Pensieve header: The category of Gaussian Differential Operators w/o denominators.

```
In[•]:= {t*, y*, a*, x*, z*} = {τ, η, α, ξ, ζ};
        {τ*, η*, α*, ξ*, ζ*} = {t, y, a, x, z};  (u_i_)* := (u*)_i;

In[•]:= {x*, α_3*, z_2*}
Out[•]= {ξ, a_3, ζ_2}

In[•]:= Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
```

Zip

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In[•]:= Zip_{}[P_] := P; Zip_{ξ_,ξs___}[P_] := (Expand[P // Zip_{ξs}] /. f_. ξ^d_. :> ∂_{ξ*,d} f) /. ξ* → 0

In[•]:= Zip_{ξ}[(ξ^2 + ξ + 3) (x^5 e^x + 7 x) + 99 a]
Out[•]= 7 + 99 a

In[•]:= Zip_{η_2}[e^(δ x y_2) ξ η_2]
Out[•]= x δ ξ

In[•]:= Zip_{ξ,η_2}[(ξ^6 + ξ + 3 + 2 ξ η_2) (x^5 e^(b x) + 7 x) + 99 a + e^(δ x y_2) ξ η_2]
Out[•]= 7 + 99 a + 720 b + δ

In[•]:= E0 = 𝔼[Sum[a_(10 i+j) x_i ξ_j, {i, 3}, {j, 3}], 1 + ϵ Sum[f_i[x_1, x_2, x_3] ξ_i, {i, 3}] +
          ϵ^2 (Sum[g_i[x_1, x_2, x_3] ξ_i, {i, 3}] + Sum[g_(10 i+j)[x_1, x_2, x_3] ξ_i ξ_j, {i, 3}, {j, 3}])]

Out[•]= 𝔼[a_11 x_1 ξ_1 + a_21 x_2 ξ_1 + a_31 x_3 ξ_1 + a_12 x_1 ξ_2 + a_22 x_2 ξ_2 + a_32 x_3 ξ_2 + a_13 x_1 ξ_3 + a_23 x_2 ξ_3 + a_33 x_3 ξ_3,
        1 + ϵ (ξ_1 f_1[x_1, x_2, x_3] + ξ_2 f_2[x_1, x_2, x_3] + ξ_3 f_3[x_1, x_2, x_3]) +
          ϵ^2 (ξ_1 g_1[x_1, x_2, x_3] + ξ_2 g_2[x_1, x_2, x_3] + ξ_3 g_3[x_1, x_2, x_3] + ξ_1^2 g_11[x_1, x_2, x_3] +
            ξ_1 ξ_2 g_12[x_1, x_2, x_3] + ξ_1 ξ_3 g_13[x_1, x_2, x_3] + ξ_1 ξ_2 g_21[x_1, x_2, x_3] + ξ_2^2 g_22[x_1, x_2, x_3] +
            ξ_2 ξ_3 g_23[x_1, x_2, x_3] + ξ_1 ξ_3 g_31[x_1, x_2, x_3] + ξ_2 ξ_3 g_32[x_1, x_2, x_3] + ξ_3^2 g_33[x_1, x_2, x_3])]

In[•]:= 𝔼 /: Zip_ξs_List@𝔼[Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
          zs = Table[ξ*, {ξ, ξs}];
          c = Q /. Alternatives @@ (ξs ⋃ zs) → 0;
          ys = Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}];
          ηs = Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
          qt = ω0^-1 Inverse@Table[Kδ_(z,ξ*) - ∂_(z,ξ) Q, {ξ, ξs}, {z, zs}];
          zrule = Thread[zs → qt.(zs + ys)];
          Q1 = c + ηs.zs /. zrule;
          Q2 = Q1 /. Alternatives @@ zs → 0;
          Simplify /@ 𝔼[Q2, Det[qt] e^-Q2 ωZip_ξs[ω1, e^Q1 (P /. zrule)]]];

In[•]:= lhs = Collect[#, {ϵ, ξ_3}, Factor] & /@ (Zip_{ξ_1,ξ_2}@E0)
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```
Collect[Last@lhs, {ϵ, ξ3}, H @@ ω0^Exponent[#,ω0,List] &]
```

Out[•]=
$$H\left[\frac{1}{\omega 0^2}\right]+\epsilon\left(H\left[\frac{1}{\omega 0^3}\right]+H\left[\frac{1}{\omega 0^3},\frac{1}{\omega 0^2}\right]\xi_3\right)+\epsilon^2\left(H\left[\frac{1}{\omega 0^4},\frac{1}{\omega 0^3}\right]+H\left[\frac{1}{\omega 0^4},\frac{1}{\omega 0^3},\frac{1}{\omega 0^2}\right]\xi_3+H\left[\frac{1}{\omega 0^4},\frac{1}{\omega 0^3},\frac{1}{\omega 0^2}\right]\xi_3^2\right)$$

In[•]:=
```
Collect[Last@lhs, {ϵ, ξ3, ω0, ω1}, 1 &]
```

Out[•]=
$$\frac{1}{\omega 0^2}+\epsilon\left(\frac{\omega 1}{\omega 0^3}+\left(\frac{1}{\omega 0^2}+\frac{\omega 1}{\omega 0^3}\right)\xi_3\right)+\epsilon^2\left(\frac{\omega 1}{\omega 0^3}+\frac{\omega 1^2}{\omega 0^4}+\left(\frac{1}{\omega 0^2}+\frac{\omega 1}{\omega 0^3}+\frac{\omega 1^2}{\omega 0^4}\right)\xi_3+\left(\frac{1}{\omega 0^2}+\frac{\omega 1}{\omega 0^3}+\frac{\omega 1^2}{\omega 0^4}\right)\xi_3^2\right)$$

In[•]:=
```
Collect[Last@lhs, {ϵ, ξ3, ω0, ω1}, 1 &] /. ω1 → ω0
```

Out[•]=
$$\frac{1}{\omega 0^2}+\epsilon\left(\frac{1}{\omega 0^2}+\frac{2\,\xi_3}{\omega 0^2}\right)+\epsilon^2\left(\frac{2}{\omega 0^2}+\frac{3\,\xi_3}{\omega 0^2}+\frac{3\,\xi_3^2}{\omega 0^2}\right)$$

In[•]:=
```
lhs2 = Together /@
  Simplify /@ (lhs /. 𝔼[Q_, P_] :> 𝔼[ω, ω Q, ω P /. {x_i_ :> ω x_i, ϵ → ω^1 ϵ, ξ_i_ :> ω^0 ξ_i}] /.
    ω → -1 + a12 a21 - a11 (-1 + a22) + a22)
```

Out[•]=
```
𝔼[-1 + a11 + a12 a21 + a22 - a11 a22,
  (-a13 a31 + a13 a22 a31 - a12 a23 a31 - a13 a21 a32 - a23 a32 + a11 a23 a32 -
     a33 + a11 a33 + a12 a21 a33 + a22 a33 - a11 a22 a33) x3 ξ3,
  -1 + ··· 459 ··· + ϵ^2 a21 a22 g21^(2,0,0) [-(-(-1 + a22) a31 + a21 a32) x3,
     -(a12 a31 - (-1 + a11) a32) x3, (-1 + a12 a21 - a11 (-1 + a22) + a22) x3] -
   ϵ^2 a21^2 g22^(2,0,0) [-(-(-1 + a22) a31 + a21 a32) x3, -(a12 a31 - (-1 + a11) a32) x3,
     (-1 + a12 a21 - a11 (-1 + a22) + a22) x3]]
```

large output | show less | show more | show all | set size limit...

```
lhs2 /. ϕ_[-(-(-1 + a22) a31 + a21 a32) x3,
    -(a12 a31 - (-1 + a11) a32) x3, (-1 + a12 a21 - a11 (-1 + a22) + a22) x3] :> ϕ
```

In[•]:=
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Denominator@Last@lhs2
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Out[•]= 1

In[•]:=
```
lhs3 = Together /@
  Simplify /@ (lhs /. 𝔼[Q_, P_] :> 𝔼[ω, ω Q, ω P /. {x_i_ :> ω x_i, ϵ → ω^0 ϵ, ξ_i_ :> ω^1 ξ_i}] /.
    ω → -1 + a12 a21 - a11 (-1 + a22) + a22)
```

Out[•]=
```
𝔼[-1 + a11 + a12 a21 + a22 - a11 a22,
  (-a13 a31 + a13 a22 a31 - a12 a23 a31 - a13 a21 a32 - a23 a32 + a11 a23 a32 -
     a33 + a11 a33 + a12 a21 a33 + a22 a33 - a11 a22 a33) x3 ξ3, ··· 1 ··· / (-1+a11+a12 a21+a22-a11 a22)^2]
```

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```
In[•]:= lhs4 = Together /@
          Simplify /@ (lhs /. 𝔼[Q_, P_] :> 𝔼[ω, ω Q, ω P /. {x_i_ :> ω x_i, ϵ → ω^2 ϵ, ξ_i_ :> ω^-1 ξ_i}] /.
            ω → -1 + a12 a21 - a11 (-1 + a22) + a22)

In[•]:= Collect[Last@lhs4, {ϵ, ξ3}, Factor]

In[•]:= rhs12 = Zip{ξ1}@Zip{ξ2}@E0

Out[•]= 𝔼[((a13 ((-1 + a22) a31 - a21 a32) + a12 (-a23 a31 + a21 a33) + (-1 + a11) (a23 a32 - (-1 + a22) a33))
          x3 ξ3) / (-1 + a12 a21 - a11 (-1 + a22) + a22), (1 - 2 a22 + ··· 137 ··· + ··· 1 ···) / (1 - a12 a21 + ··· 1 ··· - a22)^3]

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In[•]:= lhs == rhs12

Out[•]= True

In[•]:= rhs21 = Zip{ξ2}@Zip{ξ1}@E0

Out[•]= 𝔼[((a13 ((-1 + a22) a31 - a21 a32) + a12 (-a23 a31 + a21 a33) + (-1 + a11) (a23 a32 - (-1 + a22) a33))
          x3 ξ3) / (-1 + a12 a21 - a11 (-1 + a22) + a22), (1 - 2 a22 + ··· 137 ··· + ··· 1 ···) / (1 - a12 a21 + ··· 1 ··· - a22)^3]

  large output | show less | show more | show all | set size limit...

In[•]:= rhs12 == rhs21

Out[•]= True

In[•]:= Eh = 𝔼[h Sum[a10 i+j x_i ξ_j, {i, 3}, {j, 3}],
          1 + ϵ Sum[f_i[x1, x2, x3] ξ_i, {i, 3}] + ϵ Sum[f10 i+j[x1, x2, x3] ξ_i ξ_j, {i, 3}, {j, 3}]]

Out[•]= 𝔼[h (a11 x1 ξ1 + a21 x2 ξ1 + a31 x3 ξ1 + a12 x1 ξ2 + a22 x2 ξ2 + a32 x3 ξ2 + a13 x1 ξ3 + a23 x2 ξ3 + a33 x3 ξ3),
          1 + ϵ (ξ1 f1[x1, x2, x3] + ξ2 f2[x1, x2, x3] + ξ3 f3[x1, x2, x3]) +
            ϵ (ξ1^2 f11[x1, x2, x3] + ξ1 ξ2 f12[x1, x2, x3] + ξ1 ξ3 f13[x1, x2, x3] +
              ξ1 ξ2 f21[x1, x2, x3] + ξ2^2 f22[x1, x2, x3] + ξ2 ξ3 f23[x1, x2, x3] +
              ξ1 ξ3 f31[x1, x2, x3] + ξ2 ξ3 f32[x1, x2, x3] + ξ3^2 f33[x1, x2, x3]))]

Short[lhs == Normal[Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 1}]] // Zip{ξ1,ξ2}]

In[•]:= rhs0 = Zip{ξ1,ξ2}[Eh];
        rhs1 = Normal[rhs0 /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 1}]]

In[•]:= Simplify[lhs == rhs1]
```

ωZip

```
In[•]:= ωZip{}[ω_, P_] := P;
        ωZip{ξ_,ξs___}[ω_, P_] := (Expand[ωZip{ξs}[ω, P]] /. f_. ξ^d_. :> ω^d ∂{ξ^*,d} f) /. ξ^* → 0
```

```
In[•]:= 𝔼 /: ωZip_{ξs_List}@𝔼[ω_, Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, m, ω1, qt, zrule, Q1, Q2},
           zs = Table[ξ*, {ξ, ξs}];
           c = Q /. Alternatives @@ (ξs ⋃ zs) → 0;
           ys = Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}];
           ηs = Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
           m = Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}];
           ω1 = Det[m];
           qt = ω1 Inverse[m];
           zrule = Thread[zs → qt.(zs + ys)];
           Q1 = c + ηs.zs /. zrule;
           Q2 = Q1 /. Alternatives @@ zs → 0;
           Simplify /@ 𝔼[ω / ω1, Q2, e^(-Q2) ωZip_ξs[ω, e^Q1 (P /. zrule)]] ];

In[•]:= E1 = Prepend[E0, 1]

Out[•]= 𝔼[1, a11 x1 ξ1 + a21 x2 ξ1 + a31 x3 ξ1 + a12 x1 ξ2 + a22 x2 ξ2 + a32 x3 ξ2 + a13 x1 ξ3 + a23 x2 ξ3 + a33 x3 ξ3,
  1 + ∈ (ξ1 f1[x1, x2, x3] + ξ2 f2[x1, x2, x3] + ξ3 f3[x1, x2, x3]) +
   ∈ (ξ1^2 f11[x1, x2, x3] + ξ1 ξ2 f12[x1, x2, x3] + ξ1 ξ3 f13[x1, x2, x3] +
      ξ1 ξ2 f21[x1, x2, x3] + ξ2^2 f22[x1, x2, x3] + ξ2 ξ3 f23[x1, x2, x3] +
      ξ1 ξ3 f31[x1, x2, x3] + ξ2 ξ3 f32[x1, x2, x3] + ξ3^2 f33[x1, x2, x3])]

In[•]:= Short[lhs = ωZip_{ξ1,ξ2}@E1]

Out[•]//Short= 𝔼[1 / (1 - a12 a21 + a11 (-1 + a22) - a22),
  (a23 (a12 a31 - (-1 + a11) a32) + a13 (-(-1 + a22) a31 + a<<2>> <<1>>) + a33) <<1>> ξ3, <<1>>]

In[•]:= Short[rhs12 = ωZip_{ξ1}@ωZip_{ξ2}@E1]

Out[•]//Short= 𝔼[1 / ((-1 + a11 + a12 a21) (-1 + a22)), <<1>>, <<141>> + ∈ <<1>> <<1>>^<<1>> [<<1>>] / (-1 + a22)^2]

In[•]:= Simplify[1 / (1 - a12 a21 + a11 (-1 + a22) - a22) / (1 / ((-1 + a11 + a12 a21) (-1 + a22)))]

Out[•]= ((-1 + a11 + a12 a21) (-1 + a22)) / (1 - a12 a21 + a11 (-1 + a22) - a22)
```