IMPLEMENTING THE QUANTUM sl_2 PORTFOLIO OF OPERATIONS

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Abstract. Building up from some new or lightly used theoretical tools, especially "solvable approximation" and "Gaussian differential operators", we give a clean and efficient computer implementation of the quantum sl_2 portfolio of operations. Beyond the theoretical interest and the satisfaction that one obtains when complicated formulas come to life, become specific, and check, we explain (and implement and prove) why our results are valuable in knot theory.

We mean business! Page ?? displays a program which is a complete implementation of the quantum sl_2 portfolio of operations. Page ?? displays a variant of that program tailored to efficiently compute the "Rozansky-Overbay" invariants. Appendix A contains a tabulation of some of these invariants on knots with up to 10 crossings. Much more is at ωεβ/:=http://drorbn.net/SL2PO/.

Contents

1. Introduction

In Section 1.1 of the introduction we briefly and schematically recall how certain algebras lead to knot invariants, only so as to explain what exactly it is that we aim to implement and why. Section 1.2 of the introduction is the abstract of this paper, expanded from one

computations below

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paragraph to a few pages. Section 1.3 of the introduction is an introduction to the rest of the paper — a summary of what happens in it, and in what order.

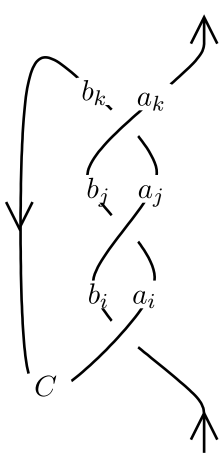
1.1. **A Quick Reminder of Algebras and R-matrices.** A “Hopf Algebra” is a vector space $(U, +, \cdot)$ (over $\mathbb{Q}$, for simplicity) along with a number of further operations: a “product” $m\colon U \otimes U \to U$, a “coproduct” $\Delta\colon U \to U \otimes U$, an “antipode” $S\colon U \to U$, a “unit” $\eta\colon \mathbb{Q} \to U$ and a “counit” $\epsilon\colon U \to \mathbb{Q}$ (which of course are required to satisfy some axioms). If U is also equipped with a “braiding” $R \in U \otimes U$ and a “cuap element” $C \in U$ and these satify a few further axioms, then U is a “ribbon Hopf algebra”. It is sometimes (but not always) useful to add to the mix a “pairing element” $P \in U^* \otimes U^*$, which is dual to the element R.

Ribbon Hopf algebras are immensely useful in low dimensional topology, as they lead to knot and tangle invariants which are well-behaved under “strand stitching”, “strand doubling”, “strand reversal”, and a few lesser operations. See e.g. [Oh, Section 4.2] and our quick summary in Aside 1.1 and in Aside **??**.

Yet from the perspective of topology, the algebras U that one uses seem like great wastelands with a few pearls hidden within. From the perspective of Aside 1.1 and Aside **??** the vector space structure of U is completely irrelevant as the operations of addition $(+)$ and multiplication by a scalar $(\cdot)$ are never used. All that matters are those elements (the

[1] It does not matter whther or not Δ, S, η, and ϵ are used for the generation of the “pearls”, as the axioms of a ribbon Hopf algebra imply that anything that can be generated with them can also be generated without them.

Draw K as a long knot in the plane so that at each crossing the two crossing strands are flowing up, and so that the two ends of K are flowing up.
Put a copy of $R = \sum a_i \otimes b_i$ on every positive crossing of K with the “a” side on the over-strand and the “b” side on the under-strand, labeling these a's and b's with distinct indices $i, j, k, \ldots$ (similarly put copies of $R^{-1} = \sum a'_i \otimes b'_i$ on the negative crossings; these are absent in our example). Put a copy of $C^{\pm 1}$ on every cuap where the tangent to the knot is pointing to the right (meaning, a C on every such cup and a C^{-1} on every such cap).
Form an expression $z(K)$ in U by multiplying all the a, b, C letters as they are seen when traveling along K and then summing over all the indices, as shown.



$$z(K) = \sum_{i,j,k} b_i a_j b_k C a_i b_j a_k$$

If R and C satisfy some conditions dictated by the standard Reidemeister moves of knot theory, the resulting $z(K)$ is a knot invariant.
Abstractly, $z(K)$ is obtained by tensoring together several copies of $R^{\pm 1} \in U^{\otimes 2}$ and $C^{\pm 1} \in U$ to get an intermediate result $z_0 \in U^{\otimes S}$, where S is a finite set with two elements for each crossing of K and one element for each right-pointing cuap. We then multiply the different tensor factors in z_0 in an order dictated by K to get an output in a single copy of U.

ASIDE 1.1. The standard methodology on an example knot.

"pearls") within tensor powers $U^{\otimes S}$ of U that can be written using the "generators" R and C, using tensor products $U^{\otimes S_1} \times U^{\otimes S_2} \to U^{\otimes(S_1 \sqcup S_2)}$, and using the multiplication m (extended to tensor powers) *yet without using* $+$ *and* $\cdot$[1].

MORE.

1.2. An Expansion of the Abstract. MORE.

1.3. Plan of the paper. MORE.

1.4. Acknowledgement. MORE.

2. Rotational Virtual Tangles

Sections 2–5 of this paper can be regarded as independet paperlets which can be read in any order.

MORE.

3. Meta-Hopf Algebras and the Drinfel'd Double Construction

Sections 2–5 of this paper can be regarded as independet paperlets which can be read in any order.

A Hopf algebra is a vector space U with some operations which satisfy some axioms (see Aside 3.1). These axioms, labeled (1)–(9) in the aside, never directly mention the vector space structure of U; that structure is only used for the formation of the spaces $U^{\otimes n}$ on which the operations are defined and in which the axioms are stated. But this calls for a generalization — why not replace $U^{\otimes n}$ with sets U_n that do not need to be vector spaces, and replace operations such as $Id \otimes \cdots \otimes m \otimes \cdots \otimes Id\colon U^{\otimes n} \to U^{\otimes(n-1)}$ with arbitrary maps $m_n\colon U_n \to U_{n-1}$ in such a manner that the axioms (1)–(9) would still make sense?

If U is a vector space over $\mathbb{Q}$ (or another field) we form $U^{\otimes n} = U \otimes \cdots \otimes U$ (n times), and in particular $U^{\otimes 0} = \mathbb{Q}$, $U^{\otimes 1} = U$, $U^{\otimes 2} = U \otimes U$, etc. We identify $U^{\otimes 1} \cong U^{\otimes 1} \otimes U^{\otimes 0} \cong U^{\otimes 0} \otimes U^{\otimes 1}$. With these conventions, a Hopf Algebra *is a vector space U endowed with maps $m\colon U^{\otimes 2} \to U^{\otimes 1}$, $\Delta\colon U^{\otimes 1} \to U^{\otimes 2}$, $\eta\colon U^{\otimes 0} \to U^{\otimes 1}$, $\epsilon\colon U^{\otimes 1} \to U^{\otimes 0}$, and an invertible $S\colon U^{\otimes 1} \to U^{\otimes 1}$ such that:*

(1) $(m \otimes Id)/\!\!/m = (Id \otimes m)/\!\!/m$.
(2) $(\eta \otimes Id)/\!\!/m = (Id \otimes \eta)/\!\!/m = Id$.
(3) $\Delta/\!\!/(\Delta \otimes Id) = \Delta/\!\!/(Id \otimes \Delta)$.
(4) $\Delta/\!\!/(\epsilon \otimes Id) = \Delta/\!\!/(Id \otimes \epsilon) = Id$.
(5) $m/\!\!/\Delta = (\Delta \otimes \delta)/\!\!/(Id \otimes \sigma \otimes Id)/\!\!/(m \otimes m)$, *where $\sigma\colon U_2 \to U_2$ is the transposition.*
(6) $\eta = (\eta \otimes \eta)/\!\!/m$.
(7) $\epsilon = \Delta/\!\!/(\epsilon \otimes \epsilon)$.
(8) $\Delta/\!\!/(S \otimes Id)/\!\!/m = \Delta/\!\!/(Id \otimes S)/\!\!/m = \epsilon/\!\!/\eta$.

Note that we are not assuming $m = \sigma/\!\!/m$ ("commutativity") or $\Delta = \Delta/\!\!/\sigma$ ("co-commutativity").

Aside 3.1. Ordinary Hopf Algebras.

A bit of further reflection[2] leads one to realize that m should be replaced with a family of operations m^{ij}_k which generalize "multiply the content of the ith tensor factor with the content of the jth tensor factor putting the result as a kth tensor factor", and that in fact, the restriction that the labels of the tensor factors would be natural numbers is a (minor) handicap. Hence we come to the following convention and definition:

Convention 3.1. If S, A, and B are finite sets, $A \subset S$, and $(S\backslash A) \cap B = \varnothing$ we write $S\backslash A \cup B$ for $(S\backslash A) \sqcup B$. In fact, whenever we write $S\backslash A \cup B$ we automatically add the assumptions that $A \subset S$ and $(S\backslash A) \cap B = \varnothing$, even if this is not explicitly stated. In this context we often suppress braces and commas when referring to sets with a small number of elements, and automatically assume that these elements are distinct. Hence for example $S\backslash ij \cup kl = (S\backslash\{i,j\}) \sqcup \{k,l\}$, and the assumptions $i \neq j$, $k \neq l$, $\{i,j\} \subset S$, and $(S\backslash\{i,j\}) \cap \{k,l\} = \varnothing$ are silently made. Finally, if A or B are omitted from the notation, the omitted set is assumed to be the empty set and all further conventions still apply.

Definition 3.2. A meta-Hopf algebra (in the category of sets) is an assignment $U\colon S \mapsto U_S$ that assigns a (possibly big) set U_S to every finite set S (we ignore the easily-resolved issues that come with the likes of "the set of all finite sets"), along with the following families of operations and axioms:

Most Interesting. For any finite S, operations $m^{ij}_k\colon U_S \to U_{S\backslash ij \cup k}$ called "meta-multiplications" or "stitchings", $\Delta^i_{jk}\colon U_S \to U_{S\backslash i \cup jk}$ "meta-comultiplications" or "doublings", $\eta_i\colon U_S \to U_{S\cup i}$ "meta-units", $\epsilon^i\colon U_S \to U_{S\backslash i}$ "meta-counits", and $S_i\colon U_S \to U_S$ "meta-antipodes", satisfying the following axioms (compare with Aside 3.1):

(1) $m^{ij}_i /\!\!/ m^{ik}_i = m^{jk}_j /\!\!/ m^{ij}_i$.
(2) $\eta_i /\!\!/ m^{ij}_j = \eta_j /\!\!/ m^{ij}_i = Id$.
(3) $\Delta^i_{ik} /\!\!/ \Delta^i_{ij} = \Delta^i_{ij} /\!\!/ \Delta^j_{jk}$.
(4) $\Delta^j_{ij} /\!\!/ \epsilon^i = \Delta^i_{ij} /\!\!/ \epsilon^j = Id$.
(5) $m^{ij}_k /\!\!/ \Delta^k_{ij} = \Delta^i_{kl} /\!\!/ \Delta^j_{mn} /\!\!/ m^{km}_i /\!\!/ m^{ln}_j$.
(6) $\eta_k = \eta_i /\!\!/ \eta_j /\!\!/ m^{ij}_k$.
(7) $\epsilon^k = \Delta^k_{ij} /\!\!/ \epsilon_i /\!\!/ \epsilon_j$.
(8) $\Delta^i_{jk} /\!\!/ S_j /\!\!/ m^{jk}_i = \Delta^i_{jk} /\!\!/ S_k /\!\!/ m^{jk}_i = \epsilon^i /\!\!/ \eta_i$.

Note that we are not assuming $m^{ij}_k = m^{ji}_k$ ("commutativity") or $\Delta^i_{jk} = \Delta^i_{kj}$ ("cocommutativity").

MORE.

3.1. Meta-monoids vs. monoid objectss in a monoidal category. MORE.

4. The Category of Perturbed Gaussian Differential Operators

Sections 2–5 of this paper can be regarded as independet paperlets which can be read in any order.

MORE.

[2]A fuller but longer explanation is at [BN, Section 10.3].

5. Solavble Approximations of Semi-Simple Lie Algebras

Sections 2–5 of this paper can be regarded as independet paperlets which can be read in any order.

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6. Odds and Ends

MORE.

Appendix A. Tables

MORE.

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