,a_8^2\,\left(1-T_1+T_1^2\right)^2\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,\hbar\,a_7\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-$$

$$8\,\gamma^2\,\hbar\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1+4\,\gamma\,\hbar\,a_7\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-8\,\gamma\,\hbar\,a_7\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1+4\,\gamma\,\hbar\,a_7\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-$$

$$8\,\gamma^2\,\hbar\,T_1^5\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-4\,\gamma\,\hbar\,a_7\,T_1^5\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-4\,\gamma\,\hbar\,a_7\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1-8\,\gamma^2\,\hbar\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+$$

$$8\,\gamma\,\hbar\,a_7\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+8\,\gamma^2\,\hbar\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1-8\,\gamma\,\hbar\,a_7\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1-8\,\gamma^2\,\hbar\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1+$$

$$4\,\gamma\,\hbar\,a_7\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_7\,y_1-\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+2\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-4\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+$$

$$4\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-6\,\gamma^2\,\hbar^2\,T_1^5\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+3\,\gamma^2\,\hbar^2\,T_1^6\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+4\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1^2-$$

$$4\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1^2+8\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1^2-4\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1^2+$$

$$4\,\gamma^2\,\hbar^2\,T_1^5\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,x_7\,y_1^2-\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2+2\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2-7\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2+$$

$$6\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2-5\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_7^2\,y_1^2-4\,a_1\,\left(1-T_1+T_1^2\right)\,T_8^2\,T_9^2\,T_{10}^2\,\Big(a_7\,\left(-1+T_1-T_1^2\right)+$$

$$\gamma\,\left(-1+\hbar\,x_1\,y_1+2\,\hbar\,T_1^3\,x_1\,y_1+T_1\,(2+\hbar\,x_1\,y_1-\hbar\,x_7\,y_1)+T_1^2\,(-4-2\,\hbar\,x_1\,y_1+\hbar\,x_7\,y_1)\right)\Big)-$$

$$3\,\gamma^2\,\hbar^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+6\,\gamma^2\,\hbar^2\,T_1\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2-9\,\gamma^2\,\hbar^2\,T_1^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+6\,\gamma^2\,\hbar^2\,T_1^3\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2-$$

$$3\,\gamma^2\,\hbar^2\,T_1^4\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2-4\,\gamma\,a_8\,\left(1-T_1+T_1^2\right)^2\,T_8\,T_9^2\,T_{10}^2\,(T_8+2\,\hbar\,x_8\,y_8)-8\,\gamma\,\hbar\,a_9\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+$$

$$16\,\gamma\,\hbar\,a_9\,T_1\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9-24\,\gamma\,\hbar\,a_9\,T_1^2\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+16\,\gamma\,\hbar\,a_9\,T_1^3\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9-$$

$$8\,\gamma\,\hbar\,a_9\,T_1^4\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9-3\,\gamma^2\,\hbar^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+6\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2-9\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+$$

$$6\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2-3\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2-8\,\gamma\,\hbar\,a_{10}\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+$$

$$16\,\gamma\,\hbar\,a_{10}\,T_1\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}-24\,\gamma\,\hbar\,a_{10}\,T_1^2\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+16\,\gamma\,\hbar\,a_{10}\,T_1^3\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}-$$

$$8\,\gamma\,\hbar\,a_{10}\,T_1^4\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}-3\,\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2+6\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2-$$

$$9\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2+6\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2-3\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2\Big)\,\epsilon+O[\epsilon]^2\Big]$$

» $\mathbb{E}\Big[\frac{\hbar\,(-3\,a_1\,t_1+a_8\,t_8+a_9\,t_9+a_{10}\,t_{10})}{\gamma},\ \hbar\left(\left(1+T_1+T_1^2\right)x_1\,y_1-\frac{x_8\,y_8}{T_8}-\frac{x_9\,y_9}{T_9}-\frac{x_{10}\,y_{10}}{T_{10}}\right),$

$$\frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_1}\,\left(1-T_1+T_1^2\right)}+\frac{1}{4\,\gamma\,\sqrt{T_1}\,\left(1-T_1+T_1^2\right)^3\,T_8^{3/2}\,T_9^{3/2}\,T_{10}^{3/2}}$$
$$\hbar\,\Big(-4\,\gamma\,a_9\,T_8^2\,T_9^2\,T_{10}^2-4\,a_9^2\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_{10}\,T_8^2\,T_9^2\,T_{10}^2-4\,a_{10}^2\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_9\,T_1\,T_8^2\,T_9^2\,T_{10}^2+$$
$$8\,a_9^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_{10}\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,a_{10}^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,\gamma\,a_9\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-$$
$$12\,a_9^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,\gamma\,a_{10}\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,a_{10}^2\,T_1^2\,T_8^2\,T_9^2\,T_{10}^2-12\,\gamma^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_9\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+$$
$$8\,a_9^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma\,a_{10}\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,a_{10}^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2+8\,\gamma^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_9\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-$$
$$4\,a_9^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,\gamma\,a_{10}\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2-4\,a_{10}^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2+12\,a_1^2\,\left(1-T_1+T_1^2\right)^2\,T_8^2\,T_9^2\,T_{10}^2-$$
$$4\,a_8^2\,\left(1-T_1+T_1^2\right)^2\,T_8^2\,T_9^2\,T_{10}^2-8\,\gamma^2\,\hbar\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-8\,\gamma^2\,\hbar\,T_1^6\,T_8^2\,T_9^2\,T_{10}^2\,x_1\,y_1-$$
$$\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+2\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+4\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+$$
$$4\,\gamma^2\,\hbar^2\,T_1^5\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+8\,\gamma^2\,\hbar^2\,T_1^6\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-6\,\gamma^2\,\hbar^2\,T_1^7\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+7\,\gamma^2\,\hbar^2\,T_1^8\,T_8^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-$$
$$4\,\gamma\,a_1\,\left(1-T_1+T_1^2\right)\,T_8^2\,T_9^2\,T_{10}^2\,\left(-1-2\,\hbar\,T_1^3\,x_1\,y_1+4\,\hbar\,T_1^4\,x_1\,y_1+T_1^2\,(-5+2\,\hbar\,x_1\,y_1)+T_1\,(3+2\,\hbar\,x_1\,y_1)\right)-$$
$$3\,\gamma^2\,\hbar^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+6\,\gamma^2\,\hbar^2\,T_1\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2-9\,\gamma^2\,\hbar^2\,T_1^2\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2+6\,\gamma^2\,\hbar^2\,T_1^3\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2-$$
$$3\,\gamma^2\,\hbar^2\,T_1^4\,T_9^2\,T_{10}^2\,x_8^2\,y_8^2-4\,\gamma\,a_8\,\left(1-T_1+T_1^2\right)^2\,T_8\,T_9^2\,T_{10}^2\,(T_8+2\,\hbar\,x_8\,y_8)-8\,\gamma\,\hbar\,a_9\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+$$
$$16\,\gamma\,\hbar\,a_9\,T_1\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9-24\,\gamma\,\hbar\,a_9\,T_1^2\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9+16\,\gamma\,\hbar\,a_9\,T_1^3\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9-$$
$$8\,\gamma\,\hbar\,a_9\,T_1^4\,T_8^2\,T_9\,T_{10}^2\,x_9\,y_9-3\,\gamma^2\,\hbar^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+6\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2-9\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2+$$
$$6\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2-3\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_{10}^2\,x_9^2\,y_9^2-8\,\gamma\,\hbar\,a_{10}\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+$$
$$16\,\gamma\,\hbar\,a_{10}\,T_1\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}-24\,\gamma\,\hbar\,a_{10}\,T_1^2\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+16\,\gamma\,\hbar\,a_{10}\,T_1^3\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}-$$
$$8\,\gamma\,\hbar\,a_{10}\,T_1^4\,T_8^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}-3\,\gamma^2\,\hbar^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2+6\,\gamma^2\,\hbar^2\,T_1\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2-$$
$$9\,\gamma^2\,\hbar^2\,T_1^2\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2+6\,\gamma^2\,\hbar^2\,T_1^3\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2-3\,\gamma^2\,\hbar^2\,T_1^4\,T_8^2\,T_9^2\,x_{10}^2\,y_{10}^2\Big)\,\epsilon+O[\epsilon]^2\Big]$$

» $\mathbb{E}\Big[\frac{\hbar\,(-2\,a_1\,t_1+a_9\,t_9+a_{10}\,t_{10})}{\gamma},\ \hbar\left((1+T_1)\,x_1\,y_1-\frac{x_9\,y_9}{T_9}-\frac{x_{10}\,y_{10}}{T_{10}}\right),$

$$\frac{\sqrt{T_9}\,\sqrt{T_{10}}}{1-T_1+T_1^2}+\frac{1}{4\,\gamma\,\left(1-T_1+T_1^2\right)^3\,T_9^{3/2}\,T_{10}^{3/2}}\,\hbar\,\Big(-4\,\gamma\,a_{10}\,T_9^2\,T_{10}^2-4\,a_{10}^2\,T_9^2\,T_{10}^2-4\,\gamma^2\,T_1\,T_9^2\,T_{10}^2+8\,\gamma\,a_{10}\,T_1\,T_9^2\,T_{10}^2+$$
$$8\,a_{10}^2\,T_1\,T_9^2\,T_{10}^2+8\,\gamma^2\,T_1^2\,T_9^2\,T_{10}^2-12\,\gamma\,a_{10}\,T_1^2\,T_9^2\,T_{10}^2-12\,a_{10}^2\,T_1^2\,T_9^2\,T_{10}^2-12\,\gamma^2\,T_1^3\,T_9^2\,T_{10}^2+8\,\gamma\,a_{10}\,T_1^3\,T_9^2\,T_{10}^2+$$
$$8\,a_{10}^2\,T_1^3\,T_9^2\,T_{10}^2+8\,\gamma^2\,T_1^4\,T_9^2\,T_{10}^2-4\,\gamma\,a_{10}\,T_1^4\,T_9^2\,T_{10}^2-4\,a_{10}^2\,T_1^4\,T_9^2\,T_{10}^2+8\,a_1^2\,\left(1-T_1+T_1^2\right)^2\,T_9^2\,T_{10}^2-$$
$$4\,a_9^2\,\left(1-T_1+T_1^2\right)^2\,T_9^2\,T_{10}^2-8\,\gamma^2\,\hbar\,T_1^2\,T_9^2\,T_{10}^2\,x_1\,y_1-8\,\gamma^2\,\hbar\,T_1^5\,T_9^2\,T_{10}^2\,x_1\,y_1-\gamma^2\,\hbar^2\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+$$
$$2\,\gamma^2\,\hbar^2\,T_1\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-4\,\gamma^2\,\hbar^2\,T_1^3\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+8\,\gamma^2\,\hbar^2\,T_1^4\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-6\,\gamma^2\,\hbar^2\,T_1^5\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2+$$
$$3\,\gamma^2\,\hbar^2\,T_1^6\,T_9^2\,T_{10}^2\,x_1^2\,y_1^2-8\,\gamma\,a_1\,T_1\,\left(1-T_1+T_1^2\right)\,T_9^2\,T_{10}^2\,\left(1+\hbar\,x_1\,y_1+\hbar\,T_1^2\,x_1\,y_1-T_1\,(2+\hbar\,x_1\,y_1)\right)-$$
$$3\,\gamma^2\,\hbar^2\,T_{10}^2\,x_9^2\,y_9^2+6\,\gamma^2\,\hbar^2\,T_1\,T_{10}^2\,x_9^2\,y_9^2-9\,\gamma^2\,\hbar^2\,T_1^2\,T_{10}^2\,x_9^2\,y_9^2+6\,\gamma^2\,\hbar^2\,T_1^3\,T_{10}^2\,x_9^2\,y_9^2-3\,\gamma^2\,\hbar^2\,T_1^4\,T_{10}^2\,x_9^2\,y_9^2-$$
$$4\,\gamma\,a_9\,\left(1-T_1+T_1^2\right)^2\,T_9\,T_{10}^2\,(T_9+2\,\hbar\,x_9\,y_9)-8\,\gamma\,\hbar\,a_{10}\,T_9^2\,T_{10}\,x_{10}\,y_{10}+16\,\gamma\,\hbar\,a_{10}\,T_1\,T_9^2\,T_{10}\,x_{10}\,y_{10}-$$
$$24\,\gamma\,\hbar\,a_{10}\,T_1^2\,T_9^2\,T_{10}\,x_{10}\,y_{10}+16\,\gamma\,\hbar\,a_{10}\,T_1^3\,T_9^2\,T_{10}\,x_{10}\,y_{10}-8\,\gamma\,\hbar\,a_{10}\,T_1^4\,T_9^2\,T_{10}\,x_{10}\,y_{10}-3\,\gamma^2\,\hbar^2\,T_9^2\,x_{10}^2\,y_{10}^2+$$
$$6\,\gamma^2\,\hbar^2\,T_1\,T_9^2\,x_{10}^2\,y_{10}^2-9\,\gamma^2\,\hbar^2\,T_1^2\,T_9^2\,x_{10}^2\,y_{10}^2+6\,\gamma^2\,\hbar^2\,T_1^3\,T_9^2\,x_{10}^2\,y_{10}^2-3\,\gamma^2\,\hbar^2\,T_1^4\,T_9^2\,x_{10}^2\,y_{10}^2\Big)\,\epsilon+O[\epsilon]^2\Big]$$

» $\mathbb{E}\Big[\frac{-\hbar\,a_1\,t_1+\hbar\,a_{10}\,t_{10}}{\gamma},\ \hbar\,x_1\,y_1-\frac{\hbar\,x_{10}\,y_{10}}{T_{10}},$

$$\frac{\sqrt{T_1}\,\sqrt{T_{10}}}{1-T_1+T_1^2}-\frac{1}{4\,\left(\gamma\,\left(1-T_1+T_1^2\right)^3\,T_{10}^{3/2}\right)}\,\Big(\hbar\,\sqrt{T_1}\,\Big(-4\,a_1^2\,\left(1-T_1+T_1^2\right)^2\,T_{10}^2+4\,a_{10}^2\,\left(1-T_1+T_1^2\right)^2\,T_{10}^2-$$
$$4\,\gamma\,a_1\,\left(-1+3\,T_1^2-4\,T_1^3+3\,T_1^4\right)\,T_{10}^2+4\,\gamma\,a_{10}\,\left(1-T_1+T_1^2\right)^2\,T_{10}\,(T_{10}+2\,\hbar\,x_{10}\,y_{10})+$$
$$\gamma^2\,\Big(\hbar^2\,\left(T_{10}^2\,x_1^2\,y_1^2+3\,x_{10}^2\,y_{10}^2\right)+T_1\,\left(T_{10}^2\,\left(4+8\,\hbar\,x_1\,y_1-2\,\hbar^2\,x_1^2\,y_1^2\right)-6\,\hbar^2\,x_{10}^2\,y_{10}^2\right)-$$
$$2\,T_1^3\,\left(T_{10}^2\,\left(-6+\hbar^2\,x_1^2\,y_1^2\right)+3\,\hbar^2\,x_{10}^2\,y_{10}^2\right)+T_1^4\,\left(T_{10}^2\,\left(-8+8\,\hbar\,x_1\,y_1+\hbar^2\,x_1^2\,y_1^2\right)+3\,\hbar^2\,x_{10}^2\,y_{10}^2\right)+$$
$$T_1^2\,\left(T_{10}^2\,\left(-8+3\,\hbar^2\,x_1^2\,y_1^2\right)+9\,\hbar^2\,x_{10}^2\,y_{10}^2\right)\Big)\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

Out[∘]= $\left\{230.047, \mathbb{E}\left[0, 0, \frac{T_1}{1 - T_1 + T_1^2} - \left(\hbar\, T_1 \left(-2\, a_1 \left(-1 + T_1 - T_1^3 + T_1^4\right) + \gamma \left(T_1 - 2\, T_1^2 - 2\, T_1^4 + 2\, \hbar\, x_1\, y_1 + T_1^3 \left(3 + 2\, \hbar\, x_1\, y_1\right)\right)\right) \epsilon\right) \Big/ \left(1 - T_1 + T_1^2\right)^3 + O[\epsilon]^2\right]\right\}$

Alternative Algorithms

In[∘]:= `{`$\lambda_{alt,2}$`[CU], HL@Simplify@Normal[`$\lambda_{alt,2}$`[CU] == Last[`$\Lambda_{CU,2}$`[{`ξ`, `η`}, {x, y}]]]]}`

Out[∘]= $\left\{1 + \left(2\, a\, \eta\, \xi - y\, \gamma\, \eta^2\, \xi - x\, \gamma\, \eta\, \xi^2 + \frac{1}{2}\, t\, \gamma\, \eta^2\, \xi^2\right) \epsilon + \frac{1}{2} \left(\left(2\, a\, \eta\, \xi - y\, \gamma\, \eta^2\, \xi - x\, \gamma\, \eta\, \xi^2 + \frac{1}{2}\, t\, \gamma\, \eta^2\, \xi^2\right)^2 + 2 \left(-a\, \gamma\, \eta^2\, \xi^2 + y\, \gamma^2\, \eta^3\, \xi^2 + x\, \gamma^2\, \eta^2\, \xi^3 - \frac{1}{3}\, t\, \gamma^2\, \eta^3\, \xi^3\right)\right) \epsilon^2 + O[\epsilon]^3, \text{True}\right\}$