

Pensieve header: A unified verification testing suite for the \$sl_2\$-portfolio project, Uxi version.
Continues pensieve://Projects/SL2Portfolio/nb/Verification.pdf.

Also continues pensieve://Projects/PPSA/nb/Verification.pdf and pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
In[*]:= wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
<< "SL2PortfolioProgram.m"
```

```
In[*]:= $p = 2; $k = 1; $U = QU;
```

```
In[*]:= HL[ε_] := Style[ε, Background → Yellow];
```

DocileQ

```
In[*]:= DQ /@ {ε² x y a₂, ε² x² y³}
```

```
Out[*]:= {True, False}
```

Initialization / Utilities

```
HL[DPx→Dε, y→Dη[x² y³] [eδ ε η]] == 6 eδ η ε δ³ ε + 6 eδ η ε δ⁴ η ε² + eδ η ε δ⁵ η² ε³]
```

True

```
SP{ξ→x} [(ξ² + ξ + 3) (x⁵ ex + 7 x) + 99 a]
7 + 99 a + 21 x + 20 ex x³ + 15 ex x⁴ + 5 ex x⁵
```

```
SP{ξ→x, η→y} [(ξ² + ξ + 3 + 2 ξ η) (x⁵ ex + 7 x) + 99 a + eδ x y ξ η]
7 + 99 a + 21 x + 20 ex x³ + 15 ex x⁴ + 5 ex x⁵ + ex y δ δ + ex y δ x y δ²
```

Implementing CU = $\mathcal{U}(sl_2^{\vee \epsilon})$

Verify σ and Δ ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\dots}$.

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp // HL
}] // Timing
{1.32813,
{(28 t^2 \gamma^4 + 116 t \gamma^5 \epsilon) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Implementing $QU = \mathcal{U}_q(\mathfrak{sl}_2^{\gamma \epsilon})$

```
Series[(1 - T e^{-2 \epsilon a \hbar}) / \hbar, {a, 0, 3}]
\frac{1 - T}{\hbar} + 2 T \epsilon a - 2 (T \epsilon^2 \hbar) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4
```

```
In[*]:= HL /@ DQ /@ Series[{{(1 - T e^{-2 \epsilon a \hbar}) / \hbar, e^{\hbar \epsilon a}}, {\epsilon, 0, 5}]
```

```
Out[*]:= {True, True}
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} -> Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}]]
{{QU[y], QU[y]} -> 0, {QU[y], QU[a]} -> \gamma QU[y],
{QU[y], QU[x]} -> \frac{(-1 + T) QU[]}{\hbar} - 2 T \epsilon QU[a] - \gamma \epsilon \hbar QU[y, x]},
{{QU[a], QU[y]} -> -\gamma QU[y], {QU[a], QU[a]} -> 0, {QU[a], QU[x]} -> \gamma QU[x]},
{{QU[x], QU[y]} -> \frac{(1 - T) QU[]}{\hbar} + 2 T \epsilon QU[a] + \gamma \epsilon \hbar QU[y, x],
{QU[x], QU[a]} -> -\gamma QU[x], {QU[x], QU[x]} -> 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
{z1, bas}, {z2, bas}, {z3, bas}]]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ \$p=5, \$k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  HL[z1 ** (z2 ** z3) - rhs // Simp]
}] // Timing
{3.78125, {( \frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \epsilon - 280 T \epsilon + 198 T^2 \gamma^5 \epsilon}{\hbar} ) QU[y, y, y, x, x] +
<<18>> + (1 + 8 \gamma \epsilon \hbar) QU[y, <<11>>, x], 0}}
```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs @ \$p=5, \$k=2):

```

With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  Short[lhs = z1 ** (z2 ** z3)],
  Short[rhs = (QU@@z1) ** ((QU@@z2) ** (QU@@z3))],
  Expand[Limit[rhs /. T2t ∪ {QU → CU}, ħ → 0] - lhs] // HL
}] // Timing
{10.125, {28 t^2 γ^4 CU[y, y, y, x, x] +
  116 t γ^5 ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, a, a, a, a, x, x, x, x],
  2 (γ^4/h^2 - 2 T γ^4/h^2 + T^2 γ^4/h^2 + γ^5/h - 2 T γ^5 ∈/h + T^2 γ^5 ∈/h) QU[y, y, y, x, x] +
  <<209>> + (1 + 8 γ ∈ ħ) QU[y, y, y, <<7>>, x, x, x], 0}}

```

Verifying σ , m , S , and Δ .

Verifying $\sigma_{i \rightarrow j, k \rightarrow l}$:

```
In[ ]:= CU@x1 + CU@x2 // σ1→3,2→4
```

```
Out[ ]:= CU[x3] + CU[x4]
```

Verifying relabeling using m :

```
In[ ]:= t1 t3 CU[y1, a1, x2] + t1 t1 CU[y1, a2, x2] // m1→3
```

```
Out[ ]:= CU[a2, x2, y3] t3^2 + CU[x2, y3, a3] t3^2
```

Verifying the meta-associativity of m :

```
In[ ]:= Module[{z, u},
  Table[u = CU[z[[1]]1, z[[2]]2, z[[3]]3]; HL[m1,3→3@m2,3→3@u == m2,3→3@m1,2→2@u],
  {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
```

```
Out[ ]:= {{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
  {True, True}, {True, True}, {True, True}, {True, True}, {True, True}}
```

Verifying the involutivity of S on CU on products of triples:

```
In[ ]:= With[{bas = CU /@ {y1, a1, x1}},
  Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas}]]
```

```
Out[ ]:= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying that S is an anti-homomorphism on CU/QU:

```
In[ ]:= With[{bas = U /@ {y1, a1, x1}},
  Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas}, {U, {CU, QU}}]]
```

```
Out[ ]:= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}
```

Verifying the co-associativity of Δ :

```
In[ ]:= Block[{bas = U /@ {y1, a1, x1}},
  Table[(z1 ** z2 ** z3 // Δ1→1,2 // Δ2→2,3) - (z1 ** z2 ** z3 // Δ1→1,3 // Δ1→1,2) // Simp // HL,
    {z1, bas}, {z2, bas}, {z3, bas}, {U, {CU, QU}}] ]
Out[ ]:= {{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
  {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
  {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying S-Δ compatibility:

```
In[ ]:= Block[{bas = U /@ {y1, a1, x1}},
  Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m1,2→1 // Simp // HL,
    {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]
Out[ ]:= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying S-Δ compatibility for opposite m, only for CU:

```
In[ ]:= Block[{bas = CU /@ {y1, a1, x1}},
  Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m2,1→1 // Simp // HL,
    {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]
Out[ ]:= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying m-Δ compatibility:

```
In[ ]:= Block[{bas1 = U /@ {y1, a1, x1}, bas2 = U /@ {y2, a2, x2}},
  Table[(z1 ** z2 ** z3 ** z4 // m1,2→1 // Δ1→1,2) -
    (z1 ** z2 ** z3 ** z4 // Δ1→3,4 // Δ2→5,6 // m3,5→1 // m4,6→2) // Simp // HL,
    {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2}] ]
Out[ ]:= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Implementing θ

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
{QU[y] → - $\frac{QU[x]}{\sqrt{T}}$  -  $\frac{\epsilon \hbar QU[a, x]}{\sqrt{T}}$  → QU[y], QU[a] → -QU[a] → QU[a],
  QU[x] →  $\left(-\frac{1}{\sqrt{T}} + \frac{\gamma \epsilon \hbar}{\sqrt{T}}\right) QU[y]$  -  $\frac{\epsilon \hbar QU[y, a]}{\sqrt{T}}$  → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Docility of AD\$f:

```
In[ ]:= HL@DQ@Block[{$p = 4}, Collect[SS@AD$f /. ω → a1, ε]]
```

```
Out[ ]:= True
```

Scaling behaviour of AD\$f:

```
HL@Simplify[AD$f == ((AD$f /. γ → 1) /. {ε → γ ε, a → γ-1 a, ω → γ-1 ω})]
```

```
True
```

```
HL@FullSimplify[
  AD$f == ((AD$f /. γ → 1) /. {ħ → γ2 ħ, ε → ε / γ, a → a / γ, t → γ-2 t, ω → γ-3 ω})]
```

```
True
```

Verifying that the asymmetric dequantizer is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizer

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
In[ ]:= {SD$P = 
$$\frac{\text{Cosh}\left[\hbar\left(\frac{\epsilon-t}{2} + \epsilon a\right)\right] - \text{Cosh}\left[\hbar\sqrt{\frac{t^2+\epsilon^2}{4} + \epsilon w}\right]}{\hbar \text{Sinh}\left[\frac{-\epsilon\hbar}{2}\right] (w - \epsilon a^2 + (t - \epsilon) a + t/2)},$$

  Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
  PowerExpand@Simplify[(SD$P /. {ħ → γ² ħ, ε → ε / γ, a → a / γ, t → γ⁻² t, w → γ⁻³ w}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
  SD$Q = Simplify[SD$P /. {a → c - 1/2}],
  Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
  FullSimplify[SD$g == FullSimplify[
    √SD$Q /. c → a + 1/2 /. {ħ → γ² ħ, ε → ε / γ, a → a / γ, t → γ⁻² t, w → γ⁻³ w}]] // HL,
  HL@DQ@Block[{$p = 4}, Collect[SS@SD$g /. w → a₁, ε]],
  HL@DQ@Block[{$p = 4}, Collect[SS@SD$f /. w → a₁, ε]]
}
```

$$\text{Out[]} = \left\{ - \left(\left(\left(\text{Cosh}\left[\left(a + \frac{1}{2}\right)(-t + \epsilon)\right]\hbar\right) - \text{Cosh}\left[\sqrt{\frac{1}{4}(t^2 + \epsilon^2) + \epsilon w}\right]\hbar \right) \text{Csch}\left[\frac{\epsilon\hbar}{2}\right] \right) / \right. \\ \left. \left(\left(\frac{t}{2} + a(t - \epsilon) - a^2\epsilon + w \right)\hbar \right) \right\}, \text{True, True,} \\ - \left(\left(4 \left(\text{Cosh}\left[\frac{1}{2}(t - 2c\epsilon)\right]\hbar\right) - \text{Cosh}\left[\frac{1}{2}\sqrt{t^2 + \epsilon^2 + 4\epsilon w}\right]\hbar \right) \text{Csch}\left[\frac{\epsilon\hbar}{2}\right] \right) / \right. \\ \left. \left((4ct + \epsilon - 4c^2\epsilon + 4w)\hbar \right) \right\}, \text{True, True, True, True}$$

Verifying the θ -symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
{True, True, True}
```

Verifying that the symmetric dequantizer is a homomorphism:

```

With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas} ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

```

The representation ρ

Verifying that ρ represents CU and QU:

```

Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ek. /; k > $k → 0],
 {U, {CU, QU}}, {z1, U/@ {y, a, x}}, {z2, U/@ {y, a, x} }
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {True, True, True}, {True, True, True}, {True, True, True}}]

```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```

Table[HL[ρ[eξ Uex].ρ[eα Uea] == ρ[eα Uea].ρ[ee-γ α ξ Uex]]], {U, {CU, QU}}]
{True, True}

```

\mathbb{C} and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from

Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

```

Table[
  {ΛU,1[{α, β}], {u, u}},
  lhs = U@CU[{u1, u2], ħ (α u1 + β u2), 1], HL[lhs == U@ΛU,1[ħ {α, β}, {u, u}]]],
  {U, {CU, QU}}, {u, {y, a, x}}]
{{{CCU[{y}, y (α + β), 1 + 0[ε]2],
  CU[] + (α ħ + β ħ) CU[y] + (α2 ħ2 / 2 + α β ħ2 + β2 ħ2 / 2) CU[y, y], True},
 {CCU[{a}, a (α + β), 1 + 0[ε]2], CU[] + (α ħ + β ħ) CU[a] + (α2 ħ2 / 2 + α β ħ2 + β2 ħ2 / 2) CU[a, a],
 True}, {CCU[{x}, x (α + β), 1 + 0[ε]2],
  CU[] + (α ħ + β ħ) CU[x] + (α2 ħ2 / 2 + α β ħ2 + β2 ħ2 / 2) CU[x, x], True}},
 {{CQU[{y}, y (α + β), 1 + 0[ε]2], QU[] + (α ħ + β ħ) QU[y] + (α2 ħ2 / 2 + α β ħ2 + β2 ħ2 / 2) QU[y, y],
 True}, {CQU[{a}, a (α + β), 1 + 0[ε]2], QU[] + (α ħ + β ħ) QU[a] +
 (α2 ħ2 / 2 + α β ħ2 + β2 ħ2 / 2) QU[a, a], True}, {CQU[{x}, x (α + β), 1 + 0[ε]2],
  QU[] + (α ħ + β ħ) QU[x] + (α2 ħ2 / 2 + α β ħ2 + β2 ħ2 / 2) QU[x, x], True}}}

```

$$\{\Lambda_{\#1}[\{\xi, \alpha\}, \{x, a\}], \text{lhs} = \#@\mathbb{E}_{\#}[\{x, a\}, \hbar(\xi x + \alpha a), 1],$$

$$\text{HL}[\text{lhs} = \#@\Lambda_{\#1}[\hbar\{\xi, \alpha\}, \{x, a\}]] \& /@ \{\text{CU}, \text{QU}\}$$

$$\{\{\mathbb{E}_{\text{CU}}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x \xi, 1 + 0[\epsilon]^2],$$

$$\text{CU}[] + \alpha \hbar \text{CU}[a] + (\xi \hbar - \alpha \gamma \xi \hbar^2) \text{CU}[x] + \frac{1}{2} \alpha^2 \hbar^2 \text{CU}[a, a] + \alpha \xi \hbar^2 \text{CU}[a, x] + \frac{1}{2} \xi^2 \hbar^2 \text{CU}[x, x],$$

$$\text{True}\}, \{\mathbb{E}_{\text{QU}}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x \xi, 1 + 0[\epsilon]^2], \text{QU}[] + \alpha \hbar \text{QU}[a] +$$

$$(\xi \hbar - \alpha \gamma \xi \hbar^2) \text{QU}[x] + \frac{1}{2} \alpha^2 \hbar^2 \text{QU}[a, a] + \alpha \xi \hbar^2 \text{QU}[a, x] + \frac{1}{2} \xi^2 \hbar^2 \text{QU}[x, x], \text{True}\}\}$$

$$\{\Lambda_{\#2}[\{\alpha, \eta\}, \{a, y\}], \text{lhs} = \#@\mathbb{E}_{\#}[\{a, y\}, \hbar(\eta y + \alpha a), 1],$$

$$\text{HL}[\text{lhs} = \#@\Lambda_{\#2}[\hbar\{\alpha, \eta\}, \{a, y\}]] \& /@ \{\text{CU}, \text{QU}\}$$

$$\{\{\mathbb{E}_{\text{CU}}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y \eta, 1 + 0[\epsilon]^3],$$

$$\text{CU}[] + \alpha \hbar \text{CU}[a] + (\eta \hbar - \alpha \gamma \eta \hbar^2) \text{CU}[y] + \frac{1}{2} \alpha^2 \hbar^2 \text{CU}[a, a] + \alpha \eta \hbar^2 \text{CU}[y, a] + \frac{1}{2} \eta^2 \hbar^2 \text{CU}[y, y],$$

$$\text{True}\}, \{\mathbb{E}_{\text{QU}}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y \eta, 1 + 0[\epsilon]^3], \text{QU}[] + \alpha \hbar \text{QU}[a] +$$

$$(\eta \hbar - \alpha \gamma \eta \hbar^2) \text{QU}[y] + \frac{1}{2} \alpha^2 \hbar^2 \text{QU}[a, a] + \alpha \eta \hbar^2 \text{QU}[y, a] + \frac{1}{2} \eta^2 \hbar^2 \text{QU}[y, y], \text{True}\}\}$$

In[*]:= Timing@LambdaQu2[{xi, eta}, {x, y}]

$$\text{Out[*]} = \{1.64063, \mathbb{E}_{\text{QU}}[\{y, a, x\}, y \eta + x \xi + \frac{(1-T) \eta \xi}{\hbar}, 1 + \frac{1}{4 \hbar}$$

$$\eta \xi (\gamma \eta \xi - 4 T \gamma \eta \xi + 3 T^2 \gamma \eta \xi + 8 a T \hbar + 2 y \gamma \eta \hbar - 6 T y \gamma \eta \hbar + 2 x \gamma \xi \hbar - 6 T x \gamma \xi \hbar + 4 x y \gamma \hbar^2) \epsilon +$$

$$\left(-a T y \gamma \eta^2 \xi (-\eta \xi + 3 T \eta \xi - 3 \hbar) - a T x \gamma \eta \xi^2 (-\eta \xi + 3 T \eta \xi - 3 \hbar) + 2 a^2 T \eta \xi (T \eta \xi - \hbar) +$$

$$2 a T x y \gamma \eta^2 \xi^2 \hbar - \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi (-\eta \xi + 3 T \eta \xi - \hbar) \hbar - \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 (-\eta \xi + 3 T \eta \xi - \hbar) \hbar +$$

$$\frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 + \frac{1}{24} y^2 \gamma^2 \eta^3 \xi (3 \eta \xi - 18 T \eta \xi + 27 T^2 \eta \xi + 4 \hbar - 28 T \hbar) + \frac{1}{24} x^2 \gamma^2 \eta \xi^3$$

$$(3 \eta \xi - 18 T \eta \xi + 27 T^2 \eta \xi + 4 \hbar - 28 T \hbar) + \frac{1}{2 \hbar} a T \gamma \eta^2 \xi^2 (\eta \xi - 4 T \eta \xi + 3 T^2 \eta \xi + 4 \hbar - 6 T \hbar) +$$

$$\frac{1}{4} x y \gamma^2 \eta \xi (2 \eta^2 \xi^2 - 10 T \eta^2 \xi^2 + 12 T^2 \eta^2 \xi^2 + 5 \eta \xi \hbar - 21 T \eta \xi \hbar + 2 \hbar^2) -$$

$$\frac{1}{24 \hbar} y \gamma^2 \eta^2 \xi (-3 \eta^2 \xi^2 + 21 T \eta^2 \xi^2 - 45 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 10 \eta \xi \hbar +$$

$$68 T \eta \xi \hbar - 82 T^2 \eta \xi \hbar - 6 \hbar^2 + 30 T \hbar^2) - \frac{1}{24 \hbar} x \gamma^2 \eta \xi^2 (-3 \eta^2 \xi^2 + 21 T \eta^2 \xi^2 -$$

$$45 T^2 \eta^2 \xi^2 + 27 T^3 \eta^2 \xi^2 - 10 \eta \xi \hbar + 68 T \eta \xi \hbar - 82 T^2 \eta \xi \hbar - 6 \hbar^2 + 30 T \hbar^2) +$$

$$\frac{1}{288 \hbar^2} (-1 + T) \gamma^2 \eta^2 \xi^2 (-9 \eta^2 \xi^2 + 63 T \eta^2 \xi^2 - 135 T^2 \eta^2 \xi^2 + 81 T^3 \eta^2 \xi^2 -$$

$$40 \eta \xi \hbar + 272 T \eta \xi \hbar - 328 T^2 \eta \xi \hbar - 36 \hbar^2 + 180 T \hbar^2) \epsilon^2 + 0[\epsilon]^3 \}$$

`{ $\Delta_{CU,1}$ [{ ξ , η }, { x , y }], lhs = CU@ \mathbb{C}_{CU} [{ x , y }, \hbar ($\xi x + \eta y$), 1],
HL[lhs = CU@ $\Delta_{CU,1}$ [\hbar { ξ , η }, { x , y }]]}`

$$\{\mathbb{C}_{CU}[\{y, a, x\}, y\eta + x\xi - t\eta\xi, 1 + \frac{1}{2}\eta\xi(4a - 2y\gamma\eta - 2x\gamma\xi + t\gamma\eta\xi)] \epsilon + O[\epsilon^2],$$

$$(1 - t\eta\xi\hbar^2) CU[] + 2\epsilon\eta\xi\hbar^2 CU[a] + \xi\hbar CU[x] + \eta\hbar CU[y] +$$

$$\frac{1}{2}\xi^2\hbar^2 CU[x, x] + \eta\xi\hbar^2 CU[y, x] + \frac{1}{2}\eta^2\hbar^2 CU[y, y], \text{True}\}$$

`In[*]:= { $\Delta_{QU,1}$ [{ ξ , η }, { x , y }], lhs = QU@ \mathbb{C}_{QU} [{ x , y }, \hbar ($\xi x + \eta y$), 1],
HL@SimpT[lhs = QU@ $\Delta_{QU,1}$ [\hbar { ξ , η }, { x , y }]]}`

`Out[*]:=` $\{\mathbb{C}_{QU}[\{y, a, x\}, y\eta + x\xi + \frac{(1-T)\eta\xi}{\hbar}, 1 + \frac{1}{4\hbar}$

$$\eta\xi(\gamma\eta\xi - 4T\gamma\eta\xi + 3T^2\gamma\eta\xi + 8aT\hbar + 2y\gamma\eta\hbar - 6Ty\gamma\eta\hbar + 2x\gamma\xi\hbar - 6Tx\gamma\xi\hbar + 4xy\gamma\hbar^2) \epsilon +$$

$$O[\epsilon^2], (1 + \eta\xi\hbar - T\eta\xi\hbar) QU[] + 2T\epsilon\eta\xi\hbar^2 QU[a] + \xi\hbar QU[x] +$$

$$\eta\hbar QU[y] + \frac{1}{2}\xi^2\hbar^2 QU[x, x] + \eta\xi\hbar^2 QU[y, x] + \frac{1}{2}\eta^2\hbar^2 QU[y, y], \text{True}\}$$

`{tt = Last[$\Delta_{CU,2}$ [{ ξ , η }, { x , y }], Log[tt],
Exponent[Normal@Log[tt] /. { $\xi \rightarrow \hbar \xi$, $\eta \rightarrow \hbar \eta$, $x \rightarrow \hbar x$, $y \rightarrow \hbar y$ }, \hbar]} // Expand`

$$\{1 + \left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right) \epsilon +$$

$$\left(2a^2\eta^2\xi^2 - a\gamma\eta^2\xi^2 - 2a y\gamma\eta^3\xi^2 + y\gamma^2\eta^3\xi^2 + \frac{1}{2}y^2\gamma^2\eta^4\xi^2 - 2ax\gamma\eta^2\xi^3 + x\gamma^2\eta^2\xi^3 + at\gamma\eta^3\xi^3 -$$

$$\frac{1}{3}t\gamma^2\eta^3\xi^3 + xy\gamma^2\eta^3\xi^3 - \frac{1}{2}ty\gamma^2\eta^4\xi^3 + \frac{1}{2}x^2\gamma^2\eta^2\xi^4 - \frac{1}{2}tx\gamma^2\eta^3\xi^4 + \frac{1}{8}t^2\gamma^2\eta^4\xi^4\right) \epsilon^2 + O[\epsilon]^3,$$

$$\left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right) \epsilon + \left(-a\gamma\eta^2\xi^2 + y\gamma^2\eta^3\xi^2 + x\gamma^2\eta^2\xi^3 - \frac{1}{3}t\gamma^2\eta^3\xi^3\right) \epsilon^2 +$$

$$O[\epsilon]^3, 6\}$$

`{tt = Last[$\Delta_{QU,2}$ [{ ξ , η }, { x , y }], Log[tt],
Exponent[Normal@Log[tt] /. { $\xi \rightarrow d \xi$, $\eta \rightarrow d \eta$, $x \rightarrow d x$, $y \rightarrow d y$ }, d]} // Expand`

$$\begin{aligned}
& \left\{ 1 + \left(2 a T \eta \xi + \frac{1}{2} y \gamma \eta^2 \xi - \frac{3}{2} T y \gamma \eta^2 \xi + \right. \right. \\
& \quad \left. \frac{1}{2} x \gamma \eta \xi^2 - \frac{3}{2} T x \gamma \eta \xi^2 + \frac{\gamma \eta^2 \xi^2}{4 \hbar} - \frac{T \gamma \eta^2 \xi^2}{\hbar} + \frac{3 T^2 \gamma \eta^2 \xi^2}{4 \hbar} + x y \gamma \eta \xi \hbar \right) \epsilon + \\
& \left(2 a^2 T^2 \eta^2 \xi^2 + 2 a T \gamma \eta^2 \xi^2 - 3 a T^2 \gamma \eta^2 \xi^2 + \frac{1}{8} \gamma^2 \eta^2 \xi^2 - \frac{3}{4} T \gamma^2 \eta^2 \xi^2 + \frac{5}{8} T^2 \gamma^2 \eta^2 \xi^2 + \right. \\
& \quad a T y \gamma \eta^3 \xi^2 - 3 a T^2 y \gamma \eta^3 \xi^2 + \frac{5}{12} y \gamma^2 \eta^3 \xi^2 - \frac{17}{6} T y \gamma^2 \eta^3 \xi^2 + \frac{41}{12} T^2 y \gamma^2 \eta^3 \xi^2 + \frac{1}{8} y^2 \gamma^2 \eta^4 \xi^2 - \\
& \quad \frac{3}{4} T y^2 \gamma^2 \eta^4 \xi^2 + \frac{9}{8} T^2 y^2 \gamma^2 \eta^4 \xi^2 + a T x \gamma \eta^2 \xi^3 - 3 a T^2 x \gamma \eta^2 \xi^3 + \frac{5}{12} x \gamma^2 \eta^2 \xi^3 - \frac{17}{6} T x \gamma^2 \eta^2 \xi^3 + \\
& \quad \frac{41}{12} T^2 x \gamma^2 \eta^2 \xi^3 + \frac{1}{2} x y \gamma^2 \eta^3 \xi^3 - \frac{5}{2} T x y \gamma^2 \eta^3 \xi^3 + 3 T^2 x y \gamma^2 \eta^3 \xi^3 + \frac{1}{8} x^2 \gamma^2 \eta^2 \xi^4 - \\
& \quad \frac{3}{4} T x^2 \gamma^2 \eta^2 \xi^4 + \frac{9}{8} T^2 x^2 \gamma^2 \eta^2 \xi^4 + \frac{\gamma^2 \eta^4 \xi^4}{32 \hbar^2} - \frac{T \gamma^2 \eta^4 \xi^4}{4 \hbar^2} + \frac{11 T^2 \gamma^2 \eta^4 \xi^4}{16 \hbar^2} - \frac{3 T^3 \gamma^2 \eta^4 \xi^4}{4 \hbar^2} + \\
& \quad \frac{9 T^4 \gamma^2 \eta^4 \xi^4}{32 \hbar^2} + \frac{a T \gamma \eta^3 \xi^3}{2 \hbar} - \frac{2 a T^2 \gamma \eta^3 \xi^3}{\hbar} + \frac{3 a T^3 \gamma \eta^3 \xi^3}{2 \hbar} + \frac{5 \gamma^2 \eta^3 \xi^3}{36 \hbar} - \frac{13 T \gamma^2 \eta^3 \xi^3}{12 \hbar} + \\
& \quad \frac{25 T^2 \gamma^2 \eta^3 \xi^3}{12 \hbar} - \frac{41 T^3 \gamma^2 \eta^3 \xi^3}{36 \hbar} + \frac{y \gamma^2 \eta^4 \xi^3}{8 \hbar} - \frac{7 T y \gamma^2 \eta^4 \xi^3}{8 \hbar} + \frac{15 T^2 y \gamma^2 \eta^4 \xi^3}{8 \hbar} - \frac{9 T^3 y \gamma^2 \eta^4 \xi^3}{8 \hbar} + \\
& \quad x \gamma^2 \eta^3 \xi^4 - \frac{7 T x \gamma^2 \eta^3 \xi^4}{8 \hbar} + \frac{15 T^2 x \gamma^2 \eta^3 \xi^4}{8 \hbar} - \frac{9 T^3 x \gamma^2 \eta^3 \xi^4}{8 \hbar} - 2 a^2 T \eta \xi \hbar + 3 a T y \gamma \eta^2 \xi \hbar + \\
& \quad \frac{1}{4} y \gamma^2 \eta^2 \xi \hbar - \frac{5}{4} T y \gamma^2 \eta^2 \xi \hbar + \frac{1}{6} y^2 \gamma^2 \eta^3 \xi \hbar - \frac{7}{6} T y^2 \gamma^2 \eta^3 \xi \hbar + 3 a T x \gamma \eta \xi^2 \hbar + \frac{1}{4} x \gamma^2 \eta \xi^2 \hbar - \\
& \quad \frac{5}{4} T x \gamma^2 \eta \xi^2 \hbar + 2 a T x y \gamma \eta^2 \xi^2 \hbar + \frac{5}{4} x y \gamma^2 \eta^2 \xi^2 \hbar - \frac{21}{4} T x y \gamma^2 \eta^2 \xi^2 \hbar + \frac{1}{2} x y^2 \gamma^2 \eta^3 \xi^2 \hbar - \\
& \quad \frac{3}{2} T x y^2 \gamma^2 \eta^3 \xi^2 \hbar + \frac{1}{6} x^2 \gamma^2 \eta \xi^3 \hbar - \frac{7}{6} T x^2 \gamma^2 \eta \xi^3 \hbar + \frac{1}{2} x^2 y \gamma^2 \eta^2 \xi^3 \hbar - \frac{3}{2} T x^2 y \gamma^2 \eta^2 \xi^3 \hbar + \\
& \quad \left. \frac{1}{2} x y \gamma^2 \eta \xi \hbar^2 + \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi \hbar^2 + \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 \hbar^2 + \frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 \right) \epsilon^2 + 0[\epsilon]^3, \\
& \left(2 a T \eta \xi + \frac{1}{2} y \gamma \eta^2 \xi - \frac{3}{2} T y \gamma \eta^2 \xi + \frac{1}{2} x \gamma \eta \xi^2 - \frac{3}{2} T x \gamma \eta \xi^2 + \frac{\gamma \eta^2 \xi^2}{4 \hbar} - \right. \\
& \quad \left. \frac{T \gamma \eta^2 \xi^2}{\hbar} + \frac{3 T^2 \gamma \eta^2 \xi^2}{4 \hbar} + x y \gamma \eta \xi \hbar \right) \epsilon + \\
& \left(2 a T \gamma \eta^2 \xi^2 - 3 a T^2 \gamma \eta^2 \xi^2 + \frac{1}{8} \gamma^2 \eta^2 \xi^2 - \frac{3}{4} T \gamma^2 \eta^2 \xi^2 + \frac{5}{8} T^2 \gamma^2 \eta^2 \xi^2 + \frac{5}{12} y \gamma^2 \eta^3 \xi^2 - \right. \\
& \quad \frac{17}{6} T y \gamma^2 \eta^3 \xi^2 + \frac{41}{12} T^2 y \gamma^2 \eta^3 \xi^2 + \frac{5}{12} x \gamma^2 \eta^2 \xi^3 - \frac{17}{6} T x \gamma^2 \eta^2 \xi^3 + \frac{41}{12} T^2 x \gamma^2 \eta^2 \xi^3 + \\
& \quad \frac{5 \gamma^2 \eta^3 \xi^3}{36 \hbar} - \frac{13 T \gamma^2 \eta^3 \xi^3}{12 \hbar} + \frac{25 T^2 \gamma^2 \eta^3 \xi^3}{12 \hbar} - \frac{41 T^3 \gamma^2 \eta^3 \xi^3}{36 \hbar} - 2 a^2 T \eta \xi \hbar + 3 a T y \gamma \eta^2 \xi \hbar + \\
& \quad \frac{1}{4} y \gamma^2 \eta^2 \xi \hbar - \frac{5}{4} T y \gamma^2 \eta^2 \xi \hbar + \frac{1}{6} y^2 \gamma^2 \eta^3 \xi \hbar - \frac{7}{6} T y^2 \gamma^2 \eta^3 \xi \hbar + 3 a T x \gamma \eta \xi^2 \hbar + \\
& \quad \frac{1}{4} x \gamma^2 \eta \xi^2 \hbar - \frac{5}{4} T x \gamma^2 \eta \xi^2 \hbar + \frac{5}{4} x y \gamma^2 \eta^2 \xi^2 \hbar - \frac{21}{4} T x y \gamma^2 \eta^2 \xi^2 \hbar + \frac{1}{6} x^2 \gamma^2 \eta \xi^3 \hbar - \\
& \quad \left. \frac{7}{6} T x^2 \gamma^2 \eta \xi^3 \hbar + \frac{1}{2} x y \gamma^2 \eta \xi \hbar^2 + \frac{1}{2} x y^2 \gamma^2 \eta^2 \xi \hbar^2 + \frac{1}{2} x^2 y \gamma^2 \eta \xi^2 \hbar^2 + \frac{1}{2} x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^2 \right) \epsilon^2 + 0[\epsilon]^3, 6 \}
\end{aligned}$$

```
Block[{$p = 4, $k = 1},
  {Delta_CU, $k [h {xi, eta, delta}, {x, y}],
   Short[lhs = CU@E_CU[{x, y}, h (xi x + eta y + delta x y), 1, $k], 5],
   HL@Simp[lhs - CU@Delta_CU, $k [h {xi, eta, delta}, {x, y}]]]
]
```

$$\left\{ \mathbb{E}_{\text{CU}} \left[\{y, a, x\}, \frac{xy \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar}, \right. \right.$$

$$\frac{1}{1 + t \delta \hbar} + \left((4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 - \right.$$

$$4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 +$$

$$4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 -$$

$$6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 +$$

$$8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 +$$

$$4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 +$$

$$4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 +$$

$$4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4) \epsilon) /$$

$$(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5) + 0[\epsilon]^2],$$

$$\left(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 + \right.$$

$$2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 -$$

$$\left. 9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4 \right) \text{CU}[\epsilon] +$$

$$(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 +$$

$$18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4) \text{CU}[a] +$$

$$\llcorner 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 \text{CU}[y, y, y, y, x, x, x] +$$

$$\frac{1}{24} \delta^4 \hbar^4$$

$$\text{CU}[y, y, y, y, x, x, x], \mathbf{0} \}$$

```
{Delta_QU, 2[{xi, eta, delta}, {x, y}], lhs = QU@E_QU[{x, y}, h (xi x + eta y + delta x y), 1],
 HL@SimpT[lhs == QU@Delta_QU, 1[h {xi, eta, delta}, {x, y}]]}
```

$$\left\{ \mathbb{E}_{\text{QU}} \left[\{y, a, x\}, \frac{\dots 1 \dots}{\dots 1 \dots}, \right. \right.$$

$$\frac{\hbar}{-\delta + T \delta \hbar} + \left((-8 a T \delta^4 \hbar^2 + 24 a T^2 \delta^4 \hbar^2 - 24 a T^3 \delta^4 \hbar^2 + 8 a T^4 \delta^4 \hbar^2 + \dots 149 \dots + \right.$$

$$4 x^2 y^2 \gamma \delta^2 \hbar^6 + 4 x y^2 \gamma \delta \eta \hbar^6 + 4 x^2 y \gamma \delta \xi \hbar^6 + 4 x y \gamma \eta \xi \hbar^6) \epsilon) /$$

$$(-4 \delta^5 + 20 T \delta^5 - 40 T^2 \delta^5 + 40 T^3 \delta^5 - 20 T^4 \delta^5 + 4 T^5 \delta^5 + \dots 12 \dots + 40 T^3 \delta^3 \hbar^2 +$$

$$40 \delta^2 \hbar^3 - 80 T \delta^2 \hbar^3 + 40 T^2 \delta^2 \hbar^3 - 20 \delta \hbar^4 + 20 T \delta \hbar^4 + 4 \hbar^5) +$$

$$\frac{(\dots 1 \dots)(\dots 1 \dots)}{\dots 1 \dots} + 0[\epsilon]^3, \dots 1 \dots, \mathbf{True} \}$$

large output
show less
show more
show all
set size limit...

```
{tt = ComposeSeries[(1 + t δ) Last[Δcu,2[{ξ, η, δ}, {x, y}]], (1 + t δ)4 e + 0[ε]18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
} // Expand

{
  2 a δ + 6 a t δ2 + 2 a x y δ2 + t γ δ2 - 4 x y γ δ2 + 6 a t2 δ3 + 4 a t x y δ3 + 2 t2 γ δ3 - 6 t x y γ δ3 -
  2 x2 y2 γ δ3 + 2 a t3 δ4 + 2 a t2 x y δ4 + t3 γ δ4 - 2 t2 x y γ δ4 -  $\frac{3}{2}$  t x2 y2 γ δ4 + 2 a y δ η -
  2 y γ δ η + 4 a t y δ2 η - 2 t y γ δ2 η - 3 x y2 γ δ2 η + 2 a t2 y δ3 η - 2 t x y2 γ δ3 η -
  y2 γ δ η2 -  $\frac{1}{2}$  t y2 γ δ2 η2 + 2 a x δ ξ - 2 x γ δ ξ + 4 a t x δ2 ξ - 2 t x γ δ2 ξ - 3 x2 y γ δ2 ξ +
  2 a t2 x δ3 ξ - 2 t x2 y γ δ3 ξ + 2 a η ξ + 4 a t δ η ξ + 2 t γ δ η ξ - 4 x y γ δ η ξ + 2 a t2 δ2 η ξ +
  2 t2 γ δ2 η ξ - 2 t x y γ δ2 η ξ - y γ η2 ξ - x2 γ δ ξ2 -  $\frac{1}{2}$  t x2 γ δ2 ξ2 - x γ η ξ2 +  $\frac{1}{2}$  t γ η2 ξ2
} ε +

{
  2 a2 δ2 - 2 a γ δ2 + 12 a2 t δ3 + 4 a2 x y δ3 - 8 a t γ δ3 - 20 a x y γ δ3 - 2 t γ2 δ3 + 18 x y γ2 δ3 +
  30 a2 t2 δ4 + 20 a2 t x y δ4 - 10 a t2 γ δ4 - 88 a t x y γ δ4 - 13 a x2 y2 γ δ4 -  $\frac{15}{2}$  t2 γ2 δ4 +
  64 t x y γ2 δ4 + 34 x2 y2 γ2 δ4 + 40 a2 t3 δ5 + 40 a2 t2 x y δ5 - 152 a t2 x y γ δ5 - 48 a t x2 y2 γ δ5 -
  10 t3 γ2 δ5 + 86 t2 x y γ2 δ5 + 107 t x2 y2 γ2 δ5 + 11 x3 y3 γ2 δ5 + 30 a2 t4 δ6 + 40 a2 t3 x y δ6 +
  10 a t4 γ δ6 - 128 a t3 x y γ δ6 - 66 a t2 x2 y2 γ δ6 - 5 t4 γ2 δ6 + 54 t3 x y γ2 δ6 +  $\frac{247}{2}$  t2 x2 y2 γ2 δ6 +
   $\frac{80}{3}$  t x3 y3 γ2 δ6 + 12 a2 t5 δ7 + 20 a2 t4 x y δ7 + 8 a t5 γ δ7 - 52 a t4 x y γ δ7 - 40 a t3 x2 y2 γ δ7 +
  16 t4 x y γ2 δ7 + 62 t3 x2 y2 γ2 δ7 +  $\frac{64}{3}$  t2 x3 y3 γ2 δ7 + 2 a2 t6 δ8 + 4 a2 t5 x y δ8 + 2 a t6 γ δ8 -
  8 a t5 x y γ δ8 - 9 a t4 x2 y2 γ δ8 +  $\frac{1}{2}$  t6 γ2 δ8 + 2 t5 x y γ2 δ8 +  $\frac{23}{2}$  t4 x2 y2 γ2 δ8 +  $\frac{17}{3}$  t3 x3 y3 γ2 δ8 +
  4 a2 y δ2 η - 12 a y γ δ2 η + 6 y γ2 δ2 η + 20 a2 t y δ3 η - 48 a t y γ δ3 η - 20 a x y2 γ δ3 η +
  14 t y γ2 δ3 η + 40 x y2 γ2 δ3 η + 40 a2 t2 y δ4 η - 72 a t2 y γ δ4 η - 72 a t x y2 γ δ4 η + 6 t2 y γ2 δ4 η +
  115 t x y2 γ2 δ4 η + 23 x2 y3 γ2 δ4 η + 40 a2 t3 y δ5 η - 48 a t3 y γ δ5 η - 96 a t2 x y2 γ δ5 η -
  6 t3 y γ2 δ5 η + 118 t2 x y2 γ2 δ5 η + 53 t x2 y3 γ2 δ5 η + 20 a2 t4 y δ6 η - 12 a t4 y γ δ6 η -
  56 a t3 x y2 γ δ6 η - 4 t4 y γ2 δ6 η + 51 t3 x y2 γ2 δ6 η + 40 t2 x2 y3 γ2 δ6 η + 4 a2 t5 y δ7 η -
  12 a t4 x y2 γ δ7 η + 8 t4 x y2 γ2 δ7 η + 10 t3 x2 y3 γ2 δ7 η - 7 a y2 γ δ2 η2 + 10 y2 γ2 δ2 η2 -
  24 a t y2 γ δ3 η2 + 24 t y2 γ2 δ3 η2 + 15 x y3 γ2 δ3 η2 - 30 a t2 y2 γ δ4 η2 +  $\frac{37}{2}$  t2 y2 γ2 δ4 η2 +
  32 t x y3 γ2 δ4 η2 - 16 a t3 y2 γ δ5 η2 + 5 t3 y2 γ2 δ5 η2 + 22 t2 x y3 γ2 δ5 η2 - 3 a t4 y2 γ δ6 η2 +
   $\frac{1}{2}$  t4 y2 γ2 δ6 η2 + 5 t3 x y3 γ2 δ6 η2 + 3 y3 γ2 δ2 η3 +  $\frac{17}{3}$  t y3 γ2 δ3 η3 +  $\frac{10}{3}$  t2 y3 γ2 δ4 η3 +
   $\frac{2}{3}$  t3 y3 γ2 δ5 η3 + 4 a2 x δ2 ξ - 12 a x γ δ2 ξ + 6 x γ2 δ2 ξ + 20 a2 t x δ3 ξ - 48 a t x γ δ3 ξ -
  20 a x2 y γ δ3 ξ + 14 t x γ2 δ3 ξ + 40 x2 y γ2 δ3 ξ + 40 a2 t2 x δ4 ξ - 72 a t2 x γ δ4 ξ - 72 a t x2 y γ δ4 ξ +
  6 t2 x γ2 δ4 ξ + 115 t x2 y γ2 δ4 ξ + 23 x3 y2 γ2 δ4 ξ + 40 a2 t3 x δ5 ξ - 48 a t3 x γ δ5 ξ -
  96 a t2 x2 y γ δ5 ξ - 6 t3 x γ2 δ5 ξ + 118 t2 x2 y γ2 δ5 ξ + 53 t x3 y2 γ2 δ5 ξ + 20 a2 t4 x δ6 ξ -
  12 a t4 x γ δ6 ξ - 56 a t3 x2 y γ δ6 ξ - 4 t4 x γ2 δ6 ξ + 51 t3 x2 y γ2 δ6 ξ + 40 t2 x3 y2 γ2 δ6 ξ +
  4 a2 t5 x δ7 ξ - 12 a t4 x2 y γ δ7 ξ + 8 t4 x2 y γ2 δ7 ξ + 10 t3 x3 y2 γ2 δ7 ξ + 4 a2 δ η ξ - 4 a γ δ η ξ +
  20 a2 t δ2 η ξ - 8 a t γ δ2 η ξ - 28 a x y γ δ2 η ξ - 6 t γ2 δ2 η ξ + 38 x y γ2 δ2 η ξ + 40 a2 t3 δ3 η ξ +

```

$$\begin{aligned}
 & 8 a^2 t^2 \gamma \delta^3 \eta \xi - 96 a t x y \gamma \delta^3 \eta \xi - 14 t^2 \gamma^2 \delta^3 \eta \xi + 88 t x y \gamma^2 \delta^3 \eta \xi + 44 x^2 y^2 \gamma^2 \delta^3 \eta \xi + \\
 & 40 a^2 t^3 \delta^4 \eta \xi + 32 a t^3 \gamma \delta^4 \eta \xi - 120 a t^2 x y \gamma \delta^4 \eta \xi - 6 t^3 \gamma^2 \delta^4 \eta \xi + 62 t^2 x y \gamma^2 \delta^4 \eta \xi + \\
 & 93 t x^2 y^2 \gamma^2 \delta^4 \eta \xi + 20 a^2 t^4 \delta^5 \eta \xi + 28 a t^4 \gamma \delta^5 \eta \xi - 64 a t^3 x y \gamma \delta^5 \eta \xi + 6 t^4 \gamma^2 \delta^5 \eta \xi + \\
 & 12 t^3 x y \gamma^2 \delta^5 \eta \xi + 63 t^2 x^2 y^2 \gamma^2 \delta^5 \eta \xi + 4 a^2 t^5 \delta^6 \eta \xi + 8 a t^5 \gamma \delta^6 \eta \xi - 12 a t^4 x y \gamma \delta^6 \eta \xi + \\
 & 4 t^5 \gamma^2 \delta^6 \eta \xi + 14 t^3 x^2 y^2 \gamma^2 \delta^6 \eta \xi - 8 a y \gamma \delta \eta^2 \xi + 6 y \gamma^2 \delta \eta^2 \xi - 24 a t y \gamma \delta^2 \eta^2 \xi + \\
 & 5 t y \gamma^2 \delta^2 \eta^2 \xi + 25 x y^2 \gamma^2 \delta^2 \eta^2 \xi - 24 a t^2 y \gamma \delta^3 \eta^2 \xi - 8 t^2 y \gamma^2 \delta^3 \eta^2 \xi + 45 t x y^2 \gamma^2 \delta^3 \eta^2 \xi - \\
 & 8 a t^3 y \gamma \delta^4 \eta^2 \xi - 7 t^3 y \gamma^2 \delta^4 \eta^2 \xi + 24 t^2 x y^2 \gamma^2 \delta^4 \eta^2 \xi + 4 t^3 x y^2 \gamma^2 \delta^5 \eta^2 \xi + 4 y^2 \gamma^2 \delta \eta^3 \xi + \\
 & 5 t y^2 \gamma^2 \delta^2 \eta^3 \xi + t^2 y^2 \gamma^2 \delta^3 \eta^3 \xi - 7 a x^2 \gamma \delta^2 \xi^2 + 10 x^2 \gamma^2 \delta^2 \xi^2 - 24 a t x^2 \gamma \delta^3 \xi^2 + \\
 & 24 t x^2 \gamma^2 \delta^3 \xi^2 + 15 x^3 y \gamma^2 \delta^3 \xi^2 - 30 a t^2 x^2 \gamma \delta^4 \xi^2 + \frac{37}{2} t^2 x^2 \gamma^2 \delta^4 \xi^2 + 32 t x^3 y \gamma^2 \delta^4 \xi^2 - \\
 & 16 a t^3 x^2 \gamma \delta^5 \xi^2 + 5 t^3 x^2 \gamma^2 \delta^5 \xi^2 + 22 t^2 x^3 y \gamma^2 \delta^5 \xi^2 - 3 a t^4 x^2 \gamma \delta^6 \xi^2 + \frac{1}{2} t^4 x^2 \gamma^2 \delta^6 \xi^2 + \\
 & 5 t^3 x^3 y \gamma^2 \delta^6 \xi^2 - 8 a x \gamma \delta \eta \xi^2 + 6 x \gamma^2 \delta \eta \xi^2 - 24 a t x \gamma \delta^2 \eta \xi^2 + 5 t x \gamma^2 \delta^2 \eta \xi^2 + \\
 & 25 x^2 y \gamma^2 \delta^2 \eta \xi^2 - 24 a t^2 x \gamma \delta^3 \eta \xi^2 - 8 t^2 x \gamma^2 \delta^3 \eta \xi^2 + 45 t x^2 y \gamma^2 \delta^3 \eta \xi^2 - 8 a t^3 x \gamma \delta^4 \eta \xi^2 - \\
 & 7 t^3 x \gamma^2 \delta^4 \eta \xi^2 + 24 t^2 x^2 y \gamma^2 \delta^4 \eta \xi^2 + 4 t^3 x^2 y \gamma^2 \delta^5 \eta \xi^2 - a \gamma \eta^2 \xi^2 - 3 t \gamma^2 \delta \eta^2 \xi^2 + \\
 & 11 x y \gamma^2 \delta \eta^2 \xi^2 + 6 a t^2 \gamma \delta^2 \eta^2 \xi^2 - \frac{5}{2} t^2 \gamma^2 \delta^2 \eta^2 \xi^2 + 12 t x y \gamma^2 \delta^2 \eta^2 \xi^2 + 8 a t^3 \gamma \delta^3 \eta^2 \xi^2 + \\
 & 4 t^3 \gamma^2 \delta^3 \eta^2 \xi^2 + 3 a t^4 \gamma \delta^4 \eta^2 \xi^2 + \frac{7}{2} t^4 \gamma^2 \delta^4 \eta^2 \xi^2 - t^3 x y \gamma^2 \delta^4 \eta^2 \xi^2 + y \gamma^2 \eta^3 \xi^2 - \\
 & t y \gamma^2 \delta \eta^3 \xi^2 - 2 t^2 y \gamma^2 \delta^2 \eta^3 \xi^2 + 3 x^3 \gamma^2 \delta^2 \xi^3 + \frac{17}{3} t x^3 \gamma^2 \delta^3 \xi^3 + \frac{10}{3} t^2 x^3 \gamma^2 \delta^4 \xi^3 + \\
 & \frac{2}{3} t^3 x^3 \gamma^2 \delta^5 \xi^3 + 4 x^2 \gamma^2 \delta \eta \xi^3 + 5 t x^2 \gamma^2 \delta^2 \eta \xi^3 + t^2 x^2 \gamma^2 \delta^3 \eta \xi^3 + x \gamma^2 \eta^2 \xi^3 - t x \gamma^2 \delta \eta^2 \xi^3 - \\
 & 2 t^2 x \gamma^2 \delta^2 \eta^2 \xi^3 - \frac{1}{3} t \gamma^2 \eta^3 \xi^3 + \frac{1}{3} t^2 \gamma^2 \delta \eta^3 \xi^3 + \frac{2}{3} t^3 \gamma^2 \delta^2 \eta^3 \xi^3 \Big) \epsilon^2 + \mathbf{0}[\epsilon]^3, 6, 6\}
 \end{aligned}$$

```

tt = Last[AQu,2[{{ξ, η, δ}, {x, y}}]];
Log[tt],
Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d] // Expand
    
```

$$\left\{ \text{Log} \left[\frac{\hbar}{-\delta + T \delta + \hbar} \right] + \left(\frac{2 a T \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} - \frac{8 a T^2 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} + \frac{12 a T^3 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} - \frac{8 a T^4 \delta^5 \hbar}{(-\delta + T \delta + \hbar)^5} + \dots 267 \dots + \frac{x^2 y^2 \gamma \delta^2 \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x y^2 \gamma \delta \eta \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x^2 y \gamma \delta \xi \hbar^6}{(-\delta + T \delta + \hbar)^5} + \frac{x y \gamma \eta \xi \hbar^6}{(-\delta + T \delta + \hbar)^5} \right) \epsilon + \left(- \frac{32 a^2 T^2 \delta^{10} \hbar^2}{(\dots 1 \dots)^2} + \dots 8307 \dots + \dots 1 \dots + \frac{144 x^2 y^2 \gamma^2 \eta^2 \xi^2 \hbar^{11}}{\dots 1 \dots} \right) \epsilon^2 + \mathbf{0}[\epsilon]^3, 6\}$$

large output
show less
show more
show all
set size limit...

Reorderings with Rord

```

With[{co = CCU[{y1, x1}1, {x2, a2, y2}2, h t1 a2 + h t1^-1 (e^t1 - 1) y1 x2, 12 + e x1 y2}],
{Short[rhs = co // Rord[x2, a2 -> 3, 3], HL[CU[co] == CU[rhs]]}]
{CCU[{y1, x1}1, {a3, x3, y2}2,
frac[e^-gamma h t1 (e^gamma h t1 h a3 t1^2 - h x3 y1 + e^t1 h x3 y1)]{t1}, 1 + x1 y2} \in + \mathbf{0}[\epsilon]^3, True}
    
```

With [{c0 = CU [{y1, a1, a2}1, {x2, x1, y2}2,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] },$
{Short[rhs = c0 // Rord_{a1,a2→3} // Rord_{x2,x1→4}, 3], HL[CU[c0] == CU[rhs]] }]
{CU [{y1, a3}1, {x4, y2}2,
 $\hbar a_3 l_{11} t_1 + \hbar a_3 l_{12} t_1 + \hbar a_3 l_{21} t_2 + \hbar a_3 l_{22} t_2 + \hbar x_4 y_1 \gamma_{11} + \hbar x_4 y_2 \gamma_{12} + \hbar x_4 y_1 \gamma_{21} + \hbar x_4 y_2 \gamma_{22},$
 $1 + (a_3 l_1 + a_3 l_2 + p_{11} x_4 y_1 + p_{21} x_4 y_1 + p_{12} x_4 y_2 + p_{22} x_4 y_2) \epsilon + O[\epsilon]^3$], True }

$\hbar a_3 l_{11} t_1 + \hbar a_3 l_{12} t_1 + \hbar a_3 l_{21} t_2 + \hbar a_3 l_{22} t_2 +$
 $\hbar x_4 y_1 \gamma_{11} + \hbar x_4 y_2 \gamma_{12} + \hbar x_4 y_1 \gamma_{21} + \hbar x_4 y_2 \gamma_{22}$ // Simplify
 $\hbar (a_3 (l_{11} t_1 + l_{12} t_1 + (l_{21} + l_{22}) t_2) + x_4 (y_1 (\gamma_{11} + \gamma_{21}) + y_2 (\gamma_{12} + \gamma_{22})))$

With [{c0 = CU [{y1, a1, x1}1, {x2, a2, y2}2,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] },$
{Short[rhs = c0 // Rord_{x2,a2→3}, 3], HL[CU[c0] == CU[rhs]] }]
{CU [{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
 $1 + e^{-\gamma \hbar l_{12} t_1 - \gamma \hbar l_{22} t_2} (e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_1 l_1 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_3 l_2 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +$
 $e^{\langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \epsilon + O[\epsilon]^3$], True }

With [{q0 = QU [{y1, a1, x1}1, {x2, a2, y2}2,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] },$
{Short[rhs = q0 // Rord_{x2,a2→3}, 3], HL[QU[q0] == QU[rhs]] }]
{QU [{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
 $1 + e^{-\gamma \hbar l_{12} t_1 - \gamma \hbar l_{22} t_2} (e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_1 l_1 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} a_3 l_2 + e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +$
 $e^{\langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma \hbar l_2 x_3 y_1 \gamma_{21} - \gamma \hbar l_2 x_3 y_2 \gamma_{22}) \epsilon + O[\epsilon]^3$], True }

With [{q0 = QU [{y1, a1, x1}1, {x2, a2, y2}2,
 $\hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $1_2 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] },$
{Short[rhs = q0 // Rord_{a2,y2→3}, 3], HL[QU[q0] == QU[rhs]] }]
{ <<1>>, True }

Timing@With [{q0 = QU [{x1, y1}1, {x2, a2, y2}2,
 $\hbar (l_{12} t_1 a_2 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2),$
 $\theta_2 + \epsilon (l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)] },$
{Short[rhs = q0 // Rord_{x1,y1→3}, 5] }]

{116.156, {QU [{y3, a3, x3}1, <<1>>2, $\frac{\langle\langle 1 \rangle\rangle}{1 - \langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle},$
 $((\hbar a_2 l_2 + p_{11} - p_{11} T_1 + \hbar p_{22} x_2 y_2 + \hbar p_{12} x_3 y_2 + \langle\langle 46 \rangle\rangle + 2 \hbar p_{12} T_1 x_2 y_2 \gamma_{11} \gamma_{21} -$
 $\hbar p_{12} T_1^2 x_2 y_2 \gamma_{11} \gamma_{21} + \hbar p_{11} x_2 y_2 \gamma_{12} \gamma_{21} - 2 \hbar p_{11} T_1 x_2 y_2 \gamma_{12} \gamma_{21} + \hbar p_{11} T_1^2 x_2 y_2 \gamma_{12} \gamma_{21}) \epsilon /$
 $(\hbar - 3 \hbar \gamma_{11} + 3 \hbar T_1 \gamma_{11} + 3 \hbar \gamma_{11}^2 - 6 \hbar T_1 \gamma_{11}^2 + 3 \hbar T_1^2 \gamma_{11}^2 - \hbar \gamma_{11}^3 + 3 \hbar T_1 \gamma_{11}^3 - 3 \hbar T_1^2 \gamma_{11}^3 + \hbar T_1^3 \gamma_{11}^3) +$
 $((8 a_3 p_{11} T_1 + \langle\langle 2726 \rangle\rangle + 3 \gamma \langle\langle 6 \rangle\rangle \gamma_{21}^3) \langle\langle 1 \rangle\rangle /$
 $(4 - 28 \gamma_{11} + \langle\langle 48 \rangle\rangle + 4 T_1^7 \gamma_{11}^7) + O[\epsilon]^3] } }$

$$\begin{aligned}
& \text{Timing@With}[\{\mathbf{qo} = \mathbb{C}_{\text{QU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \{\mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_2, \\
& \quad \hbar (\mathbf{l}_{12} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2), \\
& \quad \mathbf{l}_2 + \epsilon (\mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2) \}], \\
& \quad \{\text{Short}[\text{rhs} = \mathbf{qo} // \text{Rord}_{\mathbf{x}_1, \mathbf{y}_1 \rightarrow 3}, 5], \text{HL@SimpT}[\text{QU}[\mathbf{qo}] = \text{QU}[\text{rhs}]]\}] \\
& \{388.922, \\
& \quad \{\mathbb{C}_{\text{QU}}[\{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_1, \{\ll 1 \gg\}_2, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + ((4 \hbar \mathbf{a}_2 \mathbf{l}_2 + 4 \mathbf{p}_{11} - 4 \mathbf{p}_{11} T_1 + 4 \hbar \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2 + \\
& \quad \ll 339 \gg + \gamma \hbar^4 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4 \gamma \hbar^4 T_1 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2) \epsilon) / \\
& \quad (4 \hbar - 20 \hbar \gamma_{11} + 20 \hbar T_1 \gamma_{11} + 40 \hbar \gamma_{11}^2 - 80 \hbar T_1 \gamma_{11}^2 + 40 \hbar T_1^2 \gamma_{11}^2 - 40 \hbar \gamma_{11}^3 + \ll 13 \gg + \\
& \quad 20 \hbar T_1 \gamma_{11}^5 - 40 \hbar T_1^2 \gamma_{11}^5 + 40 \hbar T_1^3 \gamma_{11}^5 - 20 \hbar T_1^4 \gamma_{11}^5 + 4 \hbar T_1^5 \gamma_{11}^5) + \\
& \quad (576 \mathbf{a}_3 \mathbf{p}_{11} T_1 + \ll 8073 \gg + \ll 1 \gg) \ll 1 \gg} \ll 1 \gg + O[\epsilon]^3], \text{True}\}] \} \\
& \quad \ll 79 \gg + 288 T_1^9 \gamma_{11}^9
\end{aligned}$$

$$\begin{aligned}
& \text{Timing@With}[\{\mathbf{qo} = \mathbb{C}_{\text{QU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \{\mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_2, \\
& \quad \hbar (\mathbf{l}_{12} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2), \\
& \quad \mathbf{l}_2 + \epsilon (\mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2) \}], \\
& \quad \{\text{Short}[\text{rhs} = \mathbf{qo} // \text{Rord}_{\mathbf{x}_1, \mathbf{y}_1 \rightarrow 1}, 5], \text{HL@SimpT}[\text{QU}[\mathbf{qo}] = \text{QU}[\text{rhs}]]\}] \\
& \{336.781, \\
& \quad \{\mathbb{C}_{\text{QU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\ll 1 \gg\}_2, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + ((4 \hbar \mathbf{a}_2 \mathbf{l}_2 + 4 \mathbf{p}_{11} - 4 \mathbf{p}_{11} T_1 + 4 \hbar \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \\
& \quad 4 \hbar \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \ll 338 \gg + \gamma \hbar^4 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4 \gamma \hbar^4 T_1 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 \mathbf{x}_2^2 \mathbf{y}_2^2 \gamma_{12}^2 \gamma_{21}^2) \epsilon) / \\
& \quad (4 \hbar - 20 \hbar \gamma_{11} + 20 \hbar T_1 \gamma_{11} + 40 \hbar \gamma_{11}^2 - 80 \hbar T_1 \gamma_{11}^2 + 40 \hbar T_1^2 \gamma_{11}^2 - 40 \hbar \gamma_{11}^3 + \ll 10 \gg + \\
& \quad 20 \hbar T_1^4 \gamma_{11}^4 - 4 \hbar \gamma_{11}^5 + 20 \hbar T_1 \gamma_{11}^5 - 40 \hbar T_1^2 \gamma_{11}^5 + 40 \hbar T_1^3 \gamma_{11}^5 - 20 \hbar T_1^4 \gamma_{11}^5 + 4 \hbar T_1^5 \gamma_{11}^5) + \\
& \quad (576 \mathbf{a}_1 \mathbf{p}_{11} T_1 + \ll 8073 \gg + \ll 1 \gg) \ll 1 \gg} \ll 1 \gg + O[\epsilon]^3], \text{True}\}] \} \\
& \quad \ll 79 \gg + 288 T_1^9 \gamma_{11}^9
\end{aligned}$$

Canonical ordering with Cord

$\text{Cord@C}_{\text{CU}}[\{\mathbf{x}_1, \mathbf{y}_1\}_1, \mathbf{0}, \mathbf{0}_1 + \mathbf{x}_1 \mathbf{y}_1]$

$$\mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \mathbf{0}, (-\mathbf{t}_1 + \mathbf{x}_1 \mathbf{y}_1) + 2 \mathbf{a}_1 \epsilon + O[\epsilon]^2]$$

$\text{Block}[\{\mathbf{\$p} = 4, \mathbf{\$k} = \mathbf{0}, \mathbf{co} = \mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1, \mathbf{x}_2, \mathbf{a}_2, \mathbf{y}_2\}_1,$

$$\hbar (\mathbf{l}_{11} \mathbf{t}_1 \mathbf{a}_1 + \mathbf{l}_{21} \mathbf{t}_1 \mathbf{a}_2 + \mathbf{l}_{21} \mathbf{t}_2 \mathbf{a}_1 + \mathbf{l}_{22} \mathbf{t}_2 \mathbf{a}_2 + \gamma_{11} \mathbf{x}_1 \mathbf{y}_1 + \gamma_{12} \mathbf{x}_1 \mathbf{y}_2 + \gamma_{21} \mathbf{x}_2 \mathbf{y}_1 + \gamma_{22} \mathbf{x}_2 \mathbf{y}_2),$$

$$\mathbf{1}_0 + \epsilon (\mathbf{l}_1 \mathbf{a}_1 + \mathbf{l}_2 \mathbf{a}_2 + \mathbf{p}_{11} \mathbf{x}_1 \mathbf{y}_1 + \mathbf{p}_{12} \mathbf{x}_1 \mathbf{y}_2 + \mathbf{p}_{21} \mathbf{x}_2 \mathbf{y}_1 + \mathbf{p}_{22} \mathbf{x}_2 \mathbf{y}_2) \}],$$

$\text{Timing@}\{\text{Short}[\text{Cord}[\mathbf{co}], 8], \text{HL@Simp}[\text{CU}[\mathbf{co}] - \text{CU}[\text{Cord}[\mathbf{co}]]]\}$

$$\begin{aligned}
& \{4.53125, \\
& \quad \{\mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, (e^{\gamma \hbar \mathbf{l}_{11} \mathbf{t}_1 + 2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{21} \mathbf{t}_2 + 2 \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} \hbar \mathbf{a}_1 \mathbf{l}_{11} \mathbf{t}_1 + e^{\gamma \hbar \mathbf{l}_{11} \mathbf{t}_1 + 2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{21} \mathbf{t}_2 + 2 \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} \\
& \quad \hbar \mathbf{a}_1 \mathbf{l}_{12} \mathbf{t}_1 + \ll 12 \gg + e^{\gamma \hbar \mathbf{l}_{11} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{21} \mathbf{t}_2 + \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} \hbar^2 \mathbf{a}_1 \mathbf{l}_{22} \mathbf{t}_1 \mathbf{t}_2 \gamma_{22} + \hbar \mathbf{x}_1 \mathbf{y}_1 \gamma_{22}) / \\
& \quad (e^{\gamma \hbar \mathbf{l}_{11} \mathbf{t}_1 + 2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{21} \mathbf{t}_2 + 2 \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} + e^{\gamma \hbar \mathbf{l}_{11} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{21} \mathbf{t}_2 + \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} \hbar \mathbf{t}_1 \gamma_{12} + \\
& \quad e^{\gamma \hbar \mathbf{l}_{11} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{21} \mathbf{t}_2 + \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} \hbar \mathbf{t}_1 \gamma_{22}), \frac{e^{\gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2}}{e^{\gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \gamma \hbar \mathbf{l}_{22} \mathbf{t}_2} + \hbar \mathbf{t}_1 \gamma_{12} + \hbar \mathbf{t}_1 \gamma_{22}} + O[\epsilon]^1], \mathbf{0}\}] \}
\end{aligned}$$

$$\begin{aligned}
& \text{Block}[\{\$p = 4, \$k = 1, \text{co} = \mathbb{C}_{\text{CU}}[\{y_1, a_1, x_1, x_2, a_2, y_2\}_1, \\
& \quad \hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2), \\
& \quad \mathbf{1}_1 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)\}], \\
& \text{Timing}@\{\text{Short}[\text{Cord}[\text{co}], 8], \text{HL}@\text{Simp}[\text{CU}[\text{co}] - \text{CU}[\text{Cord}[\text{co}]]]\} \\
& \{81.2656, \{\mathbb{C}_{\text{CU}}[\{y_1, a_1, x_1\}_1, (e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + \ll 14 \gg + \hbar x_1 y_1 \gamma_{22}) / \\
& \quad (e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} + e^{\gamma \hbar l_{11} t_1 + \ll 2 \gg + \gamma \hbar \ll 1 \gg} t_2 \hbar t_1 \gamma_{12} + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{22}), \\
& \quad \frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + ((2 e^{2 \gamma \hbar l_{11} t_1 + 6 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 6 \gamma \hbar l_{22} t_2} a_1 l_1 + \ll 419 \gg) \epsilon) / \\
& \quad (2 e^{2 \gamma \hbar l_{11} t_1 + 6 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 6 \gamma \hbar l_{22} t_2} + 10 e^{2 \gamma \hbar l_{11} t_1 + 5 \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + 5 \gamma \hbar l_{22} t_2} \hbar t_1 \gamma_{12} + \\
& \quad \ll 18 \gg + 2 e^{2 \gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + 2 \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar^5 t_1^5 \gamma_{22}^5) + O[\epsilon]^2], \mathbf{0}\} \\
\end{aligned}$$

$$\begin{aligned}
& \text{With}[\{\text{qo} = \mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1, x_2, a_2, y_2\}_1, \\
& \quad \hbar (l_{11} t_1 a_1 + l_{12} t_1 a_2 + l_{21} t_2 a_1 + l_{22} t_2 a_2 + \gamma_{11} x_1 y_1 + \gamma_{12} x_1 y_2 + \gamma_{21} x_2 y_1 + \gamma_{22} x_2 y_2), \\
& \quad \mathbf{1}_0 + \epsilon (l_1 a_1 + l_2 a_2 + p_{11} x_1 y_1 + p_{12} x_1 y_2 + p_{21} x_2 y_1 + p_{22} x_2 y_2)\}], \\
& \text{Cord}[\text{qo}]
\end{aligned}$$

$$\begin{aligned}
& \mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \\
& \quad (e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 + \\
& \quad e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 + e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{11} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \gamma_{12} - \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \gamma_{12} - \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \tau_1 \gamma_{12} + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \tau_1 \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \tau_1 \gamma_{12} + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \tau_1 \gamma_{12} + \hbar x_1 y_1 \gamma_{12} + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar x_1 y_1 \gamma_{21} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \gamma_{22} - \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \gamma_{22} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \gamma_{22} - \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{11} t_1 \tau_1 \gamma_{22} + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{12} t_1 \tau_1 \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{21} t_2 \tau_1 \gamma_{22} + \\
& \quad e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \hbar a_1 l_{22} t_2 \tau_1 \gamma_{22} + \hbar x_1 y_1 \gamma_{22}) / \\
& \quad (e^{\gamma \hbar l_{11} t_1 + 2 \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + 2 \gamma \hbar l_{22} t_2} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \gamma_{12} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \\
& \quad \tau_1 \gamma_{12} - e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \gamma_{22} + e^{\gamma \hbar l_{11} t_1 + \gamma \hbar l_{12} t_1 + \gamma \hbar l_{21} t_2 + \gamma \hbar l_{22} t_2} \tau_1 \gamma_{22}), \\
& \quad \frac{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2}}{e^{\gamma \hbar l_{12} t_1 + \gamma \hbar l_{22} t_2} - \gamma_{12} + \tau_1 \gamma_{12} - \gamma_{22} + \tau_1 \gamma_{22}} + O[\epsilon]^1]
\end{aligned}$$

Stitching \mathbb{C} 's.

$$\begin{aligned}
& \text{co} = \mathbb{C}_{\text{CU}}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \\
& \quad \{y_3, a_3, x_3\}_3, \hbar \text{Sum}[l_{10 i+j} t_i a_j + \gamma_{10 i+j} y_i x_j, \{i, 3\}, \{j, 3\}], \mathbf{1}_2]; \\
& \{\text{co} // m_{3 \rightarrow 4}, \text{HL}@\text{Simp}[\text{CU}[m_{3 \rightarrow 4}[\text{co}]] - m_{3 \rightarrow 4}[\text{CU}[\text{co}]]]\} \\
& \{\mathbb{C}_{\text{CU}}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \\
& \quad \hbar (a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 + \\
& \quad a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} + \\
& \quad x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}), \mathbf{1} + O[\epsilon]^3], \mathbf{0}\}
\end{aligned}$$

Verifying that m commutes with evaluation, in CU:

$\mathbf{co} = \mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\mathbf{y}_2, \mathbf{a}_2, \mathbf{x}_2\}_2, \{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_3, \hbar \text{Sum}[\mathbf{1}_{10^{i+j}} \mathbf{t}_i \mathbf{a}_j + \gamma_{10^{i+j}} \mathbf{y}_i \mathbf{x}_j, \{\mathbf{i}, 3\}, \{\mathbf{j}, 3\}], \mathbf{1}_2];$
Timing@{ $\mathbf{co} // \mathbf{m}_{2 \rightarrow 3}, \text{HL@Simp}[\text{CU}[\mathbf{m}_{2 \rightarrow 3}[\mathbf{co}]] - \mathbf{m}_{2 \rightarrow 3}[\text{CU}[\mathbf{co}]]]$ }

$$\{513.453, \{\mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_3, \frac{\dots 1 \dots}{e^{-1 \dots} \dots 1 \dots \dots 1 \dots}, \frac{1}{1 + \hbar \mathbf{t}_3 \gamma_{32}} + \left(\left(4 e^{2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \dots 4 \dots} + 2 \gamma \hbar \mathbf{l}_{33} \mathbf{t}_3 \hbar^2 \mathbf{a}_3 \mathbf{x}_1 \mathbf{y}_1 \gamma_{12} \gamma_{31} - 2 \dots 7 \dots \gamma_{31} + \dots 154 \dots \right) \epsilon \right) / \left(2 e^{2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + 2 \gamma \hbar \mathbf{l}_{13} \mathbf{t}_1 + \dots 3 \dots} + 2 \gamma \hbar \mathbf{l}_{33} \mathbf{t}_3 + 10 e^{\dots 1 \dots} \hbar \mathbf{t}_3 \gamma_{32} + \dots 2 \dots + \dots 1 \dots + 2 e^{2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \dots 4 \dots} + \dots 1 \dots \hbar^5 \mathbf{t}_3^5 \gamma_{32}^5 \right) + \frac{(\dots 1 \dots)^2}{\dots 1 \dots} + O[\epsilon]^3, \mathbf{0} \}$$

large output show less show more show all set size limit...

Verifying that m commutes with evaluation, in QU:

$\mathbf{qo} = \mathbb{C}_{\text{QU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\mathbf{y}_2, \mathbf{a}_2, \mathbf{x}_2\}_2, \{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_3, \hbar \text{Sum}[\mathbf{1}_{10^{i+j}} \mathbf{t}_i \mathbf{a}_j + \gamma_{10^{i+j}} \mathbf{y}_i \mathbf{x}_j, \{\mathbf{i}, 3\}, \{\mathbf{j}, 3\}], \mathbf{1}_2];$
Timing@{ $\mathbf{qo} // \mathbf{m}_{2 \rightarrow 3}, \text{HL@SimpT}[\text{QU}[\mathbf{m}_{2 \rightarrow 3}[\mathbf{qo}]] - \mathbf{m}_{2 \rightarrow 3}[\text{QU}[\mathbf{qo}]]]$ }

$$\{7831.47, \{\mathbb{C}_{\text{QU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_3, \frac{\dots 1 \dots}{\dots 1 \dots}, \frac{1}{1 - \gamma_{32} + \mathbf{T}_3 \gamma_{32}} + \left(\left(8 e^{2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \dots 4 \dots} + 2 \gamma \hbar \mathbf{l}_{33} \mathbf{t}_3 \hbar^2 \mathbf{a}_3 \mathbf{T}_3 \mathbf{x}_1 \mathbf{y}_1 \gamma_{12} \gamma_{31} + 4 \dots 8 \dots \gamma_{31} + \dots 371 \dots \right) \epsilon \right) / \left(4 e^{2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + 2 \gamma \hbar \mathbf{l}_{13} \mathbf{t}_1 + \dots 3 \dots} + 2 \gamma \hbar \mathbf{l}_{33} \mathbf{t}_3 - 20 e^{\dots 1 \dots} \gamma_{32} + \dots 26 \dots + 4 e^{2 \gamma \hbar \mathbf{l}_{12} \mathbf{t}_1 + \dots 4 \dots} + \dots 1 \dots \mathbf{T}_3^5 \gamma_{32}^5 \right) + \frac{(\dots 1 \dots)^2}{\dots 79 \dots + \dots 1 \dots} + O[\epsilon]^3, \mathbf{0} \}$$

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Verifying meta-associativity in CU:

$\mathbf{co} = \mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\mathbf{y}_2, \mathbf{a}_2, \mathbf{x}_2\}_2, \{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_3, \hbar \text{Sum}[\lambda_{10^{i+j}} \mathbf{t}_i \mathbf{a}_j + \gamma_{10^{i+j}} \mathbf{y}_i \mathbf{x}_j, \{\mathbf{i}, 3\}, \{\mathbf{j}, 3\}], \mathbf{1}_0];$
Timing@HL[(lhs = $\mathbf{co} // \mathbf{m}_{1,2 \rightarrow 1} // \mathbf{m}_{1,3 \rightarrow 1}$) \equiv (rhs = $\mathbf{co} // \mathbf{m}_{2,3 \rightarrow 2} // \mathbf{m}_{1,2 \rightarrow 1}$)]
 {41.9219, **True**}

$\mathbf{co} = \mathbb{C}_{\text{CU}}[\{\mathbf{y}_1, \mathbf{a}_1, \mathbf{x}_1\}_1, \{\mathbf{y}_2, \mathbf{a}_2, \mathbf{x}_2\}_2, \{\mathbf{y}_3, \mathbf{a}_3, \mathbf{x}_3\}_3, \hbar \text{Sum}[\lambda_{10^{i+j}} \mathbf{t}_i \mathbf{a}_j + \gamma_{10^{i+j}} \mathbf{y}_i \mathbf{x}_j, \{\mathbf{i}, 3\}, \{\mathbf{j}, 3\}], \mathbf{1}_1];$
Timing@HL[(lhs = $\mathbf{co} // \mathbf{m}_{1,2 \rightarrow 1} // \mathbf{m}_{1,3 \rightarrow 1}$) \equiv (rhs = $\mathbf{co} // \mathbf{m}_{2,3 \rightarrow 2} // \mathbf{m}_{1,2 \rightarrow 1}$)]
 {30119.8, **True**}

`co = CCU[{y1, a1, x1}1, {y2, a2, x2}2, h Sum[l10 i+j ti aj + y10 i+j yi xj, {i, 2}, {j, 2}], 11];`
`Short[Simplify /@ (cexample = co // m1->2), 12]`
`Short[Simplify /@ (qexample = (qo = co /. CU -> QU) // m1->2), 12]`

$$\begin{aligned} & \text{CCU} \left[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11} + l_{12} + l_{21} + l_{22}) t_2 + \right. \\ & \frac{1}{1 + \hbar t_2 \gamma_{21}} e^{-\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \left(\gamma_{21} + e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \gamma_{12} (1 + \hbar t_2 \gamma_{21}) + \right. \\ & \left. \left. e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{11} (e^{\gamma \hbar (l_{11} + l_{21}) t_2} - e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar t_2 \gamma_{22}) \right) \right), \\ & \frac{1}{1 + \hbar t_2 \gamma_{21}} + \frac{1}{2 (1 + \hbar t_2 \gamma_{21})^5} e^{-2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar \left(4 a_2 (1 + \hbar t_2 \gamma_{21})^2 \right. \\ & \left. \left(e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} t_2 + x_2 y_2) \gamma_{21}^2 + e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{11} \gamma_{22} + \right. \right. \\ & \left. \left. \gamma_{21} (e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \hbar x_2 y_2 (e^{\gamma \hbar (2 l_{11} + l_{12} + 2 l_{21} + l_{22}) t_2} \gamma_{11} + e^{\gamma \hbar (l_{11} + 2 l_{12} + l_{21} + 2 l_{22}) t_2} \gamma_{22})) \right) \right) - \\ & \left. \gamma \hbar (-2 e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} t_2 \gamma_{21}^2 (1 + \hbar t_2 \gamma_{21})^2 + 4 \ll 5 \gg (\ll 1 \gg) + \right. \\ & \left. \hbar \ll 4 \gg (3 \hbar t_2 \gamma_{21}^2 + 2 e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{21} (4 + e^{\gamma \ll 3 \gg} \hbar t_2 \gamma_{22}) + \right. \\ & \left. \left. e^{\gamma \hbar (l_{11} + l_{21}) t_2} \gamma_{11} (2 + \hbar t_2 (\gamma_{21} - e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22})) \right) \right) \right) \in + O[\epsilon]^2 \end{aligned}$$

$$\begin{aligned} & \text{CQU} \left[\{y_2, a_2, x_2\}_2, \hbar a_2 (l_{11} + l_{12} + l_{21} + l_{22}) t_2 + \right. \\ & \frac{1}{1 + (-1 + T_2) \gamma_{21}} e^{-\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \left(\gamma_{21} + e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \gamma_{12} (1 + (-1 + T_2) \gamma_{21}) + \right. \\ & \left. \left. e^{\gamma \hbar (l_{12} + l_{22}) t_2} \gamma_{22} + \gamma_{11} (e^{\gamma \hbar (l_{11} + l_{21}) t_2} - e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (-1 + T_2) \gamma_{22}) \right) \right), \\ & \frac{1}{1 + (-1 + T_2) \gamma_{21}} + \frac{1}{4 (1 + (-1 + T_2) \gamma_{21})^5} e^{-2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar \\ & \left(8 a_2 T_2 (1 + (-1 + T_2) \gamma_{21})^2 (e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (-e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \right. \\ & \left. e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} T_2 + \hbar x_2 y_2) \gamma_{21}^2 + e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{11} \gamma_{22} + \right. \\ & \left. \gamma_{21} (e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} + \hbar x_2 y_2 (e^{\gamma \hbar (2 l_{11} + l_{12} + 2 l_{21} + l_{22}) t_2} \gamma_{11} + e^{\gamma \hbar (l_{11} + 2 l_{12} + l_{21} + 2 l_{22}) t_2} \gamma_{22})) \right) \right) + \\ & \left. \gamma (2 e^{2 \gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} (1 - 4 T_2 + 3 T_2^2) \gamma_{21}^2 (1 + (-1 + T_2) \gamma_{21})^2 + \right. \\ & \left. 4 e^{\gamma \hbar (l_{11} + l_{12} + l_{21} + l_{22}) t_2} \hbar x_2 y_2 \gamma_{21} (1 + (-1 + T_2) \gamma_{21}) (\ll 1 \gg) - \ll 1 \gg) \right) \right) \in + O[\epsilon]^2 \end{aligned}$$

R in QU.

`Table[Series[eqn,k[x], {e, 0, 4}], {k, 0, 5}] // Column`

$$\begin{aligned} & e^x \\ & e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \in + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \in^2 - \frac{1}{384} (e^x x^2 (-8 + x^4) \gamma^3 \hbar^3) \in^3 + \frac{e^x x^4 (-32 + x^4) \gamma^4 \hbar^4 \in^4}{6144} + O[\epsilon]^5 \\ & e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \in + \frac{1}{288} e^x x^3 (32 + 9 x) \gamma^2 \hbar^2 \in^2 - \\ & \frac{(e^x x^2 (-24 + 32 x^3 + 3 x^4) \gamma^3 \hbar^3) \in^3}{1152} + \frac{e^x x^3 (-4608 - 864 x + 1024 x^3 + 576 x^4 + 27 x^5) \gamma^4 \hbar^4 \in^4}{165888} + O[\epsilon]^5 \\ & e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \in + \frac{1}{288} e^x x^3 (32 + 9 x) \gamma^2 \hbar^2 \in^2 - \\ & \frac{(e^x x^2 (-24 + 72 x^2 + 32 x^3 + 3 x^4) \gamma^3 \hbar^3) \in^3}{1152} + \frac{e^x x^3 (-4608 - 864 x + 3616 x^3 + 576 x^4 + 27 x^5) \gamma^4 \hbar^4 \in^4}{165888} + O[\epsilon]^5 \\ & e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \in + \frac{1}{288} e^x x^3 (32 + 9 x) \gamma^2 \hbar^2 \in^2 - \frac{(e^x x^2 (-24 + 72 x^2 + 32 x^3 + 3 x^4) \gamma^3 \hbar^3) \in^3}{1152} + \frac{1}{4147200} \\ & e^x x^3 (-115200 - 21600 x + 165888 x^2 + 90400 x^3 + 14400 x^4 + 675 x^5) \gamma^4 \hbar^4 \in^4 + O[\epsilon]^5 \\ & e^x - \frac{1}{4} (e^x x^2 \gamma \hbar) \in + \frac{1}{288} e^x x^3 (32 + 9 x) \gamma^2 \hbar^2 \in^2 - \frac{(e^x x^2 (-24 + 72 x^2 + 32 x^3 + 3 x^4) \gamma^3 \hbar^3) \in^3}{1152} + \frac{1}{4147200} \\ & e^x x^3 (-115200 - 21600 x + 165888 x^2 + 90400 x^3 + 14400 x^4 + 675 x^5) \gamma^4 \hbar^4 \in^4 + O[\epsilon]^5 \end{aligned}$$

Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2+q^3+q^4)}\right\}$$

Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
 {1, 1, 1, 1, 1, 1}

QU[R_{3,4}] // Short

$$\text{QU}[] + \frac{\epsilon \hbar \text{QU}[a_3, a_4]}{\gamma} + \hbar \text{QU}[y_3, x_4] + \frac{\epsilon \ll 1 \gg \ll 1 \gg}{\gamma} + \frac{1}{2} \frac{\ll 1 \gg \ll 1 \gg}{\ll 1 \gg} - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon \hbar^2 \ll 1 \gg t_3}{\gamma^2} - \frac{\hbar^2 \text{QU}[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 \text{QU}[a_4, a_4] t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ \$p=4, \$k=2):

QU[R_{1,2} ** R_{1,2}⁻¹] // Simp // HL // Timing

{0.078125, QU[]}

Verifying R3 (~156 secs @ \$p=4, \$k=2):

{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]], HL@SimpT[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]} // Timing

$$\{0.203125, \left\{\text{QU}[] + \frac{\epsilon \hbar \text{QU}[a_1, a_2]}{\gamma} + \frac{\epsilon \hbar \text{QU}[a_1, a_3]}{\gamma} + \ll 73 \gg + 2 \epsilon \hbar^2 \text{QU}[y_1, a_2, x_3] T_2 + \text{QU}[y_1, x_3] (\hbar - \hbar T_2), \mathbf{0}\right\}\}$$

R in \mathbb{C}_{QU} .

{ $\mathbb{C}_{\text{QU},1}[\mathbf{R}_{1,2}]$, $\mathbb{C}_{\text{QU},2}[\mathbf{R}_{1,2}]$ }

$$\left\{\mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + \mathbf{O}[\epsilon]^2], \right. \\ \left. \mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + \frac{1}{288 \gamma^2} (144 \hbar^2 a_1^2 a_2^2 - 72 \gamma^2 \hbar^4 a_1 a_2 x_2^2 y_1^2 + 32 \gamma^4 \hbar^5 x_2^3 y_1^3 + 9 \gamma^4 \hbar^6 x_2^4 y_1^4) \epsilon^2 + \mathbf{O}[\epsilon]^3]\right\}$$

The morphism $\mathbb{C}_{U,k}$.

$\mathbb{C}_{\text{QU},1}[\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6}]$

$$\mathbb{C}_{\text{QU}}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \\ \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar a_2 t_1}{\gamma} - \frac{\hbar a_4 t_3}{\gamma} - \frac{\hbar a_6 t_5}{\gamma} + \hbar x_2 y_1 + \hbar x_4 y_3 + \hbar x_6 y_5, \\ 1 + \left(\frac{\hbar a_1 a_2}{\gamma} + \frac{\hbar a_3 a_4}{\gamma} + \frac{\hbar a_5 a_6}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2 - \frac{1}{4} \gamma \hbar^3 x_4^2 y_3^2 - \frac{1}{4} \gamma \hbar^3 x_6^2 y_5^2\right) \epsilon + \mathbf{O}[\epsilon]^2]$$

$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{1,3 \rightarrow 1} // \mathbf{m}_{2,5 \rightarrow 2} // \mathbf{m}_{4,6 \rightarrow 4}]$

$$\mathbb{C}_{\text{QU}} \left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma} \right. \\ \left. \left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_2} \gamma \hbar x_4 y_1 - \gamma \hbar T_2 x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2 \right), \right. \\ \left. 1 + \frac{1}{4\gamma} \left(4 \hbar a_1 a_2 + 4 \hbar a_1 a_4 + 4 \hbar a_2 a_4 - 4 \gamma \hbar^2 a_4 x_2 y_1 - 8 e^{\hbar t_2} \gamma \hbar^2 a_2 x_4 y_1 + \right. \right. \\ \left. \left. 8 \gamma \hbar^2 a_2 T_2 x_4 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 + 4 e^{\hbar t_2} \gamma^2 \hbar^3 x_2 x_4 y_1^2 - 4 \gamma^2 \hbar^3 T_2 x_2 x_4 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - \right. \right. \\ \left. \left. e^{2\hbar t_2} \gamma^2 \hbar^3 x_4^2 y_1^2 + \gamma^2 \hbar^3 T_2^2 x_4^2 y_1^2 - 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 + \right. \right. \\ \left. \left. 4 e^{\hbar t_1 + \hbar t_2} \gamma^2 \hbar^3 x_4^2 y_1 y_2 - 4 e^{\hbar t_1} \gamma^2 \hbar^3 T_2 x_4^2 y_1 y_2 - e^{2\hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2 \right) \in + \mathcal{O}[\epsilon]^2 \right]$$

$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{3,1 \rightarrow 1} // \mathbf{m}_{5,2 \rightarrow 2} // \mathbf{m}_{6,4 \rightarrow 4}]$

$$\mathbb{C}_{\text{QU}} \left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma} \right. \\ \left. \left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2 \right), \right. \\ \left. 1 + \frac{1}{4\gamma} \left(4 \hbar a_1 a_2 + 4 \hbar a_1 a_4 + 4 \hbar a_2 a_4 - 4 \gamma \hbar^2 a_4 x_2 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - \right. \right. \\ \left. \left. 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 - e^{2\hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2 \right) \in + \mathcal{O}[\epsilon]^2 \right]$$

$\mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{1,3 \rightarrow 1} // \mathbf{m}_{2,5 \rightarrow 2} // \mathbf{m}_{4,6 \rightarrow 4}] \equiv \mathbb{C}_{\text{QU},1} [\mathbf{R}_{1,2} \mathbf{R}_{3,4} \mathbf{R}_{5,6} // \mathbf{m}_{3,1 \rightarrow 1} // \mathbf{m}_{5,2 \rightarrow 2} // \mathbf{m}_{6,4 \rightarrow 4}]$

$$\hbar \left(e^{\hbar t_2} - T_2 \right) x_4 y_1 = \theta \&\& \in \hbar \left(e^{\hbar t_2} - T_2 \right) x_4 y_1 \left(8 a_2 + \gamma \hbar \left(-4 x_2 y_1 + x_4 \left(\left(e^{\hbar t_2} + T_2 \right) y_1 - 4 e^{\hbar t_1} y_2 \right) \right) \right) = \theta$$

Exponentials as needed.

```
In[ ]:= Block[{$p = 2, $k = 2}, TableForm[StringSplit[
  "y | a | x | C@y_CU | C@a_CU | C@x_CU | Q@y_QU | Q@a_QU | Q@x_QU | AD@y_QU | AD@a_QU | AD@x_QU | SD@y_QU | SD@a_QU | SD
  @x_QU | S@y_CU | S@a_CU | S@x_CU | S@y_QU | S@a_QU | S@x_QU | Δ@y_CU | Δ@a_CU | Δ@x_CU | Δ@y_QU | Δ@a_QU | Δ@x_QU",
  "|"] /. s_String ->
  {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU -> Times, {ε, 0, $k}]]]]
```

Out[]//TableForm=

y	y
a	a
x	x
C@y_CU	-x
C@a_CU	-a
C@x_CU	-y
Q@y_QU	$-\frac{x}{\sqrt{t}} - \frac{ax\epsilon\hbar}{\sqrt{t}} - \frac{a^2x\epsilon^2\hbar^2}{2\sqrt{t}}$
Q@a_QU	-a
Q@x_QU	$-\frac{y}{\sqrt{t}} + \frac{y(-a+\gamma)\epsilon\hbar}{\sqrt{t}} - \frac{y(a-\gamma)^2\epsilon^2\hbar^2}{2\sqrt{t}}$
AD@y_QU	$\frac{2}{3}a^2y\epsilon^2\hbar^2 + \frac{1}{6}y(6+3t\hbar+t^2\hbar^2) + \frac{1}{12}y\epsilon\hbar(xy\gamma\hbar - 4a(3+2t\hbar))$
AD@a_QU	a
AD@x_QU	x
SD@y_QU	$y + \frac{1}{48}t^2y\hbar^2 + \frac{1}{24}y(-2at+xy\gamma)\epsilon\hbar^2 + \frac{1}{12}a^2y\epsilon^2\hbar^2$
SD@a_QU	a
SD@x_QU	$\frac{7}{12}a^2x\epsilon^2\hbar^2 + x\left(1 + \frac{t\hbar}{2} + \frac{7t^2\hbar^2}{48}\right) + \frac{1}{24}x\epsilon\hbar(xy\gamma\hbar - 2a(12+7t\hbar))$
S@y_CU	-y
S@a_CU	-a
S@x_CU	-x
S@y_QU	$-\frac{y}{t} + \frac{y(-a+\gamma)\epsilon\hbar}{t} - \frac{y(a-\gamma)^2\epsilon^2\hbar^2}{2t}$
S@a_QU	-a
S@x_QU	$-x - a\epsilon\hbar - \frac{1}{2}a^2x\epsilon^2\hbar^2$
Δ@y_CU	y ₁ + y ₂
Δ@a_CU	a ₁ + a ₂
Δ@x_CU	x ₁ + x ₂
Δ@y_QU	$y_1 + T_1 y_2 - \epsilon\hbar a_1 T_1 y_2 + \frac{1}{2}\epsilon^2\hbar^2 a_1^2 T_1 y_2$
Δ@a_QU	a ₁ + a ₂
Δ@x_QU	$x_1 + x_2 - \epsilon\hbar a_1 x_2 + \frac{1}{2}\epsilon^2\hbar^2 a_1^2 x_2$

```
In[*]:= Timing@Block[{ $p = 4, $k = 2}, {
  s = S1[QU[y1]] /. QU → Times,
  exps = ExpQu1,$k[η, s], (* Warning: wrong unless $p≥$k+1! *)
  HL@Simp[S1@OQu[{y1}1, SS[e^h η y1]] - QU@(exps /. η → h η)]
}]
```

$$\text{Out[*]} = \left\{ 35.8281, \left\{ a_1 \left(-\frac{\epsilon \hbar}{T_1} + \frac{\gamma \epsilon^2 \hbar^2}{T_1} \right) y_1 + \left(-\frac{1}{T_1} + \frac{\gamma \epsilon \hbar}{T_1} - \frac{\gamma^2 \epsilon^2 \hbar^2}{2 T_1} \right) y_1 - \frac{\epsilon^2 \hbar^2 a_1^2 y_1}{2 T_1}, \right. \right.$$

$$\mathbb{E}_{\text{Qu}} \left[\{y_1, a_1, x_1\}_1, -\frac{\eta y_1}{T_1}, 1 + \frac{(2 \gamma \eta \hbar T_1 y_1 - 2 \eta \hbar a_1 T_1 y_1 - \gamma \eta^2 \hbar y_1^2) \epsilon}{2 T_1^2} + \right.$$

$$\left. \left(-\frac{\gamma^2 \eta \hbar^2 y_1}{2 T_1} + \frac{\gamma \eta \hbar^2 a_1 y_1}{T_1} - \frac{\eta \hbar^2 a_1^2 y_1}{2 T_1} + \frac{7 \gamma^2 \eta^2 \hbar^2 y_1^2}{4 T_1^2} - \frac{2 \gamma \eta^2 \hbar^2 a_1 y_1^2}{T_1^2} + \right. \right.$$

$$\left. \left. \frac{\eta^2 \hbar^2 a_1^2 y_1^2}{2 T_1^2} - \frac{\gamma^2 \eta^3 \hbar^2 y_1^3}{T_1^3} + \frac{\gamma \eta^3 \hbar^2 a_1 y_1^3}{2 T_1^3} + \frac{\gamma^2 \eta^4 \hbar^2 y_1^4}{8 T_1^4} \right) \epsilon^2 + \mathcal{O}[\epsilon^3], \mathbf{0} \right\}$$

```
In[*]:= Timing@Block[{ $p = 4, $k = 2}, {
  s = S1[QU[a1]] /. QU → Times,
  exps = ExpQu1,$k[α, s], (* Warning: wrong unless $p≥$k+1! *)
  HL@Simp[S1@OQu[{a1}1, SS[e^h α a1]] - QU@(exps /. α → h α)]
}]
```

$$\text{Out[*]} = \left\{ 33.5938, \left\{ -a_1, \mathbb{E}_{\text{Qu}} \left[\{y_1, a_1, x_1\}_1, -\alpha a_1, 1 + \mathcal{O}[\epsilon^3], \mathbf{0} \right] \right\} \right\}$$

```
In[*]:= Timing@Block[{ $p = 4, $k = 2}, {
  s = S1[QU[x1]] /. QU → Times,
  exps = ExpQu1,$k[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
  HL@Simp[S1@OQu[{x1}1, SS[e^h ξ x1]] - QU@(exps /. ξ → h ξ)]
}]
```

$$\text{Out[*]} = \left\{ 34.0625, \left\{ -x_1 - \epsilon \hbar a_1 x_1 - \frac{1}{2} \epsilon^2 \hbar^2 a_1^2 x_1, \right. \right.$$

$$\mathbb{E}_{\text{Qu}} \left[\{y_1, a_1, x_1\}_1, -\xi x_1, 1 + \left(-\xi \hbar a_1 x_1 - \frac{1}{2} \gamma \xi^2 \hbar x_1^2 \right) \epsilon + \left(-\frac{1}{2} \xi \hbar^2 a_1^2 x_1 + \frac{1}{4} \gamma^2 \xi^2 \hbar^2 x_1^2 - \right. \right.$$

$$\left. \left. \gamma \xi^2 \hbar^2 a_1 x_1^2 + \frac{1}{2} \xi^2 \hbar^2 a_1^2 x_1^2 - \frac{1}{2} \gamma^2 \xi^3 \hbar^2 x_1^3 + \frac{1}{2} \gamma \xi^3 \hbar^2 a_1 x_1^3 + \frac{1}{8} \gamma^2 \xi^4 \hbar^2 x_1^4 \right) \epsilon^2 + \mathcal{O}[\epsilon^3], \mathbf{0} \right\}$$

$$S(e^{\eta y} e^{\alpha a} e^{\xi x})$$

```
In[ ]:= Timing@Block[{ $p = 3, $k = 1}, {
  (sexp = m_{3,2,1 \to 1}[Exp_{QU,1,$k}[\eta, S_1[QU[y_1]] /. QU \to Times] Exp_{QU,2,$k}[\alpha, S_2[QU[a_2]] /. QU \to Times]
    Exp_{QU,3,$k}[\xi, S_3[QU[x_3]] /. QU \to Times]]) /. u_{-1} \to u,
  HL@SimpT[QU@(sexp /. {\eta \to \hbar \eta, \alpha \to \hbar \alpha, \xi \to \hbar \xi}) -
    S_1@O_{QU}[\{y_1, a_1, x_1\}_1, SS[e^{\hbar(\eta y_1 + \alpha a_1 + \xi x_1)}]]]]
}]
```

$$\text{Out[]} = \{15.2969, \{ \mathbb{E}_{QU}[\{y_1, a_1, x_1\}, \frac{1}{T \hbar} (e^{\alpha \gamma} \eta \xi - e^{\alpha \gamma} T \eta \xi - a T \alpha \hbar - e^{\alpha \gamma} y \eta \hbar - e^{\alpha \gamma} T x \xi \hbar),$$

$$1 + \frac{1}{4 T^2 \hbar} (-3 e^{2 \alpha \gamma} \gamma \eta^2 \xi^2 + 4 e^{2 \alpha \gamma} T \gamma \eta^2 \xi^2 - e^{2 \alpha \gamma} T^2 \gamma \eta^2 \xi^2 + 8 a e^{\alpha \gamma} T \eta \xi \hbar - 4 e^{\alpha \gamma} T \gamma \eta \xi \hbar +$$

$$4 e^{\alpha \gamma} T^2 \gamma \eta \xi \hbar + 6 e^{2 \alpha \gamma} y \gamma \eta^2 \xi \hbar - 2 e^{2 \alpha \gamma} T y \gamma \eta^2 \xi \hbar + 6 e^{2 \alpha \gamma} T x \gamma \eta \xi^2 \hbar -$$

$$2 e^{2 \alpha \gamma} T^2 x \gamma \eta \xi^2 \hbar - 4 a e^{\alpha \gamma} T y \eta \hbar^2 + 4 e^{\alpha \gamma} T y \gamma \eta \hbar^2 - 2 e^{2 \alpha \gamma} y^2 \gamma \eta^2 \hbar^2 -$$

$$4 a e^{\alpha \gamma} T^2 x \xi \hbar^2 - 4 e^{2 \alpha \gamma} T x y \gamma \eta \xi \hbar^2 - 2 e^{2 \alpha \gamma} T^2 x^2 \gamma \xi^2 \hbar^2) \epsilon + O[\epsilon]^2], \mathbf{0} \}$$

$$\Delta_{1 \to 1,2}(e^{\eta y_1} e^{\alpha a_1} e^{\xi x_1})$$

```
Timing@Block[{ $p = 4, $k = 2}, {
  sexp = m_{1,3,5 \to 1}@m_{2,4,6 \to 2}@Times[ (* Warning: wrong unless $p > $k + 1! *)
    Prepend[\{y_2\}_2]@Exp_{QU,1,$k}[\eta, \Delta_{1 \to 1,2}[QU[y_1]] /. QU \to Times],
    Prepend[\{a_4\}_4]@Exp_{QU,3,$k}[\alpha, \Delta_{3 \to 3,4}[QU[a_3]] /. QU \to Times],
    Prepend[\{x_6\}_6]@Exp_{QU,5,$k}[\xi, \Delta_{5 \to 5,6}[QU[x_5]] /. QU \to Times]
  ] /. {\eta \to \hbar \eta, \alpha \to \hbar \alpha, \xi \to \hbar \xi},
  HL@SimpT[QU@sexp - \Delta_{1 \to 1,2}@O_{QU}[\{y_1, a_1, x_1\}_1, SS[e^{\hbar(\eta y_1 + \alpha a_1 + \xi x_1)}]]]]
}]
```

$$\text{Out[]} = \{162., \{ \mathbb{E}_{QU}[\{y_2, a_2, x_2\}_2, \{y_1, a_1, x_1\}_1, \alpha \hbar a_1 + \alpha \hbar a_2 + \xi \hbar x_1 + \xi \hbar x_2 + \eta \hbar y_1 + \eta \hbar T_1 y_2,$$

$$1 + \frac{1}{2} (-2 \xi \hbar^2 a_1 x_2 + \gamma \xi^2 \hbar^3 x_1 x_2 - 2 \eta \hbar^2 a_1 T_1 y_2 + \gamma \eta^2 \hbar^3 T_1 y_1 y_2) \epsilon +$$

$$\frac{1}{24} (12 \xi \hbar^3 a_1^2 x_2 + 6 \gamma^2 \xi^2 \hbar^4 x_1 x_2 - 12 \gamma \xi^2 \hbar^4 a_1 x_1 x_2 + 4 \gamma^2 \xi^3 \hbar^5 x_1^2 x_2 + 12 \xi^2 \hbar^4 a_1^2 x_2^2 +$$

$$4 \gamma^2 \xi^3 \hbar^5 x_1 x_2^2 - 12 \gamma \xi^3 \hbar^5 a_1 x_1 x_2^2 + 3 \gamma^2 \xi^4 \hbar^6 x_1^2 x_2^2 + 12 \eta \hbar^3 a_1^2 T_1 y_2 +$$

$$24 \eta \xi \hbar^4 a_1^2 T_1 x_2 y_2 - 12 \gamma \eta \xi^2 \hbar^5 a_1 T_1 x_1 x_2 y_2 + 6 \gamma^2 \eta^2 \hbar^4 T_1 y_1 y_2 - 12 \gamma \eta^2 \hbar^4 a_1 T_1 y_1 y_2 -$$

$$12 \gamma \eta^2 \xi \hbar^5 a_1 T_1 x_2 y_1 y_2 + 6 \gamma^2 \eta^2 \xi^2 \hbar^6 T_1 x_1 x_2 y_1 y_2 + 4 \gamma^2 \eta^3 \hbar^5 T_1 y_1^2 y_2 + 12 \eta^2 \hbar^4 a_1^2 T_1^2 y_2^2 +$$

$$4 \gamma^2 \eta^3 \hbar^5 T_1^2 y_1 y_2^2 - 12 \gamma \eta^3 \hbar^5 a_1 T_1^2 y_1 y_2^2 + 3 \gamma^2 \eta^4 \hbar^6 T_1^2 y_1^2 y_2^2) \epsilon^2 + O[\epsilon]^3], \mathbf{0} \}$$

Zip and Bind

QZip implements the “Q-level zips” on $E(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

```
In[*]:= Timing@{E0 = E[0, Sum[a_{10 i+j} x_i \xi_j, {i, 3}, {j, 3}],
  1 + e Sum[f_i[x_1, x_2, x_3] \xi_i, {i, 3}] + e Sum[f_{10 i+j}[x_1, x_2, x_3] \xi_i \xi_j, {i, 3}, {j, 3}]],
  lhs = QZip[\xi_1, \xi_2]@E0,
  HL[lhs == QZip[\xi_1]@QZip[\xi_2]@E0]}
```

```
Out[*]:= {38.6875, {E[0, ..., 1 + e (\xi_1 ... + ... + ...) +
  e (\xi_1^2 f_{11}[x_1, x_2, x_3] + ... + \xi_3^2 f_{33}[x_1, x_2, x_3])], ..., True}}
```

large output show less show more show all set size limit...

```
In[*]:= Timing@{
  Eh = E[0, h Sum[a_{10 i+j} x_i \xi_j, {i, 3}, {j, 3}],
  1 + e Sum[f_i[x_1, x_2, x_3] \xi_i, {i, 3}] + e Sum[f_{10 i+j}[x_1, x_2, x_3] \xi_i \xi_j, {i, 3}, {j, 3}]],
  lhs = Normal[Eh /. E[L_, Q_, P_] := Series[P e^{L+Q}, {h, 0, 2}]] // Zip[\xi_1],
  HL@Simplify[lhs == Normal[QZip[\xi_1][Eh] /. E[L_, Q_, P_] := Series[P e^{L+Q}, {h, 0, 2}]]]}
```

```
Out[*]:= {18.4375, {E[0, h ..., 1 + e (\xi_1 ... + ... + ...) +
  e (\xi_1^2 f_{11}[x_1, x_2, x_3] + ... + \xi_3^2 f_{33}[x_1, x_2, x_3])], ..., True}}
```

large output show less show more show all set size limit...

LZip implements the “L-level zips” on $E(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single “P”. Here the z’s are t and α and the ζ ’s are τ and a .

```
In[*]:= Bind_{2}[E[0, \xi (x_1 + x_2), 1], E[0, \xi_2 (x_2 + x_3), 1]]
```

```
Out[*]:= E[0, \xi (x_1 + x_2 + x_3), 1]
```

```
In[*]:= Bind_{2}[E[0, (\xi_2 + \xi_3) x_2, 1], E[0, (\xi_1 + \xi_2) x, 1]]
```

```
Out[*]:= E[0, x (\xi_1 + \xi_2 + \xi_3), 1]
```

```
In[*]:= Bind_{1,2}[E[0, (\xi_2 + \xi_3) x_2 + \xi_1 x_1, 1], E[0, (\xi_1 + \xi_2) x, 1]]
```

```
Out[*]:= E[0, x (\xi_1 + \xi_2 + \xi_3), 1]
```

An $xy \rightarrow axy \rightarrow ayx \rightarrow yax \equiv xay \rightarrow xya \rightarrow yxa \rightarrow yax$ test:

```
In[*]:= Bind[E[\alpha_1 a_1 + \tau_1 t_1, e^{\gamma \alpha_1} \xi_1 x_1 + \eta_1 y_1, 1], {1}, E[\tau_1 t_1 + \alpha_1 a_1, \xi_1 x_1 + \eta_1 y_1 + \xi_1 \eta_1 t_1, 1]]
```

```
Out[*]:= E[a_1 \alpha_1 + t_1 \tau_1, y_1 \eta_1 + e^{\gamma \alpha_1} (x_1 + t_1 \eta_1) \xi_1, 1]
```

```
In[*]:= Column@{Cord[C_{Cu}[\{x_1, a_1\}_1, \xi_1 x_1 + \alpha_1 a_1, 1 + \theta_0]],
  Cord[C_{Cu}[\{x_1, y_1\}_1, \xi_1 x_1 + \eta_1 y_1, 1 + \theta_0]],
  Cord[C_{Cu}[\{a_1, y_1\}_1, \alpha_1 a_1 + \eta_1 y_1, 1 + \theta_0]]}
```

```
Out[*]:= C_{Cu}[\{a_1, x_1\}_1, e^{-\gamma \alpha_1} (e^{\gamma \alpha_1} a_1 \alpha_1 + x_1 \xi_1), 1 + O[\epsilon]^1]
  C_{Cu}[\{y_1, a_1, x_1\}_1, y_1 \eta_1 + x_1 \xi_1 - t_1 \eta_1 \xi_1, 1 + O[\epsilon]^1]
  C_{Cu}[\{y_1, a_1\}_1, e^{-\gamma \alpha_1} (e^{\gamma \alpha_1} a_1 \alpha_1 + y_1 \eta_1), 1 + O[\epsilon]^1]
```



```

In[ ]:= { rxa = E [ t1 t1 + a1 a1, e^-gamma a1 xi1 x1 + eta1 y1, 1 ];
          rxy = E [ t1 t1 + a1 a1, xi1 x1 + eta1 y1 - xi1 eta1 t1, 1 ];
          ray = E [ t1 t1 + a1 a1, e^-gamma a1 eta1 y1 + xi1 x1, 1 ];
          lhs = Expand /@ Bind[ rxa, {1}, rxy, {1}, ray ],
          HL[ lhs == Expand /@ Bind[ ray, {1}, rxy, {1}, rxa ] ] }
Out[ ]:= { E [ a1 a1 + t1 t1, e^-gamma a1 y1 eta1 + e^-gamma a1 x1 xi1 - e^-gamma a1 t1 eta1 xi1, 1 ], True }

In[ ]:= Simplify /@ m_{i,j,k} @ CCU [ { y_i, a_i, x_i }_i, { y_j, a_j, x_j }_j, eta_i y_i + alpha_i a_i + xi_i x_i + eta_j y_j + alpha_j a_j + xi_j x_j, 1 + theta_1 ]
Out[ ]:= CCU [ { y_k, a_k, x_k }_k, a_k (alpha_i + alpha_j) + y_k (eta_i + e^-gamma a_i eta_j) + e^-gamma a_j x_k xi_i - t_k eta_j xi_i + x_k xi_j,
              1 + 1/2 eta_j xi_i (4 a_k - 2 e^-gamma a_i gamma y_k eta_j + gamma (-2 e^-gamma a_j x_k + t_k eta_j) xi_i) epsilon + O[epsilon]^2 ]

```

Tensorial Representations

Associativity of tm.

```

In[ ]:= Table[Block[{ $U = U, $k = kk },
                  { lhs = Bind[ tm_{1,2->2}, {2}, tm_{2,3->1} ];
                    { $U, $k } -> HL[ lhs == Bind[ tm_{2,3->2}, {2}, tm_{1,2->1} ] ] }
                ], { U, { CU, QU } }, { kk, 0, 1 } ]
Out[ ]:= { { { { CU, 0 } -> True }, { { CU, 1 } -> True } }, { { { QU, 0 } -> True }, { { QU, 1 } -> True } } }

In[ ]:= Block[ { $U = CU, $k = 2 }, Timing@ { lhs = Bind[ tm_{1,2->2}, {2}, tm_{2,3->1} ];
              HL[ lhs == Bind[ tm_{2,3->2}, {2}, tm_{1,2->1} ] ] } ]
Out[ ]:= { 12.1094, { True } }

```

tS is an anti-homomorphism for tm.

```

In[ ]:= HL [ ( tS_1 tS_2 ) ~ B_{1,2} ~ tm_{1,2->1} == tm_{2,1->1} ~ B_1 ~ tS_1 ]
Out[ ]:= True

```

Testing co-associativity.

```

In[ ]:= HL [ tDelta_{1->1,2} ~ B_2 ~ tDelta_{2->2,3} == tDelta_{1->1,3} ~ B_1 ~ tDelta_{1->1,2} ]
Out[ ]:= True

```

Testing S is an anti-co-homomorphism

```

In[ ]:= HL [ tS_1 ~ B_1 ~ tDelta_{1->1,2} == tDelta_{1->2,1} ~ B_{1,2} ~ ( tS_1 tS_2 ) ]
Out[ ]:= True

```

Testing convolution inverse:

```

In[ ]:= { HL [ tDelta_{1->1,2} ~ B_1 ~ tS_1 ~ B_{1,2} ~ tm_{1,2->1} == teta ~ B_{ } ~ t1 ],
          HL [ tDelta_{1->1,2} ~ B_2 ~ tS_2 ~ B_{1,2} ~ tm_{1,2->1} == teta ~ B_{ } ~ t1 ] }
Out[ ]:= { True, True }

```

Testing R2

In[]:= **HL** [($\overline{\mathbf{tR}}_{1,2}$ $\mathbf{tR}_{3,4}$) $\sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{tm}_{1,3 \rightarrow 1}$ $\mathbf{tm}_{2,4 \rightarrow 2}) \equiv \mathbf{t1}$]

Out[]:= **True**

Testing quasi-triangular axioms

In[]:= **HL** [($\mathbf{t}\Delta_{1 \rightarrow 1,2}$ $\mathbf{tR}_{3,4}$) $\sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{tm}_{1,3 \rightarrow 1}$ $\mathbf{tm}_{2,4 \rightarrow 2}) \equiv (\mathbf{t}\Delta_{1 \rightarrow 2,1}$ $\mathbf{tR}_{3,4}$) $\sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{tm}_{3,1 \rightarrow 1}$ $\mathbf{tm}_{4,2 \rightarrow 2})$]

Out[]:= **True**

In[]:= **HL** [$\mathbf{tR}_{1,3} \sim \mathbf{B}_1 \sim \mathbf{t}\Delta_{1 \rightarrow 1,2} \equiv (\mathbf{tR}_{1,4}$ $\mathbf{tR}_{2,3}) \sim \mathbf{B}_{3,4} \sim \mathbf{tm}_{3,4 \rightarrow 3}$]

Out[]:= **True**

Testing R3

In[]:= **HL** [($\mathbf{tR}_{2,3}$ $\mathbf{tR}_{1,4}$ $\mathbf{tR}_{5,6}$) $\sim \mathbf{B}_{\text{Range@6}} \sim (\mathbf{tm}_{1,5 \rightarrow 1}$ $\mathbf{tm}_{2,6 \rightarrow 2}$ $\mathbf{tm}_{3,4 \rightarrow 3}) \equiv$
 ($\mathbf{tR}_{1,2}$ $\mathbf{tR}_{5,3}$ $\mathbf{tR}_{6,4}$) $\sim \mathbf{B}_{\text{Range@6}} \sim (\mathbf{tm}_{1,5 \rightarrow 1}$ $\mathbf{tm}_{2,6 \rightarrow 2}$ $\mathbf{tm}_{3,4 \rightarrow 3})$]

Out[]:= **True**

tC is the counterclockwise spinner; $\overline{\mathbf{tC}}$ is its inverse:

In[]:= **Block** [{ $\$k = 1$ }, **HL** [(\mathbf{tC}_1 $\overline{\mathbf{tC}}_2$) $\sim \mathbf{B}_{1,2} \sim \mathbf{tm}_{1,2 \rightarrow 1} \equiv \mathbf{t1}$]]

Out[]:= **True**

Cyclic R2 as on 180419 blackboard:

In[]:= **Block** [{ $\$k = 2$ }, **HL** [($\mathbf{tR}_{1,4}$ $\overline{\mathbf{tR}}_{5,2}$ $\overline{\mathbf{tC}}_3$) $\sim \mathbf{B}_{\{1,3,2,4\}} \sim (\mathbf{tm}_{1,3 \rightarrow 1}$ $\mathbf{tm}_{2,4 \rightarrow 2}) \sim \mathbf{B}_{1,5} \sim \mathbf{tm}_{1,5 \rightarrow 1} \equiv \overline{\mathbf{tC}}_1$]] // **Timing**

Out[]:= { 22.6094, **True** }

Swirl relation as on 180419 blackboard:

In[]:= **Block** [{ $\$k = 1$ },
HL [$\mathbf{tR}_{1,2} \equiv (\mathbf{tC}_1$ \mathbf{tC}_2 $\mathbf{tR}_{3,4}$ $\overline{\mathbf{tC}}_5$ $\overline{\mathbf{tC}}_6$) $\sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{tm}_{1,3 \rightarrow 1}$ $\mathbf{tm}_{2,4 \rightarrow 2}) \sim \mathbf{B}_{1,2,5,6} \sim (\mathbf{tm}_{1,5 \rightarrow 1}$ $\mathbf{tm}_{2,6 \rightarrow 2})$]] // **Timing**

Out[]:= { 1.09375, **True** }

The Four Kinks

```

In[ ]:= Block[{$k = 2, K1, K2, K3, K4},
  Column@{K1 = (tR1,3 tC2) ~ B1,2 ~ tm1,2→1 ~ B1,3 ~ tm1,3→1, K2 = (tR3,1 tC2) ~ B1,2 ~ tm1,2→1 ~ B1,3 ~ tm1,3→1,
    K3 = (tR3,1 tC2) ~ B1,2 ~ tm1,2→1 ~ B1,3 ~ tm1,3→1, K4 = (tR1,3 tC2) ~ B1,2 ~ tm1,2→1 ~ B1,3 ~ tm1,3→1,
    HL /@ {
      K1 ≡ tKink1, K3 ≡ tKink1,
      K1 ≡ K2, K3 ≡ K4,
      (K1 (K3 ~ B1,2 ~ tm1,2→2)) ~ B1,2 ~ tm1,2→1 ≡ t1,
      K1 ~ B1 ~ tS1 ≡ K1, K3 ~ B1 ~ tS1 ≡ K3}
    ]
E[-(ħ a1 t1)/γ, ħ x1 y1,
  1/√T1 + (4 γ ħ a1 + 4 ħ a1^2 - γ^2 ħ^3 x1^2 y1^2) ε + 1/(288 γ^2 √T1) (144 γ^2 ħ^2 a1^2 + 288 γ ħ^2 a1^3 + 144 ħ^2 a1^4 - 72 γ^3 ħ^4 a1 x1^2 y1^2 -
    72 γ^2 ħ^4 a1^2 x1^2 y1^2 + 32 γ^4 ħ^5 x1^3 y1^3 + 9 γ^4 ħ^6 x1^4 y1^4) ε^2 + O[ε]^3]
E[-(ħ a1 t1)/γ, ħ x1 y1,
  1/√T1 + (4 γ ħ a1 + 4 ħ a1^2 - γ^2 ħ^3 x1^2 y1^2) ε + 1/(288 γ^2 √T1) (144 γ^2 ħ^2 a1^2 + 288 γ ħ^2 a1^3 + 144 ħ^2 a1^4 - 72 γ^3 ħ^4 a1 x1^2 y1^2 -
    72 γ^2 ħ^4 a1^2 x1^2 y1^2 + 32 γ^4 ħ^5 x1^3 y1^3 + 9 γ^4 ħ^6 x1^4 y1^4) ε^2 + O[ε]^3]
E[(ħ a1 t1)/γ, -(ħ x1 y1)/T1, √T1 + (-4 γ ħ a1 T1^2 - 4 ħ a1^2 T1^2 - 8 γ ħ^2 a1 T1 x1 y1 - 3 γ^2 ħ^3 x1^2 y1^2) ε + O[ε]^2]
E[(ħ a1 t1)/γ, -(ħ x1 y1)/T1, √T1 + (-4 γ ħ a1 T1^2 - 4 ħ a1^2 T1^2 - 8 γ ħ^2 a1 T1 x1 y1 - 3 γ^2 ħ^3 x1^2 y1^2) ε + O[ε]^2]
{True, True, True, True, True, True, True}

```

Trefoil as on 180419 blackboard:

```

In[ ]:= Timing[
  Z = tR1,5 tR6,2 tR3,7 tC4 tKink8 tKink9 tKink10;
  Do[Z; Z = Z ~ B1,k ~ tm1,k→1, {k, 2, 10}];
  Z]
Out[ ]:= {98.5313,
  E[0, 0, T1/(1 - T1 + T1^2) + ((-2 ħ a1 T1 - γ ħ T1^2 + 2 ħ a1 T1^2 + 2 γ ħ T1^3 - 3 γ ħ T1^4 - 2 ħ a1 T1^4 + 2 γ ħ T1^5 + 2 ħ a1 T1^5 -
    2 γ ħ^2 T1 x1 y1 - 2 γ ħ^2 T1^4 x1 y1) ε) / (1 - 3 T1 + 6 T1^2 - 7 T1^3 + 6 T1^4 - 3 T1^5 + T1^6) + O[ε]^2]}

```

Alternative Algorithms

```

In[ ]:= {λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[Δcu,2[{ξ, η}, {x, y}]]]}
Out[ ]:= {1 + (2 a η ξ - y γ η^2 ξ - x γ η ξ^2 + 1/2 t γ η^2 ξ^2) ε +
  1/2 ((2 a η ξ - y γ η^2 ξ - x γ η ξ^2 + 1/2 t γ η^2 ξ^2)^2 + 2 (-a γ η^2 ξ^2 + y γ^2 η^3 ξ^2 + x γ^2 η^2 ξ^3 - 1/3 t γ^2 η^3 ξ^3))
  ε^2 + O[ε]^3, True}

```