

Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

Profiling

```
In[ ]:= Once[
  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Invariant"];
  << KnotTheory` ;
  << "../Profile/Profile.m";
];
BeginProfile[];
Once@PopupWindow[Button["Show Profile Monitor"],
  Dynamic[PrintProfile[], UpdateInterval -> 3, TrackedSymbols -> {}]]]
```

Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.

Read more at <http://katlas.org/wiki/KnotTheory>.

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: June 2018. Original version: July 1994.

Out[]:=

External Utilities

```
In[ ]:= HL[ε_] := Style[ε, Background -> Yellow];
```

Program

Program

Internal Utilities

Program

Canonical Form:

Program

```
In[ ]:= CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := PPCF@ExpandDenominator@ExpandNumerator@PPTogether@Together[PPExp[
  Expand[ε] /. ex ey -> ex+y /. ex -> eCF[x]]];]
```

Program

The Kronecker δ :

Program

```
In[ ]:= Kδ /: Kδi,j := If[i == j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

```
In[*]:=
E /: E[L1_, Q1_, P1_] ≡ E[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
E /: E[L1_, Q1_, P1_] E[L2_, Q2_, P2_] := E[L1 + L2, Q1 + Q2, P1 * P2];
E[L_, Q_, P_]$k_ := E[L, Q, Series[Normal@P, {ε, 0, $k}]]];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

```
In[*]:=
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})^* := (u^*)_i;
```

Program

Finite Zips:

Program

```
In[*]:=
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ε_, ζ_] := PPCollect@Collect[ε, ζ];
Zip[_][P_] := P; Zip[ζ_, ζs___][P_] := PPZip[
  (collect[P // Zip[ζs], ζ] /. f_ . ζ^{d_} => ∂_{ζ^{*,d}} f) /. ζ^* -> 0]
```

Program

QZip implements the “Q-level zips” on $E(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Program

```
In[*]:=
QZip[ζs_List@E[L_, Q_, P_] := PPQZip@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
  zs = Table[ζ^*, {ζ, ζs}];
  c = Q /. Alternatives @@ (ζs ∪ zs) -> 0;
  ys = Table[∂_ζ (Q /. Alternatives @@ zs -> 0), {ζ, ζs}];
  ηs = Table[∂_z (Q /. Alternatives @@ ζs -> 0), {z, zs}];
  qt = Inverse@Table[Kδ_{z, ζ^*} - ∂_{z, ζ} Q, {ζ, ζs}, {z, zs}];
  zrule = Thread[zs -> qt.(zs + ys)];
  Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs -> 0;
  CF /@ E[L, Q2, Det[qt] e^{-Q2} Zip_ζs[e^{Q1} (P /. zrule)]]];
```

Program

Upper to lower and lower to Upper:

Program

```
In[*]:=
U2l = {B_{i_}^{p_} -> e^{-p ħ γ b_i}, B_{i_}^{p_} -> e^{-p ħ γ b}, T_{i_}^{p_} -> e^{p ħ t_i}, T_{i_}^{p_} -> e^{p ħ t}, A_{i_}^{p_} -> e^{p γ α_i}, A_{i_}^{p_} -> e^{p γ α}};
L2u = {e^{c_ . b_i + d_} -> B_{i_}^{-c / (ħ γ)} e^d, e^{c_ . b + d_} -> B^{-c / (ħ γ)} e^d,
  e^{c_ . t_i + d_} -> T_{i_}^{c / ħ} e^d, e^{c_ . t + d_} -> T^{c / ħ} e^d,
  e^{c_ . α_i + d_} -> A_{i_}^{c / γ} e^d, e^{c_ . α + d_} -> A^{c / γ} e^d,
  e^ε -> e^{Expand@ε}}];
```

Program

LZip implements the “L-level zips” on $E(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single “P”. Here the z ’s are b and α and the ζ ’s are β and a .

Program

```
In[ ]:=
LZip $\zeta$ s_List@E[L_, Q_, P_] :=
  PPLZip@Module[{ $\zeta$ , z, zs, Zs, c, ys,  $\eta$ s, lt, zrule, Zrule, U, L1, L2, Q1, Q2},
    zs = Table[ $\zeta^*$ , { $\zeta$ ,  $\zeta$ s}]; Zs = Echo[zs /. {b  $\rightarrow$  B, t  $\rightarrow$  T,  $\alpha$   $\rightarrow$   $\mathcal{A}$ }];
    c = L /. Alternatives@@( $\zeta$ s  $\cup$  zs)  $\rightarrow$   $\emptyset$ ;
    ys = Table[ $\partial_{\zeta}$ (L /. Alternatives@@zs  $\rightarrow$   $\emptyset$ ), { $\zeta$ ,  $\zeta$ s}];
     $\eta$ s = Table[ $\partial_z$ (L /. Alternatives@@ $\zeta$ s  $\rightarrow$   $\emptyset$ ), {z, zs}];
    lt = Inverse@Table[K $\delta_{z, \zeta^*} - \partial_{z, \zeta} L$ , { $\zeta$ ,  $\zeta$ s}, {z, zs}];
    zrule = Thread[zs  $\rightarrow$  lt.(zs + ys)];
    Zrule =
      zrule /. r_Rule  $\rightarrow$  ((U = r[[1]] /. {b  $\rightarrow$  B, t  $\rightarrow$  T,  $\alpha$   $\rightarrow$   $\mathcal{A}$ })  $\rightarrow$  (U /. U21 /. r // . 12U));
    L2 = (L1 = c +  $\eta$ s.zs /. zrule) /. Alternatives@@zs  $\rightarrow$   $\emptyset$ ;
    Q2 = (Q1 = Q /. Zrule) /. Alternatives@@Zs  $\rightarrow$  1;
    CF /@ E[L2, Q2, Det[lt] e-L2-Q2 Zip $\zeta$ s[eL1+Q1(P /. Zrule)]] // . 12U];
```

Program

```
In[ ]:=
Bind{}[L_, R_] := L R;
Bind{is_}[L_ $\mathcal{E}$ , R_ $\mathcal{E}$ ] := PPBind@Module[{n},
  Times[
    L /. Table[(v : b | B | t | T | a | x | y)i  $\rightarrow$  vn $\mathcal{E}$ i, {i, {is}}],
    R /. Table[(v :  $\beta$  |  $\tau$  |  $\alpha$  |  $\mathcal{A}$  |  $\xi$  |  $\eta$ )i  $\rightarrow$  vn $\mathcal{E}$ i, {i, {is}}]
  ] // LZipJoin@Table[{ $\beta$ n $\mathcal{E}$ i,  $\tau$ n $\mathcal{E}$ i, an $\mathcal{E}$ i}, {i, {is}}] // QZipJoin@Table[{ $\xi$ n $\mathcal{E}$ i,  $\eta$ n $\mathcal{E}$ i}, {i, {is}}] ];
  BL_List[L_, R_] := BindL[L, R]; Bis_[L_, R_] := Bind{is}[L, R];
```

Program

“Define” code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of \$k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

```
In[ ]:=
SetAttributes[Define, HoldAll];
Define[def_, defs_] := (Define[def]; Define[defs]);
Define[op_is_ =  $\mathcal{E}$ _] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[opnisp, $k_Integer, PPBoot@Block[{i, j, k}, opisp, $k =  $\mathcal{E}$ ; opnisp, $k]];
    SD[opisp, op{is}, $k]; SD[opsis, op{sis}];
  ] /. {SD  $\rightarrow$  SetDelayed,
    isp  $\rightarrow$  {is} /. {i  $\rightarrow$  i_, j  $\rightarrow$  j_, k  $\rightarrow$  k_},
    nis  $\rightarrow$  {is} /. {i  $\rightarrow$  ii, j  $\rightarrow$  jj, k  $\rightarrow$  kk},
    nisp  $\rightarrow$  {is} /. {i  $\rightarrow$  ii_, j  $\rightarrow$  jj_, k  $\rightarrow$  kk_}
  } ] ]
```

Program

Booting Up

Program

```
In[ ]:= $k = 2; (*h=gamma=1;*)
```

Program

```
In[ ]:= Define [ami,j→k = E [ (αi + αj) ak, (ξi / Aj + ξj) xk, 1 ] $k,
bmi,j→k = E [ (βi + βj) bk, (ηi + ηj) yk, e(e-εβi-1) ηj yk ] $k]
```

Program

```
In[ ]:= Define [Ri,j = E [ h aj bi, h xj yi, e∑k=2$k+1 (1 - eγ ∈ h)k (h yi xj)k ] $k,
Ri,j = E [ -h aj bi, -h xj yi / Bi, 1 + If [ $k == 0, 0, (R{i,j},$k-1) $k [3] ] -
(( (R{i,j},0) $k R1,2 (R{3,4},$k-1) $k) ~ Bi,j,1,2 ~ (bmi,1→i amj,2→j) ~ Bi,j,3,4 ~ (bmi,3→i amj,4→j) ) [3] ] ]
Pi,j = E [ βi αj / h, ηi ξj / h, 1 + If [ $k == 0, 0, (P{i,j},$k-1) $k [3] ] -
(R1,2 ~ B1,2 ~ ( (P{1,j},0) $k (P{i,2},$k-1) $k) ) [3] ] ]]
```

Program

```
In[ ]:= Define [aSj = Ri,j ~ Bi ~ Pi,j,
aSi = E [ -ai αi, -xi Ai ξi, 1 + If [ $k == 0, 0, (aS{i},$k-1) $k [3] ] -
(( (aS{i},0) $k ~ Bi ~ aSi ~ Bi ~ (aS{i},$k-1) $k ) [3] ] ]]
```

Program

```
In[ ]:= Define [bSi = Ri,1 ~ B1 ~ aS1 ~ B1 ~ Pi,1,
bSi = Ri,1 ~ B1 ~ aS1 ~ B1 ~ Pi,1,
aΔi→j,k = (R1,j R2,k) ~ B1,2 ~ bm1,2→3 ~ B3 ~ P3,i,
bΔi→j,k = (Rj,1 Rk,2) ~ B1,2 ~ am1,2→3 ~ B3 ~ Pi,3]
```

Program

```
In[ ]:= Define [dmi,j→k =
(E [ βi bi + αj aj, ηi yi + ξj xj, 1 ] (aΔi→1,2 ~ B2 ~ aΔ2→2,3 ~ B3 ~ aS3) (bΔj→-1,-2 ~ B-2 ~ bΔ-2→-2,-3) ~
B-3,-2,-1,1,2,3,i,j ~ (P-1,3 P-3,1 am2,j→k bmi,-2→k),
dSi = E [ βi b1 + αi a2, ηi y1 + ξi x2, 1 ] ~ B1,2 ~ (bS1 aS2) ~ B1,2 ~ dm2,1→i,
dΔi→j,k = (bΔi→3,1 aΔi→2,4) ~ B1,2,3,4 ~ (dm3,4→k dm1,2→j)]
```

Program

```
In[ ]:= Define [Ci = E [ 0, 0, Bi1/2 e-h ∈ ai/2 ] $k,
Ci = E [ 0, 0, Bi-1/2 eh ∈ ai/2 ] $k,
Kinki = (R1,3 C2) ~ B1,2 ~ dm1,2→1 ~ B1,3 ~ dm1,3→i,
Kinki = (R1,3 C2) ~ B1,2 ~ dm1,2→1 ~ B1,3 ~ dm1,3→i]
```

Program

Note. $t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

Program

```
In[ ]:= Define [b2ti =  $\mathbb{E} [\alpha_i a_i - \beta_i t_i / \gamma, \xi_i x_i + \eta_i y_i, e^{\epsilon \beta_i a_i / \gamma}]_{\$k}$ ,
  t2bi =  $\mathbb{E} [\alpha_i a_i - \tau_i \gamma b_i, \xi_i x_i + \eta_i y_i, e^{\epsilon \tau_i a_i}]_{\$k}$ ]
```

Testing

```
In[ ]:= Block [ {$k = 1}, {
  am → ami,j→k, bm → bmi,j→k, dm → dmi,j→k, R → Ri,j,  $\bar{R}$  →  $\bar{R}_{i,j}$ , P → Pi,j,
  aS → aSi,  $\bar{aS}$  →  $\bar{aS}_i$ , bS → bSi,  $\bar{bS}$  →  $\bar{bS}_i$ , dS → dSi, aΔ → aΔi→j,k, bΔ → bΔi→j,k,
  dΔ → dΔi→j,k, C → Ci,  $\bar{C}$  →  $\bar{C}_i$ , Kink → Kinki,  $\bar{Kink}$  →  $\bar{Kink}_i$ , b2t → b2ti, t2b → t2bi
}] //
Column
```

" { Bn\$2898[1], Tn\$2898[1], An\$2898[1], Bn\$2898[2], Tn\$2898[2], An\$2898[2] }
 " { Bn\$2918[1], Tn\$2918[1], An\$2918[1], Bn\$2918[2], Tn\$2918[2], An\$2918[2] }
 " { Bn\$2937[3], Tn\$2937[3], An\$2937[3] }
 " { Bn\$2956[2], Tn\$2956[2], An\$2956[2] }
 " { Bn\$2975[i\$], Tn\$2975[i\$], An\$2975[i\$], Bn\$2975[j\$], Tn\$2975[j\$],
 An\$2975[j\$], Bn\$2975[1], Tn\$2975[1], An\$2975[1], Bn\$2975[2], Tn\$2975[2], An\$2975[2] }
 " { Bn\$3008[i\$], Tn\$3008[i\$], An\$3008[i\$], Bn\$3008[j\$], Tn\$3008[j\$],
 An\$3008[j\$], Bn\$3008[3], Tn\$3008[3], An\$3008[3], Bn\$3008[4], Tn\$3008[4], An\$3008[4] }
 " { Bn\$3045[i], Tn\$3045[i], An\$3045[i] }
 " { Bn\$3076[i\$], Tn\$3076[i\$], An\$3076[i\$] }
 " { Bn\$3109[i\$], Tn\$3109[i\$], An\$3109[i\$] }
 " { Bn\$3142[3], Tn\$3142[3], An\$3142[3] }
 " { Bn\$3187[1], Tn\$3187[1], An\$3187[1], Bn\$3187[2], Tn\$3187[2], An\$3187[2] }
 " { Bn\$3212[3], Tn\$3212[3], An\$3212[3] }
 " { Bn\$3231[-2], Tn\$3231[-2], An\$3231[-2] }
 " { Bn\$3264[-3], Tn\$3264[-3], An\$3264[-3], Bn\$3264[-2], Tn\$3264[-2], An\$3264[-2], Bn\$3264[-1],
 Tn\$3264[-1], An\$3264[-1], Bn\$3264[1], Tn\$3264[1], An\$3264[1], Bn\$3264[2], Tn\$3264[2], An\$3264[2],
 Bn\$3264[3], Tn\$3264[3], An\$3264[3], Bn\$3264[i], Tn\$3264[i], An\$3264[i], Bn\$3264[j], Tn\$3264[j], An\$3264[j] }
 " { Bn\$3335[1], Tn\$3335[1], An\$3335[1] }
 " { Bn\$3370[1], Tn\$3370[1], An\$3370[1] }
 " { Bn\$3401[1], Tn\$3401[1], An\$3401[1] }
 " { Bn\$3436[1], Tn\$3436[1], An\$3436[1] }
 " { Bn\$3467[1], Tn\$3467[1], An\$3467[1], Bn\$3467[2], Tn\$3467[2], An\$3467[2] }
 " { Bn\$3512[1], Tn\$3512[1], An\$3512[1], Bn\$3512[2], Tn\$3512[2], An\$3512[2] }
 " { Bn\$3663[1], Tn\$3663[1], An\$3663[1], Bn\$3663[2], Tn\$3663[2],
 An\$3663[2], Bn\$3663[3], Tn\$3663[3], An\$3663[3], Bn\$3663[4], Tn\$3663[4], An\$3663[4] }
 " { Bn\$3743[1], Tn\$3743[1], An\$3743[1], Bn\$3743[2], Tn\$3743[2], An\$3743[2] }
 " { Bn\$3792[1], Tn\$3792[1], An\$3792[1], Bn\$3792[3], Tn\$3792[3], An\$3792[3] }
 " { Bn\$3841[1], Tn\$3841[1], An\$3841[1], Bn\$3841[2], Tn\$3841[2], An\$3841[2] }
 " { Bn\$3896[1], Tn\$3896[1], An\$3896[1], Bn\$3896[3], Tn\$3896[3], An\$3896[3] }
 am $\rightarrow \mathbb{E} \left[\mathbf{a}_k (\alpha_i + \alpha_j), \mathbf{x}_k \left(\frac{\xi_i}{\beta_j} + \xi_j \right), \mathbf{1} \right]$
 bm $\rightarrow \mathbb{E} \left[\mathbf{b}_k (\beta_i + \beta_j), \mathbf{y}_k (\eta_i + \eta_j), \mathbf{1} - \mathbf{y}_k \beta_i \eta_j \in + \mathbf{O}[\epsilon]^2 \right]$

$\overline{aS} \rightarrow$

$$\mathbb{E} \left[-a_i \alpha_i, -x_i \mathcal{A}_i \xi_i, 1 - \frac{1}{4 B_{n\$3045[i]}^2} \left(e^{-\hbar x_i Y_{n\$3045[i]} \mathcal{A}_{n\$3076[i]} - x_i \epsilon_{n\$3076[i]}} \left(4 e^{\frac{\hbar x_i Y_{n\$3045[i]} \mathcal{A}_{n\$3076[i]} + x_i \mathcal{A}_{n\$3076[i]} \epsilon_{n\$3076[i]}}{B_{n\$3045[i]}} \hbar a_i \right. \right. \right. \\ \left. \left. B_{n\$3045[i]} x_i \mathcal{A}_i \mathcal{A}_{n\$3076[i]} \xi_i + e^{\frac{\hbar x_i Y_{n\$3045[i]} \mathcal{A}_{n\$3076[i]} + x_i \mathcal{A}_{n\$3076[i]} \epsilon_{n\$3076[i]}}{B_{n\$3045[i]}} \gamma \hbar x_i^2 \mathcal{A}_i^2 \mathcal{A}_{n\$3076[i]}^2 \xi_i^2 + \right. \right. \\ \left. \left. e^{\hbar x_i Y_{n\$3045[i]} \mathcal{A}_{n\$3076[i]} + x_i \mathcal{A}_{n\$3076[i]} \epsilon_{n\$3076[i]}} \gamma \hbar B_{n\$3045[i]}^2 x_i^2 \mathcal{A}_i^2 \mathcal{A}_{n\$3076[i]}^2 \xi_i^2 \right) \right] \in + O[\epsilon]^2$$

$$bS \rightarrow \mathbb{E} \left[-b_i \beta_i, -\frac{y_i \eta_i}{B_i}, 1 + \frac{1}{4 B_i^2 B_{n\$3045[i]}^2} e^{-\frac{Y_{n\$3045[i]} \mathcal{A}_{n\$3335[1]} \eta_i}{B_i B_{n\$3045[1]}} - \frac{\mathcal{A}_{n\$3335[1]} \eta_i \epsilon_{n\$3335[1]}}{\hbar B_i} \right. \\ \left. \left(-4 e^{\frac{Y_{n\$3045[i]} \mathcal{A}_{n\$3335[1]} \eta_i}{B_i} + \frac{\eta_i \epsilon_{n\$3335[1]}}{\hbar B_i}} B_i B_{n\$3045[i]} y_i \mathcal{A}_{n\$3335[1]} \beta_i \eta_i + e^{\frac{Y_{n\$3045[i]} \mathcal{A}_{n\$3335[1]} \eta_i}{B_i} + \frac{\eta_i \epsilon_{n\$3335[1]}}{\hbar B_i}} \gamma \hbar y_i^2 \mathcal{A}_{n\$3335[1]}^2 \eta_i^2 + \right. \right. \\ \left. \left. e^{\frac{Y_{n\$3045[i]} \mathcal{A}_{n\$3335[1]} \eta_i}{B_i B_{n\$3045[1]}} + \frac{\eta_i \epsilon_{n\$3335[1]}}{\hbar B_i}} \gamma \hbar B_{n\$3045[i]}^2 y_i^2 \mathcal{A}_{n\$3335[1]}^2 \eta_i^2 \right) \right] \in + O[\epsilon]^2$$

$$\overline{bS} \rightarrow \mathbb{E} \left[-b_i \beta_i, -\frac{y_i \eta_i}{B_i}, 1 + \frac{1}{4 B_i^2 B_{n\$3045[i]}^2} e^{-Y_{n\$3045[i]} \mathcal{A}_{n\$3076[1]} \eta_i - \frac{\eta_i \epsilon_{n\$3076[1]} \mathcal{A}_{n\$3401[1]} \eta_i \epsilon_{n\$3401[1]}}{\hbar} - \frac{Y_{n\$3045[i]} \mathcal{A}_{n\$3076[1]} \eta_i + \mathcal{A}_{n\$3076[1]} \epsilon_{n\$3076[1]} \eta_i \epsilon_{n\$3401[1]}}{\hbar B_i} B_i B_{n\$3045[i]} \right. \\ \left. y_i \mathcal{A}_{n\$3076[1]} \mathcal{A}_{n\$3401[1]} \beta_i \eta_i - e^{\frac{Y_{n\$3045[i]} \mathcal{A}_{n\$3076[1]} \eta_i + \mathcal{A}_{n\$3076[1]} \epsilon_{n\$3076[1]} \eta_i \epsilon_{n\$3401[1]}}{\hbar B_i}} \gamma \hbar y_i^2 \mathcal{A}_{n\$3076[1]}^2 \mathcal{A}_{n\$3401[1]}^2 \eta_i^2 - \right. \\ \left. e^{Y_{n\$3045[i]} \mathcal{A}_{n\$3076[1]} \eta_i + \frac{\mathcal{A}_{n\$3076[1]} \epsilon_{n\$3076[1]} \eta_i \epsilon_{n\$3401[1]}}{\hbar} + \frac{\eta_i \epsilon_{n\$3401[1]}}{\hbar B_i}} \gamma \hbar B_{n\$3045[i]}^2 y_i^2 \mathcal{A}_{n\$3076[1]}^2 \mathcal{A}_{n\$3401[1]}^2 \eta_i^2 \right] \in + O[\epsilon]^2$$

$$\begin{aligned}
 a\Delta &\rightarrow \mathbb{E} \left[a_j \alpha_i + a_k \alpha_i, x_j \xi_i + x_k \xi_i, 1 + \frac{1}{2} \left(-2 \hbar a_j x_k \xi_i + \gamma \hbar x_j x_k \xi_i^2 \right) \in + \mathcal{O}[\epsilon]^2 \right] \\
 b\Delta &\rightarrow \mathbb{E} \left[b_j \beta_i + b_k \beta_i, B_k y_j \eta_i + y_k \eta_i, 1 + \frac{1}{2} \gamma \hbar B_k y_j y_k \eta_i^2 \in + \mathcal{O}[\epsilon]^2 \right] \\
 d\Delta &\rightarrow \mathbb{E} \left[a_j \alpha_i + a_k \alpha_i + b_j \beta_i + b_k \beta_i, y_j \eta_i + B_j y_k \eta_i + x_j \xi_i + x_k \xi_i, 1 + \frac{1}{2} e^{-B_j Y_{n\$3663[3]} \eta_i} \left(e^{B_j B_{n\$3663[1]} Y_{n\$3663[3]} \eta_i} \gamma \right) \right]
 \end{aligned}$$

$$C \rightarrow \mathbb{E} \left[\theta, \theta, \sqrt{B_i} - \frac{1}{2} \left(\hbar a_i \sqrt{B_i} \right) \in + \mathcal{O}[\epsilon]^2 \right]$$

$$\bar{C} \rightarrow \mathbb{E} \left[\theta, \theta, \frac{1}{\sqrt{B_i}} + \frac{\hbar a_i \epsilon}{2 \sqrt{B_i}} + \mathcal{O}[\epsilon]^2 \right]$$

$$Kink \rightarrow \mathbb{E} \left[\hbar a_i b_i, \hbar x_i y_i, \frac{\sqrt{B_i B_{n\$3743[2]} B_{n\$3792[1]}}}{B_i B_{n\$3743[2]} B_{n\$3792[1]}} + \left(\left(2 \hbar a_i \sqrt{B_i B_{n\$3743[2]} B_{n\$3792[1]}} - \gamma \hbar^3 \sqrt{B_i B_{n\$3743[2]} B_{n\$3792[1]}} \right) \right) \right]$$

$$\begin{aligned}
 \overline{Kink} &\rightarrow \mathbb{E} \left[-\hbar a_i b_i, -\frac{\hbar x_i y_i}{B_i}, \sqrt{B_i B_{n\$3841[2]} B_{n\$3896[1]}} + \frac{1}{4 B_i^2 B_{n\$3841[1]}^2 B_{n\$3896[1]}^2} e^{-\frac{\hbar x_i Y_{n\$3841[1]}}{B_i B_{n\$3841[1]} B_{n\$3896[1]}} - \frac{\hbar x_i Y_{n\$3896[1]}}{B_i B_{n\$3896[1]}}} \right. \\
 &\quad \left. \left(-2 e^{\frac{\hbar x_i Y_{n\$3841[1]}}{B_i B_{n\$3841[1]} B_{n\$3896[1]}} + \frac{\hbar x_i Y_{n\$3896[1]}}{B_i B_{n\$3896[1]}}} \hbar a_i B_i^2 B_{n\$3841[1]}^2 B_{n\$3896[1]}^2 \sqrt{B_i B_{n\$3841[2]} B_{n\$3896[1]}} - 4 e^{\frac{\hbar x_i Y_{n\$3841[1]}}{B_i B_{n\$3896[1]}} + \frac{\hbar x_i Y_{n\$3896[1]}}{B_i}} \hbar^2 \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 b2t &\rightarrow \mathbb{E} \left[a_i \alpha_i - \frac{t_i \beta_i}{\gamma}, y_i \eta_i + x_i \xi_i, 1 + \frac{a_i \beta_i \epsilon}{\gamma} + \mathcal{O}[\epsilon]^2 \right] \\
 t2b &\rightarrow \mathbb{E} \left[a_i \alpha_i - \gamma b_i \tau_i, y_i \eta_i + x_i \xi_i, 1 + a_i \tau_i \epsilon + \mathcal{O}[\epsilon]^2 \right]
 \end{aligned}$$

Check that on the generators this agrees with our conventions in the handout:

$$\begin{aligned}
 \text{In[]:= } & \text{Timing} @ \left\{ \left\{ "[a, x]" \rightarrow \left(\left(\mathbb{E}[\theta, \theta, a_2 x_1] \sim B_{1,2} \sim a_{m_{1,2 \rightarrow 1}} \right) [[3]] - \left(\mathbb{E}[\theta, \theta, a_1 x_2] \sim B_{1,2} \sim a_{m_{1,2 \rightarrow 1}} \right) [[3]] \right) \right\}, \right. \\
 & \quad \left. "[b, y]" \rightarrow \left(\left(\mathbb{E}[\theta, \theta, y_2 b_1] \sim B_{1,2} \sim b_{m_{1,2 \rightarrow 1}} \right) [[3]] - \left(\mathbb{E}[\theta, \theta, y_1 b_2] \sim B_{1,2} \sim b_{m_{1,2 \rightarrow 1}} \right) [[3]] \right) \right\} /. \\
 & \quad z_{-1} \rightarrow z, \\
 & \quad \{ "Delta[y]" \rightarrow \text{Last}[\mathbb{E}[\theta, \theta, y_1] \sim B_1 \sim b_{\Delta_{1 \rightarrow 1, 2}}], \\
 & \quad \quad "Delta[b]" \rightarrow \text{Last}[\mathbb{E}[\theta, \theta, b_1] \sim B_1 \sim b_{\Delta_{1 \rightarrow 1, 2}}], \\
 & \quad \quad "Delta[a]" \rightarrow \text{Last}[\mathbb{E}[\theta, \theta, a_1] \sim B_1 \sim a_{\Delta_{1 \rightarrow 1, 2}}], \\
 & \quad \quad "Delta[x]" \rightarrow \text{Last}[\mathbb{E}[\theta, \theta, x_1] \sim B_1 \sim a_{\Delta_{1 \rightarrow 1, 2}}] \}, \\
 & \quad \{ \\
 & \quad \quad "S(a)" \rightarrow \left(\left(\mathbb{E}[\theta, \theta, a_1] \sim B_1 \sim a_{S_1} \right) [[3]] \right), \\
 & \quad \quad "S(x)" \rightarrow \left(\left(\mathbb{E}[\theta, \theta, x_1] \sim B_1 \sim a_{S_1} \right) [[3]] \right), \\
 & \quad \quad "S(b)" \rightarrow \left(\left(\mathbb{E}[\theta, \theta, b_1] \sim B_1 \sim b_{S_1} \right) [[3]] \right), \\
 & \quad \quad "S(y)" \rightarrow \left(\left(\mathbb{E}[\theta, \theta, y_1] \sim B_1 \sim b_{S_1} \right) [[3]] \right) \\
 & \quad \left. \right\} /. z_{-1} \rightarrow z
 \end{aligned}$$

```

" { Bn$4002[1], Tn$4002[1], An$4002[1], Bn$4002[2], Tn$4002[2], An$4002[2] }
" { Bn$4019[1], Tn$4019[1], An$4019[1], Bn$4019[2], Tn$4019[2], An$4019[2] }
" { Bn$4036[1], Tn$4036[1], An$4036[1], Bn$4036[2], Tn$4036[2], An$4036[2] }
" { Bn$4054[1], Tn$4054[1], An$4054[1], Bn$4054[2], Tn$4054[2], An$4054[2] }
" { Bn$4065[1], Tn$4065[1], An$4065[1], Bn$4065[2], Tn$4065[2], An$4065[2] }
" { Bn$4104[1], Tn$4104[1], An$4104[1], Bn$4104[2], Tn$4104[2], An$4104[2] }
" { Bn$4129[3], Tn$4129[3], An$4129[3] }
" { Bn$4159[1], Tn$4159[1], An$4159[1] }
" { Bn$4178[1], Tn$4178[1], An$4178[1] }
" { Bn$4189[1], Tn$4189[1], An$4189[1], Bn$4189[2], Tn$4189[2], An$4189[2] }
" { Bn$4217[3], Tn$4217[3], An$4217[3] }
" { Bn$4249[1], Tn$4249[1], An$4249[1] }
" { Bn$4268[1], Tn$4268[1], An$4268[1] }
" { Bn$4310[i$], Tn$4310[i$], An$4310[i$], Bn$4310[j$], Tn$4310[j$],
  An$4310[j$], Bn$4310[1], Tn$4310[1], An$4310[1], Bn$4310[2], Tn$4310[2], An$4310[2] }
" { Bn$4375[i$], Tn$4375[i$], An$4375[i$], Bn$4375[j$], Tn$4375[j$],
  An$4375[j$], Bn$4375[3], Tn$4375[3], An$4375[3], Bn$4375[4], Tn$4375[4], An$4375[4] }
" { Bn$4492[i], Tn$4492[i], An$4492[i] }
" { Bn$4623[1], Tn$4623[1], An$4623[1] }
" { Bn$4700[1], Tn$4700[1], An$4700[1] }
" { Bn$4848[1], Tn$4848[1], An$4848[1] }
" { Bn$5033[1], Tn$5033[1], An$5033[1] }

```

Out[]:= \$Aborted

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```

In[ ]:= Timing@Block[{ $k = 3,
  HL /@ { (am1,2→1 ~ B1 ~ am1,3→1) ≡ (am2,3→2 ~ B2 ~ am1,2→1), (bm1,2→1 ~ B1 ~ bm1,3→1) ≡ (bm2,3→2 ~ B2 ~ bm1,2→1) }
}

```

- '' { $b_{n\$4635[1]} \rightarrow b_{n\$4635[1]}$, $t_{n\$4635[1]} \rightarrow t_{n\$4635[1]}$, $\alpha_{n\$4635[1]} \rightarrow \alpha_1 + \alpha_2 + \alpha_{n\$4635[1]}$ }
- '' { $B_{n\$4635[1]} \rightarrow B_{n\$4635[1]}$, $T_{n\$4635[1]} \rightarrow T_{n\$4635[1]}$, $\mathcal{A}_{n\$4635[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_{n\$4635[1]}$ }
- '' { $b_{n\$4654[2]} \rightarrow b_{n\$4654[2]}$, $t_{n\$4654[2]} \rightarrow t_{n\$4654[2]}$, $\alpha_{n\$4654[2]} \rightarrow \alpha_2 + \alpha_3 + \alpha_{n\$4654[2]}$ }
- '' { $B_{n\$4654[2]} \rightarrow B_{n\$4654[2]}$, $T_{n\$4654[2]} \rightarrow T_{n\$4654[2]}$, $\mathcal{A}_{n\$4654[2]} \rightarrow \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_{n\$4654[2]}$ }
- '' { $b_{n\$4679[1]} \rightarrow b_1 + b_{n\$4679[1]}$, $t_{n\$4679[1]} \rightarrow t_{n\$4679[1]}$, $\alpha_{n\$4679[1]} \rightarrow \alpha_{n\$4679[1]}$ }
- '' { $B_{n\$4679[1]} \rightarrow B_1 B_{n\$4679[1]}$, $T_{n\$4679[1]} \rightarrow T_{n\$4679[1]}$, $\mathcal{A}_{n\$4679[1]} \rightarrow \mathcal{A}_{n\$4679[1]}$ }
- '' { $b_{n\$4706[2]} \rightarrow b_1 + b_{n\$4706[2]}$, $t_{n\$4706[2]} \rightarrow t_{n\$4706[2]}$, $\alpha_{n\$4706[2]} \rightarrow \alpha_{n\$4706[2]}$ }
- '' { $B_{n\$4706[2]} \rightarrow B_1 B_{n\$4706[2]}$, $T_{n\$4706[2]} \rightarrow T_{n\$4706[2]}$, $\mathcal{A}_{n\$4706[2]} \rightarrow \mathcal{A}_{n\$4706[2]}$ }

Out[*]= { 0.109375, { True, True } }

R and P are inverses:

In[*]:= Timing@Block[{ \$k = 3 }, { $R_{i,j}$, $P_{i,k}$, HL[$R_{i,j} \sim B_i \sim P_{i,k} \equiv \mathbb{E}[a_j \alpha_k, x_j \xi_k, 1]$] }]

- » { $b_{n\$4745[1]} \rightarrow b_{n\$4745[1]} + \frac{\alpha_j \$}{\hbar}$, $t_{n\$4745[1]} \rightarrow t_{n\$4745[1]}$, $\alpha_{n\$4745[1]} \rightarrow \alpha_{n\$4745[1]}$,
 $b_{n\$4745[2]} \rightarrow b_{n\$4745[2]}$, $t_{n\$4745[2]} \rightarrow t_{n\$4745[2]}$, $\alpha_{n\$4745[2]} \rightarrow \hbar \left(b_{n\$4745[1]} + \frac{\alpha_j \$}{\hbar} \right) + \alpha_{n\$4745[2]}$ }
- » { $B_{n\$4745[1]} \rightarrow \frac{B_{n\$4745[1]}}{\mathcal{A}_j \$}$, $T_{n\$4745[1]} \rightarrow T_{n\$4745[1]}$, $\mathcal{A}_{n\$4745[1]} \rightarrow \mathcal{A}_{n\$4745[1]}$,
 $B_{n\$4745[2]} \rightarrow B_{n\$4745[2]}$, $T_{n\$4745[2]} \rightarrow T_{n\$4745[2]}$, $\mathcal{A}_{n\$4745[2]} \rightarrow \frac{\mathcal{A}_j \$ \mathcal{A}_{n\$4745[2]}}{B_{n\$4745[1]}}$ }
- » { $b_{n\$4791[i]} \rightarrow b_{n\$4791[i]} + \frac{\alpha_k}{\hbar}$, $t_{n\$4791[i]} \rightarrow t_{n\$4791[i]}$, $\alpha_{n\$4791[i]} \rightarrow \alpha_{n\$4791[i]}$ }
- » { $B_{n\$4791[i]} \rightarrow \frac{B_{n\$4791[i]}}{\mathcal{A}_k}$, $T_{n\$4791[i]} \rightarrow T_{n\$4791[i]}$, $\mathcal{A}_{n\$4791[i]} \rightarrow \mathcal{A}_{n\$4791[i]}$ }

Out[*]= { 0.09375, { $\mathbb{E} \left[\hbar a_j b_i, \hbar x_j y_i, 1 - \frac{1}{4} (\gamma \hbar^3 x_j^2 y_i^2) \epsilon + \left(\frac{1}{9} \gamma^2 \hbar^5 x_j^3 y_i^3 + \frac{1}{32} \gamma^2 \hbar^6 x_j^4 y_i^4 \right) \epsilon^2 + \right.$
 $\left. \frac{1}{1152} (24 \gamma^3 \hbar^5 x_j^2 y_i^2 - 72 \gamma^3 \hbar^7 x_j^4 y_i^4 - 32 \gamma^3 \hbar^8 x_j^5 y_i^5 - 3 \gamma^3 \hbar^9 x_j^6 y_i^6) \epsilon^3 + 0[\epsilon]^4 \right]$,
 $\mathbb{E} \left[\frac{\alpha_k \beta_i}{\hbar}, \frac{\eta_i \xi_k}{\hbar}, 1 + \frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4 \hbar} + \frac{(36 \gamma^2 \hbar^2 \eta_i^2 \xi_k^2 + 40 \gamma^2 \hbar \eta_i^3 \xi_k^3 + 9 \gamma^2 \eta_i^4 \xi_k^4) \epsilon^2}{288 \hbar^2} - \frac{1}{1152 \hbar^3} \right.$
 $\left. (-48 \gamma^3 \hbar^4 \eta_i^2 \xi_k^2 - 192 \gamma^3 \hbar^3 \eta_i^3 \xi_k^3 - 156 \gamma^3 \hbar^2 \eta_i^4 \xi_k^4 - 40 \gamma^3 \hbar \eta_i^5 \xi_k^5 - 3 \gamma^3 \eta_i^6 \xi_k^6) \epsilon^3 + 0[\epsilon]^4 \right]$, True } }

as and \overline{aS} are inverses, b_s and \overline{bS} are inverses:

In[*]:= Timing[HL /@ { $\overline{aS}_1 \sim B_1 \sim aS_1 \equiv \mathbb{E}[a_1 \alpha_1, x_1 \xi_1, 1]$, $\overline{bS}_1 \sim B_1 \sim bS_1 \equiv \mathbb{E}[b_1 \beta_1, y_1 \eta_1, 1]$ }]

```

" {bn$4860[i$] -> bn$4860[i$], tn$4860[i$] -> tn$4860[i$], an$4860[i$] -> -ai$ + an$4860[i$] }
» {Bn$4860[i$] -> Bn$4860[i$], Tn$4860[i$] -> Tn$4860[i$], An$4860[i$] -> An$4860[i$] / ai$ }
" {bn$4905[i$] -> bn$4905[i$], tn$4905[i$] -> tn$4905[i$], an$4905[i$] -> ai$ + an$4905[i$] }
" {Bn$4905[i$] -> Bn$4905[i$], Tn$4905[i$] -> Tn$4905[i$], An$4905[i$] -> ai$ An$4905[i$] }
" {bn$4970[1] -> bn$4970[1], tn$4970[1] -> tn$4970[1], an$4970[1] -> -a1 + an$4970[1] }
» {Bn$4970[1] -> Bn$4970[1], Tn$4970[1] -> Tn$4970[1], An$4970[1] -> An$4970[1] / a1 }
" {bn$5071[1] -> bn$5071[1], tn$5071[1] -> tn$5071[1], an$5071[1] -> h1 bi + an$5071[1] }
» {Bn$5071[1] -> Bn$5071[1], Tn$5071[1] -> Tn$5071[1], An$5071[1] -> An$5071[1] / Bi }
" {bn$5131[1] -> bn$5131[1], tn$5131[1] -> tn$5131[1], an$5131[1] -> -h1 bi + an$5131[1] }
" {Bn$5131[1] -> Bn$5131[1], Tn$5131[1] -> Tn$5131[1], An$5131[1] -> Bi An$5131[1] }
" {bn$5198[1] -> -b1 + bn$5198[1], tn$5198[1] -> tn$5198[1], an$5198[1] -> an$5198[1] }
» {Bn$5198[1] -> Bn$5198[1] / B1, Tn$5198[1] -> Tn$5198[1], An$5198[1] -> An$5198[1] }

```

Out[]:= {0.359375, {True, True}}

(co)-associativity on both sides

In[]:= Timing[HL /@

$$\left\{ \left(a\Delta_{1 \rightarrow 1, 2} \sim B_2 \sim a\Delta_{2 \rightarrow 2, 3} \right) \equiv \left(a\Delta_{1 \rightarrow 1, 3} \sim B_1 \sim a\Delta_{1 \rightarrow 1, 2} \right), \left(b\Delta_{1 \rightarrow 1, 2} \sim B_2 \sim b\Delta_{2 \rightarrow 2, 3} \right) \equiv \left(b\Delta_{1 \rightarrow 1, 3} \sim B_1 \sim b\Delta_{1 \rightarrow 1, 2} \right), \right. \\ \left. \left(am_{1, 2 \rightarrow 1} \sim B_1 \sim am_{1, 3 \rightarrow 1} \right) \equiv \left(am_{2, 3 \rightarrow 2} \sim B_2 \sim am_{1, 2 \rightarrow 1} \right), \left(bm_{1, 2 \rightarrow 1} \sim B_1 \sim bm_{1, 3 \rightarrow 1} \right) \equiv \left(bm_{2, 3 \rightarrow 2} \sim B_2 \sim bm_{1, 2 \rightarrow 1} \right) \right\}$$

```

" { b_n$5300[2] -> b_n$5300[2], t_n$5300[2] -> t_n$5300[2], alpha_n$5300[2] -> alpha_1 + alpha_n$5300[2] }
" { B_n$5300[2] -> B_n$5300[2], T_n$5300[2] -> T_n$5300[2], A_n$5300[2] -> A_1 A_n$5300[2] }
" { b_n$5332[1] -> b_n$5332[1], t_n$5332[1] -> t_n$5332[1], alpha_n$5332[1] -> alpha_1 + alpha_n$5332[1] }
" { B_n$5332[1] -> B_n$5332[1], T_n$5332[1] -> T_n$5332[1], A_n$5332[1] -> A_1 A_n$5332[1] }
" { b_n$5367[2] -> b_2 + b_3 + b_n$5367[2], t_n$5367[2] -> t_n$5367[2], alpha_n$5367[2] -> alpha_n$5367[2] }
" { B_n$5367[2] -> B_2 B_3 B_n$5367[2], T_n$5367[2] -> T_n$5367[2], A_n$5367[2] -> A_n$5367[2] }
" { b_n$5416[1] -> b_1 + b_2 + b_n$5416[1], t_n$5416[1] -> t_n$5416[1], alpha_n$5416[1] -> alpha_n$5416[1] }
" { B_n$5416[1] -> B_1 B_2 B_n$5416[1], T_n$5416[1] -> T_n$5416[1], A_n$5416[1] -> A_n$5416[1] }
" { b_n$5471[1] -> b_n$5471[1], t_n$5471[1] -> t_n$5471[1], alpha_n$5471[1] -> alpha_1 + alpha_2 + alpha_n$5471[1] }
" { B_n$5471[1] -> B_n$5471[1], T_n$5471[1] -> T_n$5471[1], A_n$5471[1] -> A_1 A_2 A_n$5471[1] }
" { b_n$5490[2] -> b_n$5490[2], t_n$5490[2] -> t_n$5490[2], alpha_n$5490[2] -> alpha_2 + alpha_3 + alpha_n$5490[2] }
" { B_n$5490[2] -> B_n$5490[2], T_n$5490[2] -> T_n$5490[2], A_n$5490[2] -> A_2 A_3 A_n$5490[2] }
" { b_n$5515[1] -> b_1 + b_n$5515[1], t_n$5515[1] -> t_n$5515[1], alpha_n$5515[1] -> alpha_n$5515[1] }
" { B_n$5515[1] -> B_1 B_n$5515[1], T_n$5515[1] -> T_n$5515[1], A_n$5515[1] -> A_n$5515[1] }
" { b_n$5538[2] -> b_1 + b_n$5538[2], t_n$5538[2] -> t_n$5538[2], alpha_n$5538[2] -> alpha_n$5538[2] }
" { B_n$5538[2] -> B_1 B_n$5538[2], T_n$5538[2] -> T_n$5538[2], A_n$5538[2] -> A_n$5538[2] }

```

Out[*]= {0.4375, {True, True, True, True}}

Δ is an algebra morphism

```

In[*]= Timing[HL /@ { am_{1,2->1} ~ B_1 ~ a_{\Delta_{1->1,2}} \equiv (a_{\Delta_{1->1,3}} a_{\Delta_{2->2,4}}) ~ B_{1,2,3,4} ~ (am_{3,4->2} am_{1,2->1}),
  bm_{1,2->1} ~ B_1 ~ b_{\Delta_{1->1,2}} \equiv (b_{\Delta_{1->1,3}} b_{\Delta_{2->2,4}}) ~ B_{1,2,3,4} ~ (bm_{3,4->2} bm_{1,2->1}) } ]

```

```

" { b_n$5568[1] -> b_n$5568[1], t_n$5568[1] -> t_n$5568[1], alpha_n$5568[1] -> alpha_1 + alpha_2 + alpha_n$5568[1] }
" { B_n$5568[1] -> B_n$5568[1], T_n$5568[1] -> T_n$5568[1], A_n$5568[1] -> A_1 A_2 A_n$5568[1] }
" { b_n$5593[1] -> b_n$5593[1], t_n$5593[1] -> t_n$5593[1], alpha_n$5593[1] -> alpha_1 + alpha_n$5593[1], b_n$5593[2] -> b_n$5593[2],
  t_n$5593[2] -> t_n$5593[2], alpha_n$5593[2] -> alpha_2 + alpha_n$5593[2], b_n$5593[3] -> b_n$5593[3], t_n$5593[3] -> t_n$5593[3],
  alpha_n$5593[3] -> alpha_1 + alpha_n$5593[3], b_n$5593[4] -> b_n$5593[4], t_n$5593[4] -> t_n$5593[4], alpha_n$5593[4] -> alpha_2 + alpha_n$5593[4] }
" { B_n$5593[1] -> B_n$5593[1], T_n$5593[1] -> T_n$5593[1], A_n$5593[1] -> A_1 A_n$5593[1], B_n$5593[2] -> B_n$5593[2],
  T_n$5593[2] -> T_n$5593[2], A_n$5593[2] -> A_2 A_n$5593[2], B_n$5593[3] -> B_n$5593[3], T_n$5593[3] -> T_n$5593[3],
  A_n$5593[3] -> A_1 A_n$5593[3], B_n$5593[4] -> B_n$5593[4], T_n$5593[4] -> T_n$5593[4], A_n$5593[4] -> A_2 A_n$5593[4] }
" { b_n$5648[1] -> b_1 + b_2 + b_n$5648[1], t_n$5648[1] -> t_n$5648[1], alpha_n$5648[1] -> alpha_n$5648[1] }
" { B_n$5648[1] -> B_1 B_2 B_n$5648[1], T_n$5648[1] -> T_n$5648[1], A_n$5648[1] -> A_n$5648[1] }
" { b_n$5685[1] -> b_1 + b_n$5685[1], t_n$5685[1] -> t_n$5685[1], alpha_n$5685[1] -> alpha_n$5685[1], b_n$5685[2] -> b_1 + b_n$5685[2],
  t_n$5685[2] -> t_n$5685[2], alpha_n$5685[2] -> alpha_n$5685[2], b_n$5685[3] -> b_2 + b_n$5685[3], t_n$5685[3] -> t_n$5685[3],
  alpha_n$5685[3] -> alpha_n$5685[3], b_n$5685[4] -> b_2 + b_n$5685[4], t_n$5685[4] -> t_n$5685[4], alpha_n$5685[4] -> alpha_n$5685[4] }
" { B_n$5685[1] -> B_1 B_n$5685[1], T_n$5685[1] -> T_n$5685[1], A_n$5685[1] -> A_n$5685[1], B_n$5685[2] -> B_1 B_n$5685[2],
  T_n$5685[2] -> T_n$5685[2], A_n$5685[2] -> A_n$5685[2], B_n$5685[3] -> B_2 B_n$5685[3], T_n$5685[3] -> T_n$5685[3],
  A_n$5685[3] -> A_n$5685[3], B_n$5685[4] -> B_2 B_n$5685[4], T_n$5685[4] -> T_n$5685[4], A_n$5685[4] -> A_n$5685[4] }

```

Out[*]= {0.625, {True, True}}

An explicit formula for aS_i

$$\begin{aligned}
 \text{In[]:= } & \text{Timing@Block}[\{\$k = 4\}, \text{HL}[\text{aSi} \equiv \mathbb{E}[-\alpha_i a_j, -\xi_i X_i, \\
 & \text{Sum}[\text{Expand}[\frac{e^{\xi_i X_i} (-\hbar \gamma \epsilon)^k}{2^k k!} \text{Nest}[\text{Expand}[X_i^2 \partial_{\{X_i, 2\}} \#] \&, e^{-\xi_i e^{\hbar \epsilon a_i} X_i}, k]], \{k, \theta, \$k\}]]_{\$k \sim} \\
 & \text{Bi}_{i,j} \sim \text{am}_{i,j \rightarrow i}]] \\
 \text{"} & \{b_{n\$5771[i\$]} \rightarrow b_{i\$} + b_{n\$5771[i\$]}, t_{n\$5771[i\$]} \rightarrow t_{n\$5771[i\$]}, \alpha_{n\$5771[i\$]} \rightarrow \alpha_{n\$5771[i\$]}, \\
 & b_{n\$5771[j\$]} \rightarrow b_{n\$5771[j\$]}, t_{n\$5771[j\$]} \rightarrow t_{n\$5771[j\$]}, \alpha_{n\$5771[j\$]} \rightarrow -\hbar (b_{i\$} + b_{n\$5771[i\$]}) + \alpha_{n\$5771[j\$]}, \\
 & b_{n\$5771[1]} \rightarrow b_{i\$} + b_{n\$5771[1]}, t_{n\$5771[1]} \rightarrow t_{n\$5771[1]}, \alpha_{n\$5771[1]} \rightarrow \alpha_{n\$5771[1]}, \\
 & b_{n\$5771[2]} \rightarrow b_{n\$5771[2]}, t_{n\$5771[2]} \rightarrow t_{n\$5771[2]}, \alpha_{n\$5771[2]} \rightarrow \hbar (b_{i\$} + b_{n\$5771[1]}) + \alpha_{n\$5771[2]}\} \\
 \text{»} & \{B_{n\$5771[i\$]} \rightarrow B_{i\$} B_{n\$5771[i\$]}, T_{n\$5771[i\$]} \rightarrow T_{n\$5771[i\$]}, \mathcal{A}_{n\$5771[i\$]} \rightarrow \mathcal{A}_{n\$5771[i\$]}, B_{n\$5771[j\$]} \rightarrow B_{n\$5771[j\$]}, \\
 & T_{n\$5771[j\$]} \rightarrow T_{n\$5771[j\$]}, \mathcal{A}_{n\$5771[j\$]} \rightarrow B_{i\$} B_{n\$5771[i\$]} \mathcal{A}_{n\$5771[j\$]}, B_{n\$5771[1]} \rightarrow B_{i\$} B_{n\$5771[1]}, T_{n\$5771[1]} \rightarrow T_{n\$5771[1]}, \\
 & \mathcal{A}_{n\$5771[1]} \rightarrow \mathcal{A}_{n\$5771[1]}, B_{n\$5771[2]} \rightarrow B_{n\$5771[2]}, T_{n\$5771[2]} \rightarrow T_{n\$5771[2]}, \mathcal{A}_{n\$5771[2]} \rightarrow \frac{\mathcal{A}_{n\$5771[2]}}{B_{i\$} B_{n\$5771[1]}}\} \\
 \text{"} & \{b_{n\$6045[i\$]} \rightarrow b_{i\$} + b_{n\$6045[i\$]}, t_{n\$6045[i\$]} \rightarrow t_{n\$6045[i\$]}, \alpha_{n\$6045[i\$]} \rightarrow \alpha_{n\$6045[i\$]}, b_{n\$6045[j\$]} \rightarrow b_{n\$6045[j\$]}, \\
 & t_{n\$6045[j\$]} \rightarrow t_{n\$6045[j\$]}, \alpha_{n\$6045[j\$]} \rightarrow \alpha_{n\$6045[j\$]}, b_{n\$6045[3]} \rightarrow b_{i\$} + b_{n\$6045[3]}, t_{n\$6045[3]} \rightarrow t_{n\$6045[3]}, \\
 & \alpha_{n\$6045[3]} \rightarrow \alpha_{n\$6045[3]}, b_{n\$6045[4]} \rightarrow b_{n\$6045[4]}, t_{n\$6045[4]} \rightarrow t_{n\$6045[4]}, \alpha_{n\$6045[4]} \rightarrow -\hbar (b_{i\$} + b_{n\$6045[3]}) + \alpha_{n\$6045[4]}\} \\
 \text{"} & \{B_{n\$6045[i\$]} \rightarrow B_{i\$} B_{n\$6045[i\$]}, T_{n\$6045[i\$]} \rightarrow T_{n\$6045[i\$]}, \mathcal{A}_{n\$6045[i\$]} \rightarrow \mathcal{A}_{n\$6045[i\$]}, B_{n\$6045[j\$]} \rightarrow B_{n\$6045[j\$]}, \\
 & T_{n\$6045[j\$]} \rightarrow T_{n\$6045[j\$]}, \mathcal{A}_{n\$6045[j\$]} \rightarrow \mathcal{A}_{n\$6045[j\$]}, B_{n\$6045[3]} \rightarrow B_{i\$} B_{n\$6045[3]}, T_{n\$6045[3]} \rightarrow T_{n\$6045[3]}, \\
 & \mathcal{A}_{n\$6045[3]} \rightarrow \mathcal{A}_{n\$6045[3]}, B_{n\$6045[4]} \rightarrow B_{n\$6045[4]}, T_{n\$6045[4]} \rightarrow T_{n\$6045[4]}, \mathcal{A}_{n\$6045[4]} \rightarrow B_{i\$} B_{n\$6045[3]} \mathcal{A}_{n\$6045[4]}\} \\
 \text{"} & \{b_{n\$6456[i\$]} \rightarrow b_{i\$} + b_{n\$6456[i\$]}, t_{n\$6456[i\$]} \rightarrow t_{n\$6456[i\$]}, \alpha_{n\$6456[i\$]} \rightarrow \alpha_{n\$6456[i\$]}, \\
 & b_{n\$6456[j\$]} \rightarrow b_{n\$6456[j\$]}, t_{n\$6456[j\$]} \rightarrow t_{n\$6456[j\$]}, \alpha_{n\$6456[j\$]} \rightarrow -\hbar (b_{i\$} + b_{n\$6456[i\$]}) + \alpha_{n\$6456[j\$]}, \\
 & b_{n\$6456[1]} \rightarrow b_{i\$} + b_{n\$6456[1]}, t_{n\$6456[1]} \rightarrow t_{n\$6456[1]}, \alpha_{n\$6456[1]} \rightarrow \alpha_{n\$6456[1]}, \\
 & b_{n\$6456[2]} \rightarrow b_{n\$6456[2]}, t_{n\$6456[2]} \rightarrow t_{n\$6456[2]}, \alpha_{n\$6456[2]} \rightarrow \hbar (b_{i\$} + b_{n\$6456[1]}) + \alpha_{n\$6456[2]}\} \\
 \text{»} & \{B_{n\$6456[i\$]} \rightarrow B_{i\$} B_{n\$6456[i\$]}, T_{n\$6456[i\$]} \rightarrow T_{n\$6456[i\$]}, \mathcal{A}_{n\$6456[i\$]} \rightarrow \mathcal{A}_{n\$6456[i\$]}, B_{n\$6456[j\$]} \rightarrow B_{n\$6456[j\$]}, \\
 & T_{n\$6456[j\$]} \rightarrow T_{n\$6456[j\$]}, \mathcal{A}_{n\$6456[j\$]} \rightarrow B_{i\$} B_{n\$6456[i\$]} \mathcal{A}_{n\$6456[j\$]}, B_{n\$6456[1]} \rightarrow B_{i\$} B_{n\$6456[1]}, T_{n\$6456[1]} \rightarrow T_{n\$6456[1]}, \\
 & \mathcal{A}_{n\$6456[1]} \rightarrow \mathcal{A}_{n\$6456[1]}, B_{n\$6456[2]} \rightarrow B_{n\$6456[2]}, T_{n\$6456[2]} \rightarrow T_{n\$6456[2]}, \mathcal{A}_{n\$6456[2]} \rightarrow \frac{\mathcal{A}_{n\$6456[2]}}{B_{i\$} B_{n\$6456[1]}}\} \\
 \text{"} & \{b_{n\$7482[i\$]} \rightarrow b_{i\$} + b_{n\$7482[i\$]}, t_{n\$7482[i\$]} \rightarrow t_{n\$7482[i\$]}, \alpha_{n\$7482[i\$]} \rightarrow \alpha_{n\$7482[i\$]}, b_{n\$7482[j\$]} \rightarrow b_{n\$7482[j\$]}, \\
 & t_{n\$7482[j\$]} \rightarrow t_{n\$7482[j\$]}, \alpha_{n\$7482[j\$]} \rightarrow \alpha_{n\$7482[j\$]}, b_{n\$7482[3]} \rightarrow b_{i\$} + b_{n\$7482[3]}, t_{n\$7482[3]} \rightarrow t_{n\$7482[3]}, \\
 & \alpha_{n\$7482[3]} \rightarrow \alpha_{n\$7482[3]}, b_{n\$7482[4]} \rightarrow b_{n\$7482[4]}, t_{n\$7482[4]} \rightarrow t_{n\$7482[4]}, \alpha_{n\$7482[4]} \rightarrow -\hbar (b_{i\$} + b_{n\$7482[3]}) + \alpha_{n\$7482[4]}\} \\
 \text{"} & \{B_{n\$7482[i\$]} \rightarrow B_{i\$} B_{n\$7482[i\$]}, T_{n\$7482[i\$]} \rightarrow T_{n\$7482[i\$]}, \mathcal{A}_{n\$7482[i\$]} \rightarrow \mathcal{A}_{n\$7482[i\$]}, B_{n\$7482[j\$]} \rightarrow B_{n\$7482[j\$]}, \\
 & T_{n\$7482[j\$]} \rightarrow T_{n\$7482[j\$]}, \mathcal{A}_{n\$7482[j\$]} \rightarrow \mathcal{A}_{n\$7482[j\$]}, B_{n\$7482[3]} \rightarrow B_{i\$} B_{n\$7482[3]}, T_{n\$7482[3]} \rightarrow T_{n\$7482[3]}, \\
 & \mathcal{A}_{n\$7482[3]} \rightarrow \mathcal{A}_{n\$7482[3]}, B_{n\$7482[4]} \rightarrow B_{n\$7482[4]}, T_{n\$7482[4]} \rightarrow T_{n\$7482[4]}, \mathcal{A}_{n\$7482[4]} \rightarrow B_{i\$} B_{n\$7482[3]} \mathcal{A}_{n\$7482[4]}\} \\
 \text{»} & \{b_{n\$8917[1]} \rightarrow b_{n\$8917[1]} + \frac{\alpha_{j\$}}{\hbar}, t_{n\$8917[1]} \rightarrow t_{n\$8917[1]}, \alpha_{n\$8917[1]} \rightarrow \alpha_{n\$8917[1]}, \\
 & b_{n\$8917[2]} \rightarrow b_{n\$8917[2]}, t_{n\$8917[2]} \rightarrow t_{n\$8917[2]}, \alpha_{n\$8917[2]} \rightarrow \hbar \left(b_{n\$8917[1]} + \frac{\alpha_{j\$}}{\hbar} \right) + \alpha_{n\$8917[2]}\} \\
 \text{»} & \{B_{n\$8917[1]} \rightarrow \frac{B_{n\$8917[1]}}{\mathcal{A}_{j\$}}, T_{n\$8917[1]} \rightarrow T_{n\$8917[1]}, \mathcal{A}_{n\$8917[1]} \rightarrow \mathcal{A}_{n\$8917[1]}, \\
 & B_{n\$8917[2]} \rightarrow B_{n\$8917[2]}, T_{n\$8917[2]} \rightarrow T_{n\$8917[2]}, \mathcal{A}_{n\$8917[2]} \rightarrow \frac{\mathcal{A}_{j\$} \mathcal{A}_{n\$8917[2]}}{B_{n\$8917[1]}}\} \\
 \text{»} & \{b_{n\$9032[i]} \rightarrow b_{n\$9032[i]} + \frac{\alpha_j}{\hbar}, t_{n\$9032[i]} \rightarrow t_{n\$9032[i]}, \alpha_{n\$9032[i]} \rightarrow \alpha_{n\$9032[i]}\}
 \end{aligned}$$

$$\begin{aligned}
 & \gg \left\{ \mathbf{B}_{n\$9032[i]} \rightarrow \frac{\mathbf{B}_{n\$9032[i]}}{\mathcal{A}_j}, \mathbf{T}_{n\$9032[i]} \rightarrow \mathbf{T}_{n\$9032[i]}, \mathcal{A}_{n\$9032[i]} \rightarrow \mathcal{A}_{n\$9032[i]} \right\} \\
 & \text{''} \left\{ \mathbf{b}_{n\$9479[i]} \rightarrow \mathbf{b}_{n\$9479[i]}, \mathbf{t}_{n\$9479[i]} \rightarrow \mathbf{t}_{n\$9479[i]}, \alpha_{n\$9479[i]} \rightarrow \alpha_{n\$9479[i]}, \right. \\
 & \quad \left. \mathbf{b}_{n\$9479[j]} \rightarrow \mathbf{b}_{n\$9479[j]}, \mathbf{t}_{n\$9479[j]} \rightarrow \mathbf{t}_{n\$9479[j]}, \alpha_{n\$9479[j]} \rightarrow -\alpha_i + \alpha_{n\$9479[j]} \right\} \\
 & \gg \left\{ \mathbf{B}_{n\$9479[i]} \rightarrow \mathbf{B}_{n\$9479[i]}, \mathbf{T}_{n\$9479[i]} \rightarrow \mathbf{T}_{n\$9479[i]}, \mathcal{A}_{n\$9479[i]} \rightarrow \mathcal{A}_{n\$9479[i]}, \right. \\
 & \quad \left. \mathbf{B}_{n\$9479[j]} \rightarrow \mathbf{B}_{n\$9479[j]}, \mathbf{T}_{n\$9479[j]} \rightarrow \mathbf{T}_{n\$9479[j]}, \mathcal{A}_{n\$9479[j]} \rightarrow \frac{\mathcal{A}_{n\$9479[j]}}{\mathcal{A}_i} \right\}
 \end{aligned}$$

Out[*]= { 3.10938, True }

S is convolution inverse of id

$$\text{In[*]= Timing[HL[# \equiv \mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{1}]] \& /@ \{ }$$

$$\begin{aligned}
 & \left(\mathbf{a}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_1 \sim \mathbf{a}\mathbf{S}_1 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{a}\mathbf{m}_{1, 2 \rightarrow 1}, \left(\mathbf{a}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_2 \sim \mathbf{a}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{a}\mathbf{m}_{1, 2 \rightarrow 1}, \\
 & \left(\mathbf{b}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_1 \sim \mathbf{b}\mathbf{S}_1 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{b}\mathbf{m}_{1, 2 \rightarrow 1}, \left(\mathbf{b}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_2 \sim \mathbf{b}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{b}\mathbf{m}_{1, 2 \rightarrow 1} \left. \right\}]
 \end{aligned}$$


```

" { b_n$9534[1] -> b_n$9534[1], t_n$9534[1] -> t_n$9534[1], alpha_n$9534[1] -> alpha_1 + alpha_n$9534[1] }
" { B_n$9534[1] -> B_n$9534[1], T_n$9534[1] -> T_n$9534[1], A_n$9534[1] -> A_1 A_n$9534[1] }
" { b_n$9611[1] -> b_n$9611[1], t_n$9611[1] -> t_n$9611[1], alpha_n$9611[1] -> -alpha_1 + alpha_n$9611[1],
  b_n$9611[2] -> b_n$9611[2], t_n$9611[2] -> t_n$9611[2], alpha_n$9611[2] -> alpha_1 + alpha_n$9611[2] }
» { B_n$9611[1] -> B_n$9611[1], T_n$9611[1] -> T_n$9611[1], A_n$9611[1] -> (A_n$9611[1])/A_1,
  B_n$9611[2] -> B_n$9611[2], T_n$9611[2] -> T_n$9611[2], A_n$9611[2] -> A_1 A_n$9611[2] }
" { b_n$9672[2] -> b_n$9672[2], t_n$9672[2] -> t_n$9672[2], alpha_n$9672[2] -> alpha_1 + alpha_n$9672[2] }
" { B_n$9672[2] -> B_n$9672[2], T_n$9672[2] -> T_n$9672[2], A_n$9672[2] -> A_1 A_n$9672[2] }
" { b_n$9732[1] -> b_n$9732[1], t_n$9732[1] -> t_n$9732[1], alpha_n$9732[1] -> alpha_1 + alpha_n$9732[1],
  b_n$9732[2] -> b_n$9732[2], t_n$9732[2] -> t_n$9732[2], alpha_n$9732[2] -> -alpha_1 + alpha_n$9732[2] }
» { B_n$9732[1] -> B_n$9732[1], T_n$9732[1] -> T_n$9732[1], A_n$9732[1] -> A_1 A_n$9732[1],
  B_n$9732[2] -> B_n$9732[2], T_n$9732[2] -> T_n$9732[2], A_n$9732[2] -> (A_n$9732[2])/A_1 }
" { b_n$9795[1] -> -b_1 + b_n$9795[1], t_n$9795[1] -> t_n$9795[1], alpha_n$9795[1] -> alpha_n$9795[1] }
» { B_n$9795[1] -> (B_n$9795[1])/B_1, T_n$9795[1] -> T_n$9795[1], A_n$9795[1] -> A_n$9795[1] }
" { b_n$9854[1] -> b_1 + b_n$9854[1], t_n$9854[1] -> t_n$9854[1], alpha_n$9854[1] -> alpha_n$9854[1],
  b_n$9854[2] -> b_1 + b_n$9854[2], t_n$9854[2] -> t_n$9854[2], alpha_n$9854[2] -> alpha_n$9854[2] }
" { B_n$9854[1] -> B_1 B_n$9854[1], T_n$9854[1] -> T_n$9854[1], A_n$9854[1] -> A_n$9854[1],
  B_n$9854[2] -> B_1 B_n$9854[2], T_n$9854[2] -> T_n$9854[2], A_n$9854[2] -> A_n$9854[2] }
" { b_n$9873[2] -> -b_2 + b_n$9873[2], t_n$9873[2] -> t_n$9873[2], alpha_n$9873[2] -> alpha_n$9873[2] }
» { B_n$9873[2] -> (B_n$9873[2])/B_2, T_n$9873[2] -> T_n$9873[2], A_n$9873[2] -> A_n$9873[2] }
" { b_n$9952[1] -> b_1 + b_n$9952[1], t_n$9952[1] -> t_n$9952[1], alpha_n$9952[1] -> alpha_n$9952[1],
  b_n$9952[2] -> b_1 + b_n$9952[2], t_n$9952[2] -> t_n$9952[2], alpha_n$9952[2] -> alpha_n$9952[2] }
" { B_n$9952[1] -> B_1 B_n$9952[1], T_n$9952[1] -> T_n$9952[1], A_n$9952[1] -> A_n$9952[1],
  B_n$9952[2] -> B_1 B_n$9952[2], T_n$9952[2] -> T_n$9952[2], A_n$9952[2] -> A_n$9952[2] }

```

Out[*]= {0.6875, {True, True, True, True}}

But not with the opposite product:

```

In[*]:= Timing[Short[# &#226;#220; [0, 0, 1]] & /@ {
  (aDelta_1->1,2 ~ B_1 ~ aS_1) ~ B_1,2 ~ am_2,1->1, (aDelta_1->1,2 ~ B_2 ~ aS_2) ~ B_1,2 ~ am_2,1->1,
  (bDelta_1->1,2 ~ B_1 ~ bS_1) ~ B_1,2 ~ bm_2,1->1, (bDelta_1->1,2 ~ B_2 ~ bS_2) ~ B_1,2 ~ bm_2,1->1 } ]

```

$$\begin{aligned}
 & \{ b_{n\$10050[1]} \rightarrow b_{n\$10050[1]}, t_{n\$10050[1]} \rightarrow t_{n\$10050[1]}, \alpha_{n\$10050[1]} \rightarrow \alpha_1 + \alpha_{n\$10050[1]} \} \\
 & \{ B_{n\$10050[1]} \rightarrow B_{n\$10050[1]}, T_{n\$10050[1]} \rightarrow T_{n\$10050[1]}, \mathcal{A}_{n\$10050[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$10050[1]} \} \\
 & \{ b_{n\$10127[1]} \rightarrow b_{n\$10127[1]}, t_{n\$10127[1]} \rightarrow t_{n\$10127[1]}, \alpha_{n\$10127[1]} \rightarrow -\alpha_1 + \alpha_{n\$10127[1]}, \\
 & \quad b_{n\$10127[2]} \rightarrow b_{n\$10127[2]}, t_{n\$10127[2]} \rightarrow t_{n\$10127[2]}, \alpha_{n\$10127[2]} \rightarrow \alpha_1 + \alpha_{n\$10127[2]} \} \\
 & \left\{ B_{n\$10127[1]} \rightarrow B_{n\$10127[1]}, T_{n\$10127[1]} \rightarrow T_{n\$10127[1]}, \mathcal{A}_{n\$10127[1]} \rightarrow \frac{\mathcal{A}_{n\$10127[1]}}{\mathcal{A}_1}, \right. \\
 & \quad \left. B_{n\$10127[2]} \rightarrow B_{n\$10127[2]}, T_{n\$10127[2]} \rightarrow T_{n\$10127[2]}, \mathcal{A}_{n\$10127[2]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$10127[2]} \right\} \\
 & \{ b_{n\$10245[2]} \rightarrow b_{n\$10245[2]}, t_{n\$10245[2]} \rightarrow t_{n\$10245[2]}, \alpha_{n\$10245[2]} \rightarrow \alpha_1 + \alpha_{n\$10245[2]} \} \\
 & \{ B_{n\$10245[2]} \rightarrow B_{n\$10245[2]}, T_{n\$10245[2]} \rightarrow T_{n\$10245[2]}, \mathcal{A}_{n\$10245[2]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$10245[2]} \} \\
 & \{ b_{n\$10305[1]} \rightarrow b_{n\$10305[1]}, t_{n\$10305[1]} \rightarrow t_{n\$10305[1]}, \alpha_{n\$10305[1]} \rightarrow \alpha_1 + \alpha_{n\$10305[1]}, \\
 & \quad b_{n\$10305[2]} \rightarrow b_{n\$10305[2]}, t_{n\$10305[2]} \rightarrow t_{n\$10305[2]}, \alpha_{n\$10305[2]} \rightarrow -\alpha_1 + \alpha_{n\$10305[2]} \} \\
 & \left\{ B_{n\$10305[1]} \rightarrow B_{n\$10305[1]}, T_{n\$10305[1]} \rightarrow T_{n\$10305[1]}, \mathcal{A}_{n\$10305[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$10305[1]}, \right. \\
 & \quad \left. B_{n\$10305[2]} \rightarrow B_{n\$10305[2]}, T_{n\$10305[2]} \rightarrow T_{n\$10305[2]}, \mathcal{A}_{n\$10305[2]} \rightarrow \frac{\mathcal{A}_{n\$10305[2]}}{\mathcal{A}_1} \right\} \\
 & \{ b_{n\$10371[1]} \rightarrow -b_1 + b_{n\$10371[1]}, t_{n\$10371[1]} \rightarrow t_{n\$10371[1]}, \alpha_{n\$10371[1]} \rightarrow \alpha_{n\$10371[1]} \} \\
 & \left\{ B_{n\$10371[1]} \rightarrow \frac{B_{n\$10371[1]}}{B_1}, T_{n\$10371[1]} \rightarrow T_{n\$10371[1]}, \mathcal{A}_{n\$10371[1]} \rightarrow \mathcal{A}_{n\$10371[1]} \right\} \\
 & \{ b_{n\$10430[1]} \rightarrow b_1 + b_{n\$10430[1]}, t_{n\$10430[1]} \rightarrow t_{n\$10430[1]}, \alpha_{n\$10430[1]} \rightarrow \alpha_{n\$10430[1]}, \\
 & \quad b_{n\$10430[2]} \rightarrow b_1 + b_{n\$10430[2]}, t_{n\$10430[2]} \rightarrow t_{n\$10430[2]}, \alpha_{n\$10430[2]} \rightarrow \alpha_{n\$10430[2]} \} \\
 & \{ B_{n\$10430[1]} \rightarrow B_1 B_{n\$10430[1]}, T_{n\$10430[1]} \rightarrow T_{n\$10430[1]}, \mathcal{A}_{n\$10430[1]} \rightarrow \mathcal{A}_{n\$10430[1]}, \\
 & \quad B_{n\$10430[2]} \rightarrow B_1 B_{n\$10430[2]}, T_{n\$10430[2]} \rightarrow T_{n\$10430[2]}, \mathcal{A}_{n\$10430[2]} \rightarrow \mathcal{A}_{n\$10430[2]} \} \\
 & \{ b_{n\$10462[2]} \rightarrow -b_2 + b_{n\$10462[2]}, t_{n\$10462[2]} \rightarrow t_{n\$10462[2]}, \alpha_{n\$10462[2]} \rightarrow \alpha_{n\$10462[2]} \} \\
 & \left\{ B_{n\$10462[2]} \rightarrow \frac{B_{n\$10462[2]}}{B_2}, T_{n\$10462[2]} \rightarrow T_{n\$10462[2]}, \mathcal{A}_{n\$10462[2]} \rightarrow \mathcal{A}_{n\$10462[2]} \right\} \\
 & \{ b_{n\$10541[1]} \rightarrow b_1 + b_{n\$10541[1]}, t_{n\$10541[1]} \rightarrow t_{n\$10541[1]}, \alpha_{n\$10541[1]} \rightarrow \alpha_{n\$10541[1]}, \\
 & \quad b_{n\$10541[2]} \rightarrow b_1 + b_{n\$10541[2]}, t_{n\$10541[2]} \rightarrow t_{n\$10541[2]}, \alpha_{n\$10541[2]} \rightarrow \alpha_{n\$10541[2]} \} \\
 & \{ B_{n\$10541[1]} \rightarrow B_1 B_{n\$10541[1]}, T_{n\$10541[1]} \rightarrow T_{n\$10541[1]}, \mathcal{A}_{n\$10541[1]} \rightarrow \mathcal{A}_{n\$10541[1]}, \\
 & \quad B_{n\$10541[2]} \rightarrow B_1 B_{n\$10541[2]}, T_{n\$10541[2]} \rightarrow T_{n\$10541[2]}, \mathcal{A}_{n\$10541[2]} \rightarrow \mathcal{A}_{n\$10541[2]} \} \\
 \text{Out[]:=} & \{ 0.734375, \left\{ \frac{1}{2} \left(-2 \gamma \in \hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2 \epsilon^2 \hbar^2 x_1 \mathcal{A}_1 \xi_1 - 2 \gamma \epsilon^2 \hbar^2 a_1 x_1 \mathcal{A}_1 \xi_1 + 2 \gamma^2 \epsilon^2 \hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2 \right) = 0, \right. \\
 & \quad \frac{1}{2} \left(-2 \gamma \in \hbar x_1 \xi_1 - \gamma^2 \epsilon^2 \hbar^2 x_1 \xi_1 + 2 \gamma^2 \epsilon^2 \hbar^2 x_1^2 \xi_1^2 \right) = 0, \\
 & \quad \frac{1}{2} \left(-2 \gamma \in \hbar y_1 \eta_1 - \gamma^2 \epsilon^2 \hbar^2 y_1 \eta_1 + 2 \gamma^2 \epsilon^2 \hbar^2 y_1^2 \eta_1^2 \right) = 0, \\
 & \quad \left. \frac{-2 \gamma \in \hbar B_1 y_1 \eta_1 + \ll 3 \gg + 2 \gamma^2 \epsilon^2 \hbar^2 y_1^2 \eta_1^2}{2 B_1^2} = 0 \right\} \}
 \end{aligned}$$

S is an algebra anti-(co)morphism

$$\text{In[]:= } \text{Timing} \left[\text{HL} / @ \left\{ \text{am}_{1,2 \rightarrow 1} \sim B_1 \sim \text{aS}_1 \equiv \left(\text{aS}_1 \text{aS}_2 \right) \sim B_{1,2} \sim \text{am}_{2,1 \rightarrow 1}, \text{bm}_{1,2 \rightarrow 1} \sim B_1 \sim \text{bS}_1 \equiv \left(\text{bS}_1 \text{bS}_2 \right) \sim B_{1,2} \sim \text{bm}_{2,1 \rightarrow 1}, \right. \right. \\
 \left. \left. \text{aS}_1 \sim B_1 \sim \text{a}\Delta_{1 \rightarrow 1,2} \equiv \text{a}\Delta_{1 \rightarrow 2,1} \sim B_{1,2} \sim \left(\text{aS}_1 \text{aS}_2 \right), \text{bS}_1 \sim B_1 \sim \text{b}\Delta_{1 \rightarrow 1,2} \equiv \text{b}\Delta_{1 \rightarrow 2,1} \sim B_{1,2} \sim \left(\text{bS}_1 \text{bS}_2 \right) \right\} \right]$$

```

" { b_n$10688[1] -> b_n$10688[1], t_n$10688[1] -> t_n$10688[1], alpha_n$10688[1] -> alpha_1 + alpha_2 + alpha_n$10688[1] }
" { B_n$10688[1] -> B_n$10688[1], T_n$10688[1] -> T_n$10688[1], A_n$10688[1] -> A_1 A_2 A_n$10688[1] }
" { b_n$10733[1] -> b_n$10733[1], t_n$10733[1] -> t_n$10733[1], alpha_n$10733[1] -> -alpha_1 + alpha_n$10733[1],
  b_n$10733[2] -> b_n$10733[2], t_n$10733[2] -> t_n$10733[2], alpha_n$10733[2] -> -alpha_2 + alpha_n$10733[2] }
» { B_n$10733[1] -> B_n$10733[1], T_n$10733[1] -> T_n$10733[1], A_n$10733[1] -> (A_n$10733[1])/A_1,
  B_n$10733[2] -> B_n$10733[2], T_n$10733[2] -> T_n$10733[2], A_n$10733[2] -> (A_n$10733[2])/A_2 }
" { b_n$10832[1] -> -b_1 + b_n$10832[1], t_n$10832[1] -> t_n$10832[1], alpha_n$10832[1] -> alpha_n$10832[1] }
» { B_n$10832[1] -> (B_n$10832[1])/B_1, T_n$10832[1] -> T_n$10832[1], A_n$10832[1] -> A_n$10832[1] }
" { b_n$10895[1] -> b_1 + b_n$10895[1], t_n$10895[1] -> t_n$10895[1], alpha_n$10895[1] -> alpha_n$10895[1],
  b_n$10895[2] -> b_1 + b_n$10895[2], t_n$10895[2] -> t_n$10895[2], alpha_n$10895[2] -> alpha_n$10895[2] }
" { B_n$10895[1] -> B_1 B_n$10895[1], T_n$10895[1] -> T_n$10895[1], A_n$10895[1] -> A_n$10895[1],
  B_n$10895[2] -> B_1 B_n$10895[2], T_n$10895[2] -> T_n$10895[2], A_n$10895[2] -> A_n$10895[2] }
" { b_n$11006[1] -> b_n$11006[1], t_n$11006[1] -> t_n$11006[1], alpha_n$11006[1] -> -alpha_1 + alpha_n$11006[1] }
» { B_n$11006[1] -> B_n$11006[1], T_n$11006[1] -> T_n$11006[1], A_n$11006[1] -> (A_n$11006[1])/A_1 }
" { b_n$11086[1] -> b_n$11086[1], t_n$11086[1] -> t_n$11086[1], alpha_n$11086[1] -> alpha_1 + alpha_n$11086[1],
  b_n$11086[2] -> b_n$11086[2], t_n$11086[2] -> t_n$11086[2], alpha_n$11086[2] -> alpha_1 + alpha_n$11086[2] }
" { B_n$11086[1] -> B_n$11086[1], T_n$11086[1] -> T_n$11086[1], A_n$11086[1] -> A_1 A_n$11086[1],
  B_n$11086[2] -> B_n$11086[2], T_n$11086[2] -> T_n$11086[2], A_n$11086[2] -> A_1 A_n$11086[2] }
" { b_n$11215[1] -> b_1 + b_2 + b_n$11215[1], t_n$11215[1] -> t_n$11215[1], alpha_n$11215[1] -> alpha_n$11215[1] }
" { B_n$11215[1] -> B_1 B_2 B_n$11215[1], T_n$11215[1] -> T_n$11215[1], A_n$11215[1] -> A_n$11215[1] }
" { b_n$11280[1] -> -b_1 + b_n$11280[1], t_n$11280[1] -> t_n$11280[1], alpha_n$11280[1] -> alpha_n$11280[1],
  b_n$11280[2] -> -b_2 + b_n$11280[2], t_n$11280[2] -> t_n$11280[2], alpha_n$11280[2] -> alpha_n$11280[2] }
» { B_n$11280[1] -> (B_n$11280[1])/B_1, T_n$11280[1] -> T_n$11280[1], A_n$11280[1] -> A_n$11280[1],
  B_n$11280[2] -> (B_n$11280[2])/B_2, T_n$11280[2] -> T_n$11280[2], A_n$11280[2] -> A_n$11280[2] }

```

Out[*]= {0.953125, {True, True, True, True}}

Pairing axioms

```

In[*]:= Timing[HL /@ { (bm_{1,2->1} E[alpha_3 a_3, xi_3 x_3, 1]) ~ B_{1,3} ~ P_{1,3} ==
  ( E[beta_1 b_1, eta_1 y_1, 1] E[beta_2 b_2, eta_2 y_2, 1] a_{Delta_{3->4,5}} ) ~ B_{1,4} ~ P_{1,4} ~ B_{2,5} ~ P_{2,5},
  (b_{Delta_{1->1,2}} E[alpha_3 a_3, xi_3 x_3, 1] E[alpha_4 a_4, xi_4 x_4, 1]) ~ B_{1,3} ~ P_{1,3} ~ B_{2,4} ~ P_{2,4} ==
  ( E[beta_1 b_1, eta_1 y_1, 1] am_{3,4->3} ) ~ B_{1,3} ~ P_{1,3} } ]

```

- » $\left\{ b_{n\$11402[1]} \rightarrow b_{n\$11402[1]} + \frac{\alpha_3 + \alpha_{n\$11402[3]}}{\hbar}, t_{n\$11402[1]} \rightarrow t_{n\$11402[1]}, \alpha_{n\$11402[1]} \rightarrow \alpha_{n\$11402[1]}, \right.$
 $\left. b_{n\$11402[3]} \rightarrow b_{n\$11402[3]}, t_{n\$11402[3]} \rightarrow t_{n\$11402[3]}, \alpha_{n\$11402[3]} \rightarrow \alpha_3 + \alpha_{n\$11402[3]} \right\}$
- » $\left\{ B_{n\$11402[1]} \rightarrow \frac{B_{n\$11402[1]}}{\mathcal{A}_3 \mathcal{A}_{n\$11402[3]}}, T_{n\$11402[1]} \rightarrow T_{n\$11402[1]}, \mathcal{A}_{n\$11402[1]} \rightarrow \mathcal{A}_{n\$11402[1]}, \right.$
 $\left. B_{n\$11402[3]} \rightarrow B_{n\$11402[3]}, T_{n\$11402[3]} \rightarrow T_{n\$11402[3]}, \mathcal{A}_{n\$11402[3]} \rightarrow \mathcal{A}_3 \mathcal{A}_{n\$11402[3]} \right\}$
- » $\left\{ b_{n\$11425[1]} \rightarrow b_{n\$11425[1]} + \frac{\alpha_3 + \alpha_{n\$11425[4]}}{\hbar}, t_{n\$11425[1]} \rightarrow t_{n\$11425[1]}, \alpha_{n\$11425[1]} \rightarrow \alpha_{n\$11425[1]}, \right.$
 $\left. b_{n\$11425[4]} \rightarrow b_{n\$11425[4]}, t_{n\$11425[4]} \rightarrow t_{n\$11425[4]}, \alpha_{n\$11425[4]} \rightarrow \alpha_3 + \alpha_{n\$11425[4]} \right\}$
- » $\left\{ B_{n\$11425[1]} \rightarrow \frac{B_{n\$11425[1]}}{\mathcal{A}_3 \mathcal{A}_{n\$11425[4]}}, T_{n\$11425[1]} \rightarrow T_{n\$11425[1]}, \mathcal{A}_{n\$11425[1]} \rightarrow \mathcal{A}_{n\$11425[1]}, \right.$
 $\left. B_{n\$11425[4]} \rightarrow B_{n\$11425[4]}, T_{n\$11425[4]} \rightarrow T_{n\$11425[4]}, \mathcal{A}_{n\$11425[4]} \rightarrow \mathcal{A}_3 \mathcal{A}_{n\$11425[4]} \right\}$
- » $\left\{ b_{n\$11455[2]} \rightarrow b_{n\$11455[2]} + \frac{\alpha_3 + \alpha_{n\$11455[5]}}{\hbar}, t_{n\$11455[2]} \rightarrow t_{n\$11455[2]}, \alpha_{n\$11455[2]} \rightarrow \alpha_{n\$11455[2]}, \right.$
 $\left. b_{n\$11455[5]} \rightarrow b_{n\$11455[5]}, t_{n\$11455[5]} \rightarrow t_{n\$11455[5]}, \alpha_{n\$11455[5]} \rightarrow \alpha_3 + \alpha_{n\$11455[5]} \right\}$
- » $\left\{ B_{n\$11455[2]} \rightarrow \frac{B_{n\$11455[2]}}{\mathcal{A}_3 \mathcal{A}_{n\$11455[5]}}, T_{n\$11455[2]} \rightarrow T_{n\$11455[2]}, \mathcal{A}_{n\$11455[2]} \rightarrow \mathcal{A}_{n\$11455[2]}, \right.$
 $\left. B_{n\$11455[5]} \rightarrow B_{n\$11455[5]}, T_{n\$11455[5]} \rightarrow T_{n\$11455[5]}, \mathcal{A}_{n\$11455[5]} \rightarrow \mathcal{A}_3 \mathcal{A}_{n\$11455[5]} \right\}$
- » $\left\{ b_{n\$11493[1]} \rightarrow b_{n\$11493[1]} + \frac{\alpha_3 + \alpha_{n\$11493[3]}}{\hbar}, t_{n\$11493[1]} \rightarrow t_{n\$11493[1]}, \alpha_{n\$11493[1]} \rightarrow \alpha_{n\$11493[1]}, \right.$
 $\left. b_{n\$11493[3]} \rightarrow b_{n\$11493[3]}, t_{n\$11493[3]} \rightarrow t_{n\$11493[3]}, \alpha_{n\$11493[3]} \rightarrow \alpha_3 + \alpha_{n\$11493[3]} \right\}$
- » $\left\{ B_{n\$11493[1]} \rightarrow \frac{B_{n\$11493[1]}}{\mathcal{A}_3 \mathcal{A}_{n\$11493[3]}}, T_{n\$11493[1]} \rightarrow T_{n\$11493[1]}, \mathcal{A}_{n\$11493[1]} \rightarrow \mathcal{A}_{n\$11493[1]}, \right.$
 $\left. B_{n\$11493[3]} \rightarrow B_{n\$11493[3]}, T_{n\$11493[3]} \rightarrow T_{n\$11493[3]}, \mathcal{A}_{n\$11493[3]} \rightarrow \mathcal{A}_3 \mathcal{A}_{n\$11493[3]} \right\}$
- » $\left\{ b_{n\$11530[2]} \rightarrow b_{n\$11530[2]} + \frac{\alpha_4 + \alpha_{n\$11530[4]}}{\hbar}, t_{n\$11530[2]} \rightarrow t_{n\$11530[2]}, \alpha_{n\$11530[2]} \rightarrow \alpha_{n\$11530[2]}, \right.$
 $\left. b_{n\$11530[4]} \rightarrow b_{n\$11530[4]}, t_{n\$11530[4]} \rightarrow t_{n\$11530[4]}, \alpha_{n\$11530[4]} \rightarrow \alpha_4 + \alpha_{n\$11530[4]} \right\}$
- » $\left\{ B_{n\$11530[2]} \rightarrow \frac{B_{n\$11530[2]}}{\mathcal{A}_4 \mathcal{A}_{n\$11530[4]}}, T_{n\$11530[2]} \rightarrow T_{n\$11530[2]}, \mathcal{A}_{n\$11530[2]} \rightarrow \mathcal{A}_{n\$11530[2]}, \right.$
 $\left. B_{n\$11530[4]} \rightarrow B_{n\$11530[4]}, T_{n\$11530[4]} \rightarrow T_{n\$11530[4]}, \mathcal{A}_{n\$11530[4]} \rightarrow \mathcal{A}_4 \mathcal{A}_{n\$11530[4]} \right\}$
- » $\left\{ b_{n\$11579[1]} \rightarrow b_{n\$11579[1]} + \frac{\alpha_3 + \alpha_4 + \alpha_{n\$11579[3]}}{\hbar}, t_{n\$11579[1]} \rightarrow t_{n\$11579[1]}, \alpha_{n\$11579[1]} \rightarrow \alpha_{n\$11579[1]}, \right.$
 $\left. b_{n\$11579[3]} \rightarrow b_{n\$11579[3]}, t_{n\$11579[3]} \rightarrow t_{n\$11579[3]}, \alpha_{n\$11579[3]} \rightarrow \alpha_3 + \alpha_4 + \alpha_{n\$11579[3]} \right\}$
- » $\left\{ B_{n\$11579[1]} \rightarrow \frac{B_{n\$11579[1]}}{\mathcal{A}_3 \mathcal{A}_4 \mathcal{A}_{n\$11579[3]}}, T_{n\$11579[1]} \rightarrow T_{n\$11579[1]}, \mathcal{A}_{n\$11579[1]} \rightarrow \mathcal{A}_{n\$11579[1]}, \right.$
 $\left. B_{n\$11579[3]} \rightarrow B_{n\$11579[3]}, T_{n\$11579[3]} \rightarrow T_{n\$11579[3]}, \mathcal{A}_{n\$11579[3]} \rightarrow \mathcal{A}_3 \mathcal{A}_4 \mathcal{A}_{n\$11579[3]} \right\}$

Out[*]= {0.4375, {True, True}}

In[*]= Timing[HL /@ { (bS1 E[α2 a2, ξ2 x2, 1]) ~B1,2 ~P1,2 ≡ (E[β1 b1, η1 y1, 1] aS2) ~B1,2 ~P1,2, (bS1 E[α2 a2, ξ2 x2, 1]) ~B1,2 ~P1,2 ≡ (E[β1 b1, η1 y1, 1] aS2) ~B1,2 ~P1,2 }]

- » $\left\{ b_{n\$11611[1]} \rightarrow b_{n\$11611[1]} + \frac{\alpha_2 + \alpha_{n\$11611[2]}}{\hbar}, t_{n\$11611[1]} \rightarrow t_{n\$11611[1]}, \alpha_{n\$11611[1]} \rightarrow \alpha_{n\$11611[1]}, \right.$
 $\left. b_{n\$11611[2]} \rightarrow b_{n\$11611[2]}, t_{n\$11611[2]} \rightarrow t_{n\$11611[2]}, \alpha_{n\$11611[2]} \rightarrow \alpha_2 + \alpha_{n\$11611[2]} \right\}$
- » $\left\{ B_{n\$11611[1]} \rightarrow \frac{B_{n\$11611[1]}}{\mathcal{A}_2 \mathcal{A}_{n\$11611[2]}}, T_{n\$11611[1]} \rightarrow T_{n\$11611[1]}, \mathcal{A}_{n\$11611[1]} \rightarrow \mathcal{A}_{n\$11611[1]}, \right.$
 $\left. B_{n\$11611[2]} \rightarrow B_{n\$11611[2]}, T_{n\$11611[2]} \rightarrow T_{n\$11611[2]}, \mathcal{A}_{n\$11611[2]} \rightarrow \mathcal{A}_2 \mathcal{A}_{n\$11611[2]} \right\}$
- » $\left\{ b_{n\$11661[1]} \rightarrow b_{n\$11661[1]} + \frac{-\alpha_2 + \alpha_{n\$11661[2]}}{\hbar}, t_{n\$11661[1]} \rightarrow t_{n\$11661[1]}, \alpha_{n\$11661[1]} \rightarrow \alpha_{n\$11661[1]}, \right.$
 $\left. b_{n\$11661[2]} \rightarrow b_{n\$11661[2]}, t_{n\$11661[2]} \rightarrow t_{n\$11661[2]}, \alpha_{n\$11661[2]} \rightarrow -\alpha_2 + \alpha_{n\$11661[2]} \right\}$
- » $\left\{ B_{n\$11661[1]} \rightarrow \frac{B_{n\$11661[1]} \mathcal{A}_2}{\mathcal{A}_{n\$11661[2]}}, T_{n\$11661[1]} \rightarrow T_{n\$11661[1]}, \mathcal{A}_{n\$11661[1]} \rightarrow \mathcal{A}_{n\$11661[1]}, \right.$
 $\left. B_{n\$11661[2]} \rightarrow B_{n\$11661[2]}, T_{n\$11661[2]} \rightarrow T_{n\$11661[2]}, \mathcal{A}_{n\$11661[2]} \rightarrow \frac{\mathcal{A}_{n\$11661[2]}}{\mathcal{A}_2} \right\}$
- » $\left\{ b_{n\$11725[1]} \rightarrow b_{n\$11725[1]} + \frac{\alpha_2 + \alpha_{n\$11725[2]}}{\hbar}, t_{n\$11725[1]} \rightarrow t_{n\$11725[1]}, \alpha_{n\$11725[1]} \rightarrow \alpha_{n\$11725[1]}, \right.$
 $\left. b_{n\$11725[2]} \rightarrow b_{n\$11725[2]}, t_{n\$11725[2]} \rightarrow t_{n\$11725[2]}, \alpha_{n\$11725[2]} \rightarrow \alpha_2 + \alpha_{n\$11725[2]} \right\}$
- » $\left\{ B_{n\$11725[1]} \rightarrow \frac{B_{n\$11725[1]}}{\mathcal{A}_2 \mathcal{A}_{n\$11725[2]}}, T_{n\$11725[1]} \rightarrow T_{n\$11725[1]}, \mathcal{A}_{n\$11725[1]} \rightarrow \mathcal{A}_{n\$11725[1]}, \right.$
 $\left. B_{n\$11725[2]} \rightarrow B_{n\$11725[2]}, T_{n\$11725[2]} \rightarrow T_{n\$11725[2]}, \mathcal{A}_{n\$11725[2]} \rightarrow \mathcal{A}_2 \mathcal{A}_{n\$11725[2]} \right\}$
- » $\left\{ b_{n\$11783[1]} \rightarrow b_{n\$11783[1]} + \frac{-\alpha_2 + \alpha_{n\$11783[2]}}{\hbar}, t_{n\$11783[1]} \rightarrow t_{n\$11783[1]}, \alpha_{n\$11783[1]} \rightarrow \alpha_{n\$11783[1]}, \right.$
 $\left. b_{n\$11783[2]} \rightarrow b_{n\$11783[2]}, t_{n\$11783[2]} \rightarrow t_{n\$11783[2]}, \alpha_{n\$11783[2]} \rightarrow -\alpha_2 + \alpha_{n\$11783[2]} \right\}$
- » $\left\{ B_{n\$11783[1]} \rightarrow \frac{B_{n\$11783[1]} \mathcal{A}_2}{\mathcal{A}_{n\$11783[2]}}, T_{n\$11783[1]} \rightarrow T_{n\$11783[1]}, \mathcal{A}_{n\$11783[1]} \rightarrow \mathcal{A}_{n\$11783[1]}, \right.$
 $\left. B_{n\$11783[2]} \rightarrow B_{n\$11783[2]}, T_{n\$11783[2]} \rightarrow T_{n\$11783[2]}, \mathcal{A}_{n\$11783[2]} \rightarrow \frac{\mathcal{A}_{n\$11783[2]}}{\mathcal{A}_2} \right\}$

Out[]:= {0.328125, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```

In[ ]:= Timing@{
  "[a,y]" -> ((E[0, 0, y2 a1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, y1 a2] ~ B1,2 ~ dm1,2->1) [[3]]),
  "[b,x]" -> ((E[0, 0, x2 b1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, x1 b2] ~ B1,2 ~ dm1,2->1) [[3]]),
  "xy-qyx" -> ((E[0, 0, x1 y2] ~ B1,2 ~ dm1,2->1) [[3]] - (1 + e) (E[0, 0, y1 x2] ~ B1,2 ~ dm1,2->1) [[3]])
} /. {z_1 -> z} // Expand // Factor,
{
  "Δ(a)" -> ((E[0, 0, a1] ~ B1 ~ dΔ1->1,2) [[3]]),
  "Δ(x)" -> ((E[0, 0, x1] ~ B1 ~ dΔ1->1,2) [[3]]),
  "Δ(b)" -> ((E[0, 0, b1] ~ B1 ~ dΔ1->1,2) [[3]]),
  "Δ(y)" -> ((E[0, 0, y1] ~ B1 ~ dΔ1->1,2) [[3]])
} // Simplify,
{
  "S(a)" -> ((E[0, 0, a1] ~ B1 ~ dS1) [[3]]),
  "S(x)" -> ((E[0, 0, x1] ~ B1 ~ dS1) [[3]]),
  "S(b)" -> ((E[0, 0, b1] ~ B1 ~ dS1) [[3]]),
  "S(y)" -> ((E[0, 0, y1] ~ B1 ~ dS1) [[3]])
} /. {z_1 -> z} // Simplify
}

"
{bn$11858[2] -> bn$11858[2], tn$11858[2] -> tn$11858[2], αn$11858[2] -> αi + αn$11858[2]}
"
{Bn$11858[2] -> Bn$11858[2], Tn$11858[2] -> Tn$11858[2], An$11858[2] -> Ai An$11858[2]}
"
{bn$11890[3] -> bn$11890[3], tn$11890[3] -> tn$11890[3], αn$11890[3] -> αi + αn$11890[3]}
"
{Bn$11890[3] -> Bn$11890[3], Tn$11890[3] -> Tn$11890[3], An$11890[3] -> Ai An$11890[3]}
"
{bn$11986[-2] -> b_-3 + b_-2 + bn$11986[-2], tn$11986[-2] -> tn$11986[-2], αn$11986[-2] -> αn$11986[-2]}
"
{Bn$11986[-2] -> B_-3 B_-2 Bn$11986[-2], Tn$11986[-2] -> Tn$11986[-2], An$11986[-2] -> An$11986[-2]}
"
{bn$12035[-3] -> bn$12035[-3] + (αi + αn$12035[1])/ħ, tn$12035[-3] -> tn$12035[-3], αn$12035[-3] -> αn$12035[-3],
bn$12035[-2] -> bk + bn$12035[-2], tn$12035[-2] -> tn$12035[-2], αn$12035[-2] -> αn$12035[-2],
bn$12035[-1] -> bn$12035[-1] + (-αi + αn$12035[3])/ħ, tn$12035[-1] -> tn$12035[-1], αn$12035[-1] -> αn$12035[-1],
bn$12035[1] -> bn$12035[1], tn$12035[1] -> tn$12035[1], αn$12035[1] -> αi + αn$12035[1], bn$12035[2] -> bn$12035[2],
tn$12035[2] -> tn$12035[2], αn$12035[2] -> αi + αn$12035[2], bn$12035[3] -> bn$12035[3], tn$12035[3] -> tn$12035[3],
αn$12035[3] -> -αi + αn$12035[3], bn$12035[i] -> bk + bn$12035[i], tn$12035[i] -> tn$12035[i],
αn$12035[i] -> αn$12035[i], bn$12035[j] -> bn$12035[j], tn$12035[j] -> tn$12035[j], αn$12035[j] -> αj + αn$12035[j]}
"
{Bn$12035[-3] -> (Bn$12035[-3])/Ai An$12035[1], Tn$12035[-3] -> Tn$12035[-3], An$12035[-3] -> An$12035[-3], Bn$12035[-2] -> Bk Bn$12035[-2],
Tn$12035[-2] -> Tn$12035[-2], An$12035[-2] -> An$12035[-2], Bn$12035[-1] -> (Bn$12035[-1] Ai)/An$12035[3], Tn$12035[-1] -> Tn$12035[-1],
An$12035[-1] -> An$12035[-1], Bn$12035[1] -> Bn$12035[1], Tn$12035[1] -> Tn$12035[1], An$12035[1] -> Ai An$12035[1],
Bn$12035[2] -> Bn$12035[2], Tn$12035[2] -> Tn$12035[2], An$12035[2] -> Ai An$12035[2], Bn$12035[3] -> Bn$12035[3],
Tn$12035[3] -> Tn$12035[3], An$12035[3] -> (An$12035[3])/Ai, Bn$12035[i] -> Bk Bn$12035[i], Tn$12035[i] -> Tn$12035[i],
An$12035[i] -> An$12035[i], Bn$12035[j] -> Bn$12035[j], Tn$12035[j] -> Tn$12035[j], An$12035[j] -> Aj An$12035[j]}
"
{bn$12429[1] -> b1 + bn$12429[1], tn$12429[1] -> tn$12429[1], αn$12429[1] -> αn$12429[1],
bn$12429[2] -> b1 + bn$12429[2], tn$12429[2] -> tn$12429[2], αn$12429[2] -> αn$12429[2]}

```

$\{ B_{n\$12429[1]} \rightarrow B_1 B_{n\$12429[1]}, T_{n\$12429[1]} \rightarrow T_{n\$12429[1]}, \mathcal{A}_{n\$12429[1]} \rightarrow \mathcal{A}_{n\$12429[1]}, B_{n\$12429[2]} \rightarrow B_1 B_{n\$12429[2]}, T_{n\$12429[2]} \rightarrow T_{n\$12429[2]}, \mathcal{A}_{n\$12429[2]} \rightarrow \mathcal{A}_{n\$12429[2]} \}$

$\{ b_{n\$12628[1]} \rightarrow b_1 + b_{n\$12628[1]}, t_{n\$12628[1]} \rightarrow t_{n\$12628[1]}, \alpha_{n\$12628[1]} \rightarrow \alpha_{n\$12628[1]}, b_{n\$12628[2]} \rightarrow b_1 + b_{n\$12628[2]}, t_{n\$12628[2]} \rightarrow t_{n\$12628[2]}, \alpha_{n\$12628[2]} \rightarrow \alpha_{n\$12628[2]} \}$

$\{ B_{n\$12628[1]} \rightarrow B_1 B_{n\$12628[1]}, T_{n\$12628[1]} \rightarrow T_{n\$12628[1]}, \mathcal{A}_{n\$12628[1]} \rightarrow \mathcal{A}_{n\$12628[1]}, B_{n\$12628[2]} \rightarrow B_1 B_{n\$12628[2]}, T_{n\$12628[2]} \rightarrow T_{n\$12628[2]}, \mathcal{A}_{n\$12628[2]} \rightarrow \mathcal{A}_{n\$12628[2]} \}$

$\{ b_{n\$12827[1]} \rightarrow b_1 + b_{n\$12827[1]}, t_{n\$12827[1]} \rightarrow t_{n\$12827[1]}, \alpha_{n\$12827[1]} \rightarrow \alpha_{n\$12827[1]}, b_{n\$12827[2]} \rightarrow b_1 + b_{n\$12827[2]}, t_{n\$12827[2]} \rightarrow t_{n\$12827[2]}, \alpha_{n\$12827[2]} \rightarrow \alpha_{n\$12827[2]} \}$

$\{ B_{n\$12827[1]} \rightarrow B_1 B_{n\$12827[1]}, T_{n\$12827[1]} \rightarrow T_{n\$12827[1]}, \mathcal{A}_{n\$12827[1]} \rightarrow \mathcal{A}_{n\$12827[1]}, B_{n\$12827[2]} \rightarrow B_1 B_{n\$12827[2]}, T_{n\$12827[2]} \rightarrow T_{n\$12827[2]}, \mathcal{A}_{n\$12827[2]} \rightarrow \mathcal{A}_{n\$12827[2]} \}$

$\{ b_{n\$12967[1]} \rightarrow b_1 + b_{n\$12967[1]}, t_{n\$12967[1]} \rightarrow t_{n\$12967[1]}, \alpha_{n\$12967[1]} \rightarrow \alpha_{n\$12967[1]}, b_{n\$12967[2]} \rightarrow b_1 + b_{n\$12967[2]}, t_{n\$12967[2]} \rightarrow t_{n\$12967[2]}, \alpha_{n\$12967[2]} \rightarrow \alpha_{n\$12967[2]} \}$

$\{ B_{n\$12967[1]} \rightarrow B_1 B_{n\$12967[1]}, T_{n\$12967[1]} \rightarrow T_{n\$12967[1]}, \mathcal{A}_{n\$12967[1]} \rightarrow \mathcal{A}_{n\$12967[1]}, B_{n\$12967[2]} \rightarrow B_1 B_{n\$12967[2]}, T_{n\$12967[2]} \rightarrow T_{n\$12967[2]}, \mathcal{A}_{n\$12967[2]} \rightarrow \mathcal{A}_{n\$12967[2]} \}$

$\{ b_{n\$13109[1]} \rightarrow b_1 + b_{n\$13109[1]}, t_{n\$13109[1]} \rightarrow t_{n\$13109[1]}, \alpha_{n\$13109[1]} \rightarrow \alpha_{n\$13109[1]}, b_{n\$13109[2]} \rightarrow b_1 + b_{n\$13109[2]}, t_{n\$13109[2]} \rightarrow t_{n\$13109[2]}, \alpha_{n\$13109[2]} \rightarrow \alpha_{n\$13109[2]} \}$

$\{ B_{n\$13109[1]} \rightarrow B_1 B_{n\$13109[1]}, T_{n\$13109[1]} \rightarrow T_{n\$13109[1]}, \mathcal{A}_{n\$13109[1]} \rightarrow \mathcal{A}_{n\$13109[1]}, B_{n\$13109[2]} \rightarrow B_1 B_{n\$13109[2]}, T_{n\$13109[2]} \rightarrow T_{n\$13109[2]}, \mathcal{A}_{n\$13109[2]} \rightarrow \mathcal{A}_{n\$13109[2]} \}$

$\{ b_{n\$13263[1]} \rightarrow b_1 + b_{n\$13263[1]}, t_{n\$13263[1]} \rightarrow t_{n\$13263[1]}, \alpha_{n\$13263[1]} \rightarrow \alpha_{n\$13263[1]}, b_{n\$13263[2]} \rightarrow b_1 + b_{n\$13263[2]}, t_{n\$13263[2]} \rightarrow t_{n\$13263[2]}, \alpha_{n\$13263[2]} \rightarrow \alpha_{n\$13263[2]} \}$

$\{ B_{n\$13263[1]} \rightarrow B_1 B_{n\$13263[1]}, T_{n\$13263[1]} \rightarrow T_{n\$13263[1]}, \mathcal{A}_{n\$13263[1]} \rightarrow \mathcal{A}_{n\$13263[1]}, B_{n\$13263[2]} \rightarrow B_1 B_{n\$13263[2]}, T_{n\$13263[2]} \rightarrow T_{n\$13263[2]}, \mathcal{A}_{n\$13263[2]} \rightarrow \mathcal{A}_{n\$13263[2]} \}$

$\{ b_{n\$13386[1]} \rightarrow b_j + b_{n\$13386[1]}, t_{n\$13386[1]} \rightarrow t_{n\$13386[1]}, \alpha_{n\$13386[1]} \rightarrow \alpha_{n\$13386[1]}, b_{n\$13386[2]} \rightarrow b_j + b_{n\$13386[2]}, t_{n\$13386[2]} \rightarrow t_{n\$13386[2]}, \alpha_{n\$13386[2]} \rightarrow \alpha_i + \alpha_{n\$13386[2]}, b_{n\$13386[3]} \rightarrow b_k + b_{n\$13386[3]}, t_{n\$13386[3]} \rightarrow t_{n\$13386[3]}, \alpha_{n\$13386[3]} \rightarrow \alpha_{n\$13386[3]}, b_{n\$13386[4]} \rightarrow b_k + b_{n\$13386[4]}, t_{n\$13386[4]} \rightarrow t_{n\$13386[4]}, \alpha_{n\$13386[4]} \rightarrow \alpha_i + \alpha_{n\$13386[4]} \}$

$\{ B_{n\$13386[1]} \rightarrow B_j B_{n\$13386[1]}, T_{n\$13386[1]} \rightarrow T_{n\$13386[1]}, \mathcal{A}_{n\$13386[1]} \rightarrow \mathcal{A}_{n\$13386[1]}, B_{n\$13386[2]} \rightarrow B_j B_{n\$13386[2]}, T_{n\$13386[2]} \rightarrow T_{n\$13386[2]}, \mathcal{A}_{n\$13386[2]} \rightarrow \mathcal{A}_i \mathcal{A}_{n\$13386[2]}, B_{n\$13386[3]} \rightarrow B_k B_{n\$13386[3]}, T_{n\$13386[3]} \rightarrow T_{n\$13386[3]}, \mathcal{A}_{n\$13386[3]} \rightarrow \mathcal{A}_{n\$13386[3]}, B_{n\$13386[4]} \rightarrow B_k B_{n\$13386[4]}, T_{n\$13386[4]} \rightarrow T_{n\$13386[4]}, \mathcal{A}_{n\$13386[4]} \rightarrow \mathcal{A}_i \mathcal{A}_{n\$13386[4]} \}$

$\{ b_{n\$14146[1]} \rightarrow b_1 + b_2 + b_{n\$14146[1]}, t_{n\$14146[1]} \rightarrow t_{n\$14146[1]}, \alpha_{n\$14146[1]} \rightarrow \alpha_{n\$14146[1]} \}$

$\{ B_{n\$14146[1]} \rightarrow B_1 B_2 B_{n\$14146[1]}, T_{n\$14146[1]} \rightarrow T_{n\$14146[1]}, \mathcal{A}_{n\$14146[1]} \rightarrow \mathcal{A}_{n\$14146[1]} \}$

$\{ b_{n\$14195[1]} \rightarrow b_1 + b_2 + b_{n\$14195[1]}, t_{n\$14195[1]} \rightarrow t_{n\$14195[1]}, \alpha_{n\$14195[1]} \rightarrow \alpha_{n\$14195[1]} \}$

$\{ B_{n\$14195[1]} \rightarrow B_1 B_2 B_{n\$14195[1]}, T_{n\$14195[1]} \rightarrow T_{n\$14195[1]}, \mathcal{A}_{n\$14195[1]} \rightarrow \mathcal{A}_{n\$14195[1]} \}$

$\{ b_{n\$14257[1]} \rightarrow b_1 + b_2 + b_{n\$14257[1]}, t_{n\$14257[1]} \rightarrow t_{n\$14257[1]}, \alpha_{n\$14257[1]} \rightarrow \alpha_{n\$14257[1]} \}$

$\{ B_{n\$14257[1]} \rightarrow B_1 B_2 B_{n\$14257[1]}, T_{n\$14257[1]} \rightarrow T_{n\$14257[1]}, \mathcal{A}_{n\$14257[1]} \rightarrow \mathcal{A}_{n\$14257[1]} \}$

$\{ b_{n\$14306[1]} \rightarrow b_1 + b_2 + b_{n\$14306[1]}, t_{n\$14306[1]} \rightarrow t_{n\$14306[1]}, \alpha_{n\$14306[1]} \rightarrow \alpha_{n\$14306[1]} \}$

$\{ B_{n\$14306[1]} \rightarrow B_1 B_2 B_{n\$14306[1]}, T_{n\$14306[1]} \rightarrow T_{n\$14306[1]}, \mathcal{A}_{n\$14306[1]} \rightarrow \mathcal{A}_{n\$14306[1]} \}$

$\{ b_{n\$14341[1]} \rightarrow -b_1 + b_{n\$14341[1]}, t_{n\$14341[1]} \rightarrow t_{n\$14341[1]}, \alpha_{n\$14341[1]} \rightarrow \alpha_{n\$14341[1]}, b_{n\$14341[2]} \rightarrow b_{n\$14341[2]}, t_{n\$14341[2]} \rightarrow t_{n\$14341[2]}, \alpha_{n\$14341[2]} \rightarrow \alpha_i + \alpha_{n\$14341[2]} \}$

$\{ B_{n\$14341[1]} \rightarrow \frac{B_{n\$14341[1]}}{B_1}, T_{n\$14341[1]} \rightarrow T_{n\$14341[1]}, \mathcal{A}_{n\$14341[1]} \rightarrow \mathcal{A}_{n\$14341[1]}, B_{n\$14341[2]} \rightarrow B_{n\$14341[2]}, T_{n\$14341[2]} \rightarrow T_{n\$14341[2]}, \mathcal{A}_{n\$14341[2]} \rightarrow \mathcal{A}_i \mathcal{A}_{n\$14341[2]} \}$

```

" {bn$14422[1] -> b_i + bn$14422[1], tn$14422[1] -> tn$14422[1], an$14422[1] -> an$14422[1],
  bn$14422[2] -> b_i + bn$14422[2], tn$14422[2] -> tn$14422[2], an$14422[2] -> -a_i + an$14422[2]}
" {Bn$14422[1] -> B_i Bn$14422[1], Tn$14422[1] -> Tn$14422[1], An$14422[1] -> An$14422[1],
  Bn$14422[2] -> B_i Bn$14422[2], Tn$14422[2] -> Tn$14422[2], An$14422[2] -> An$14422[2]}
" {bn$14869[1] -> -b_1 + bn$14869[1], tn$14869[1] -> tn$14869[1], an$14869[1] -> an$14869[1]}
" {Bn$14869[1] -> Bn$14869[1], Tn$14869[1] -> Tn$14869[1], An$14869[1] -> An$14869[1]}
" {bn$15330[1] -> -b_1 + bn$15330[1], tn$15330[1] -> tn$15330[1], an$15330[1] -> an$15330[1]}
" {Bn$15330[1] -> Bn$15330[1], Tn$15330[1] -> Tn$15330[1], An$15330[1] -> An$15330[1]}
" {bn$15550[1] -> -b_1 + bn$15550[1], tn$15550[1] -> tn$15550[1], an$15550[1] -> an$15550[1]}
" {Bn$15550[1] -> Bn$15550[1], Tn$15550[1] -> Tn$15550[1], An$15550[1] -> An$15550[1]}
" {bn$15817[1] -> -b_1 + bn$15817[1], tn$15817[1] -> tn$15817[1], an$15817[1] -> an$15817[1]}
" {Bn$15817[1] -> Bn$15817[1], Tn$15817[1] -> Tn$15817[1], An$15817[1] -> An$15817[1]}
Out[ ]:= {7.89063, { { [a, y] -> -y y + O[ ]^3, [b, x] -> x x + O[ ]^3,
  xy - qyx -> (-x y + (1 - B + x y h) / h) + (a B - x y + x y y h) e + 1/2 (-a^2 B h + x y y^2 h^2) e^2 + O[ ]^3 },
  { Delta(a) -> (a_1 + a_2) + O[ ]^3, Delta(x) -> (x_1 + x_2) - h a_1 x_2 e + 1/2 h^2 a_1^2 x_2 e^2 + O[ ]^3,
  Delta(b) -> (b_1 + b_2) + O[ ]^3, Delta(y) -> (y_1 + B_1 y_2) + O[ ]^3 },
  { S(a) -> -a + O[ ]^3, S(x) -> -x - a x h e - 1/2 (a^2 x h^2) e^2 + O[ ]^3,
  S(b) -> -b + O[ ]^3, S(y) -> -y / B + y y h e / B - (y y^2 h^2) e^2 / (2 B) + O[ ]^3 } }

```

(co)-associativity

```

In[ ]:= Timing[HL /@
  { (dDelta1->1,2 ~ B2 ~ dDelta2->2,3) == (dDelta1->1,3 ~ B1 ~ dDelta1->1,2), (dm1,2->1 ~ B1 ~ dm1,3->1) == (dm2,3->2 ~ B2 ~ dm1,2->1) } ]
" {bn$16043[2] -> b_2 + b_3 + bn$16043[2], tn$16043[2] -> tn$16043[2], an$16043[2] -> a_1 + an$16043[2]}
" {Bn$16043[2] -> B_2 B_3 Bn$16043[2], Tn$16043[2] -> Tn$16043[2], An$16043[2] -> A_1 A_n$16043[2]}
" {bn$16114[1] -> b_1 + b_2 + bn$16114[1], tn$16114[1] -> tn$16114[1], an$16114[1] -> a_1 + an$16114[1]}
" {Bn$16114[1] -> B_1 B_2 Bn$16114[1], Tn$16114[1] -> Tn$16114[1], An$16114[1] -> A_1 A_n$16114[1]}
" {bn$16193[1] -> b_1 + bn$16193[1], tn$16193[1] -> tn$16193[1], an$16193[1] -> a_1 + a_2 + an$16193[1]}
" {Bn$16193[1] -> B_1 Bn$16193[1], Tn$16193[1] -> Tn$16193[1], An$16193[1] -> A_1 A_2 A_n$16193[1]}
" {bn$16989[2] -> b_1 + bn$16989[2], tn$16989[2] -> tn$16989[2], an$16989[2] -> a_2 + a_3 + an$16989[2]}
" {Bn$16989[2] -> B_1 Bn$16989[2], Tn$16989[2] -> Tn$16989[2], An$16989[2] -> A_2 A_3 A_n$16989[2]}
Out[ ]:= {5.875, { True, True }}

```


Δ is an algebra morphism

```

In[ ]:= Timing@HL [dm1,2→1 ~ B1 ~ dΔ1→1,2 ≡ (dΔ1→1,3 dΔ2→2,4) ~ B1,2,3,4 ~ (dm3,4→2 dm1,2→1) ]
" {bn$17777[1] → b1 + b2 + bn$17777[1], tn$17777[1] → tn$17777[1], αn$17777[1] → α1 + α2 + αn$17777[1]}
" {Bn$17777[1] → B1 B2 Bn$17777[1], Tn$17777[1] → Tn$17777[1], An$17777[1] → A1 A2 An$17777[1]}
" {bn$18144[1] → b1 + bn$18144[1], tn$18144[1] → tn$18144[1], αn$18144[1] → α1 + αn$18144[1], bn$18144[2] → b1 + bn$18144[2],
tn$18144[2] → tn$18144[2], αn$18144[2] → α2 + αn$18144[2], bn$18144[3] → b2 + bn$18144[3], tn$18144[3] → tn$18144[3],
αn$18144[3] → α1 + αn$18144[3], bn$18144[4] → b2 + bn$18144[4], tn$18144[4] → tn$18144[4], αn$18144[4] → α2 + αn$18144[4]}
" {Bn$18144[1] → B1 Bn$18144[1], Tn$18144[1] → Tn$18144[1], An$18144[1] → A1 An$18144[1], Bn$18144[2] → B1 Bn$18144[2],
Tn$18144[2] → Tn$18144[2], An$18144[2] → A2 An$18144[2], Bn$18144[3] → B2 Bn$18144[3], Tn$18144[3] → Tn$18144[3],
An$18144[3] → A1 An$18144[3], Bn$18144[4] → B2 Bn$18144[4], Tn$18144[4] → Tn$18144[4], An$18144[4] → A2 An$18144[4]}
Out[ ]:= {7.60938, True}

```

S is convolution inverse of id

```

In[ ]:= Timing [
HL [# ≡ E [0, 0, 1] ] & /@ { (dΔ1→1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2→1, (dΔ1→1,2 ~ B2 ~ dS2) ~ B1,2 ~ dm1,2→1 } ]
" {bn$19481[1] → -b1 + bn$19481[1], tn$19481[1] → tn$19481[1], αn$19481[1] → α1 + αn$19481[1]}
" {Bn$19481[1] →  $\frac{Bn$19481[1]}{B_1}$ , Tn$19481[1] → Tn$19481[1], An$19481[1] → A1 An$19481[1]}
" {bn$20356[1] → b1 + bn$20356[1], tn$20356[1] → tn$20356[1], αn$20356[1] → -α1 + αn$20356[1],
bn$20356[2] → b1 + bn$20356[2], tn$20356[2] → tn$20356[2], αn$20356[2] → α1 + αn$20356[2]}
" {Bn$20356[1] → B1 Bn$20356[1], Tn$20356[1] → Tn$20356[1], An$20356[1] →  $\frac{An$20356[1]}{A_1}$ ,
Bn$20356[2] → B1 Bn$20356[2], Tn$20356[2] → Tn$20356[2], An$20356[2] → A1 An$20356[2]}
" {bn$22204[2] → -b2 + bn$22204[2], tn$22204[2] → tn$22204[2], αn$22204[2] → α1 + αn$22204[2]}
" {Bn$22204[2] →  $\frac{Bn$22204[2]}{B_2}$ , Tn$22204[2] → Tn$22204[2], An$22204[2] → A1 An$22204[2]}
" {bn$22730[1] → b1 + bn$22730[1], tn$22730[1] → tn$22730[1], αn$22730[1] → α1 + αn$22730[1],
bn$22730[2] → b1 + bn$22730[2], tn$22730[2] → tn$22730[2], αn$22730[2] → -α1 + αn$22730[2]}
" {Bn$22730[1] → B1 Bn$22730[1], Tn$22730[1] → Tn$22730[1], An$22730[1] → A1 An$22730[1],
Bn$22730[2] → B1 Bn$22730[2], Tn$22730[2] → Tn$22730[2], An$22730[2] →  $\frac{An$22730[2]}{A_1}$ }
Out[ ]:= {9.45313, {True, True}}

```

S is a (co)-algebra anti-morphism

```

In[ ]:= Timing [HL /@
Expand /@ {dm1,2→1 ~ B1 ~ dS1 ≡ (dS1 dS2) ~ B1,2 ~ dm2,1→1, dS1 ~ B1 ~ dΔ1→1,2 ≡ dΔ1→2,1 ~ B1,2 ~ (dS1 dS2) } ]

```

$$\{ b_{n\$24379[1]} \rightarrow -b_1 + b_{n\$24379[1]}, t_{n\$24379[1]} \rightarrow t_{n\$24379[1]}, \alpha_{n\$24379[1]} \rightarrow \alpha_1 + \alpha_2 + \alpha_{n\$24379[1]} \}$$

$$\left\{ B_{n\$24379[1]} \rightarrow \frac{B_{n\$24379[1]}}{B_1}, T_{n\$24379[1]} \rightarrow T_{n\$24379[1]}, \mathcal{A}_{n\$24379[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_{n\$24379[1]} \right\}$$

$$\{ b_{n\$25808[1]} \rightarrow b_1 + b_{n\$25808[1]}, t_{n\$25808[1]} \rightarrow t_{n\$25808[1]}, \alpha_{n\$25808[1]} \rightarrow -\alpha_1 + \alpha_{n\$25808[1]}, b_{n\$25808[2]} \rightarrow b_1 + b_{n\$25808[2]}, t_{n\$25808[2]} \rightarrow t_{n\$25808[2]}, \alpha_{n\$25808[2]} \rightarrow -\alpha_2 + \alpha_{n\$25808[2]} \}$$

$$\left\{ B_{n\$25808[1]} \rightarrow B_1 B_{n\$25808[1]}, T_{n\$25808[1]} \rightarrow T_{n\$25808[1]}, \mathcal{A}_{n\$25808[1]} \rightarrow \frac{\mathcal{A}_{n\$25808[1]}}{\mathcal{A}_1}, B_{n\$25808[2]} \rightarrow B_1 B_{n\$25808[2]}, T_{n\$25808[2]} \rightarrow T_{n\$25808[2]}, \mathcal{A}_{n\$25808[2]} \rightarrow \frac{\mathcal{A}_{n\$25808[2]}}{\mathcal{A}_2} \right\}$$

$$\{ b_{n\$28790[1]} \rightarrow b_1 + b_2 + b_{n\$28790[1]}, t_{n\$28790[1]} \rightarrow t_{n\$28790[1]}, \alpha_{n\$28790[1]} \rightarrow -\alpha_1 + \alpha_{n\$28790[1]} \}$$

$$\left\{ B_{n\$28790[1]} \rightarrow B_1 B_2 B_{n\$28790[1]}, T_{n\$28790[1]} \rightarrow T_{n\$28790[1]}, \mathcal{A}_{n\$28790[1]} \rightarrow \frac{\mathcal{A}_{n\$28790[1]}}{\mathcal{A}_1} \right\}$$

$$\{ b_{n\$29421[1]} \rightarrow -b_1 + b_{n\$29421[1]}, t_{n\$29421[1]} \rightarrow t_{n\$29421[1]}, \alpha_{n\$29421[1]} \rightarrow \alpha_1 + \alpha_{n\$29421[1]}, b_{n\$29421[2]} \rightarrow -b_2 + b_{n\$29421[2]}, t_{n\$29421[2]} \rightarrow t_{n\$29421[2]}, \alpha_{n\$29421[2]} \rightarrow \alpha_1 + \alpha_{n\$29421[2]} \}$$

$$\left\{ B_{n\$29421[1]} \rightarrow \frac{B_{n\$29421[1]}}{B_1}, T_{n\$29421[1]} \rightarrow T_{n\$29421[1]}, \mathcal{A}_{n\$29421[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$29421[1]}, B_{n\$29421[2]} \rightarrow \frac{B_{n\$29421[2]}}{B_2}, T_{n\$29421[2]} \rightarrow T_{n\$29421[2]}, \mathcal{A}_{n\$29421[2]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$29421[2]} \right\}$$

Out[]:= {16.0625, {True, True}}

Quasi-triangular axiom 1:

In[]:= Timing@HL[R_{1,2} ~ B₁ ~ dΔ_{1->1,3} ≡ (R_{1,4} R_{3,2}) ~ B_{2,4} ~ dm_{2,4->2}]

$$\{ b_{n\$31824[1]} \rightarrow b_1 + b_3 + b_{n\$31824[1]}, t_{n\$31824[1]} \rightarrow t_{n\$31824[1]}, \alpha_{n\$31824[1]} \rightarrow \alpha_{n\$31824[1]} \}$$

$$\{ B_{n\$31824[1]} \rightarrow B_1 B_3 B_{n\$31824[1]}, T_{n\$31824[1]} \rightarrow T_{n\$31824[1]}, \mathcal{A}_{n\$31824[1]} \rightarrow \mathcal{A}_{n\$31824[1]} \}$$

$$\{ b_{n\$31871[2]} \rightarrow b_2 + b_{n\$31871[2]}, t_{n\$31871[2]} \rightarrow t_{n\$31871[2]}, \alpha_{n\$31871[2]} \rightarrow \hbar b_3 + \alpha_{n\$31871[2]}, b_{n\$31871[4]} \rightarrow b_2 + b_{n\$31871[4]}, t_{n\$31871[4]} \rightarrow t_{n\$31871[4]}, \alpha_{n\$31871[4]} \rightarrow \hbar b_1 + \alpha_{n\$31871[4]} \}$$

$$\left\{ B_{n\$31871[2]} \rightarrow B_2 B_{n\$31871[2]}, T_{n\$31871[2]} \rightarrow T_{n\$31871[2]}, \mathcal{A}_{n\$31871[2]} \rightarrow \frac{\mathcal{A}_{n\$31871[2]}}{B_3}, B_{n\$31871[4]} \rightarrow B_2 B_{n\$31871[4]}, T_{n\$31871[4]} \rightarrow T_{n\$31871[4]}, \mathcal{A}_{n\$31871[4]} \rightarrow \frac{\mathcal{A}_{n\$31871[4]}}{B_1} \right\}$$

Out[]:= {0.59375, True}

Quasi-triangular axiom 2:

In[]:= Timing@HL[((dΔ_{1->1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3->1} dm_{2,4->2})) ≡ ((dΔ_{1->2,1} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{3,1->1} dm_{4,2->2}))]

$$\begin{aligned}
& \{ \mathbf{b}_{n\$32082[1]} \rightarrow \mathbf{b}_1 + \mathbf{b}_{n\$32082[1]}, \mathbf{t}_{n\$32082[1]} \rightarrow \mathbf{t}_{n\$32082[1]}, \alpha_{n\$32082[1]} \rightarrow \alpha_1 + \alpha_{n\$32082[1]}, \\
& \mathbf{b}_{n\$32082[2]} \rightarrow \mathbf{b}_2 + \mathbf{b}_{n\$32082[2]}, \mathbf{t}_{n\$32082[2]} \rightarrow \mathbf{t}_{n\$32082[2]}, \alpha_{n\$32082[2]} \rightarrow \alpha_1 + \alpha_{n\$32082[2]}, \\
& \mathbf{b}_{n\$32082[3]} \rightarrow \mathbf{b}_1 + \mathbf{b}_{n\$32082[3]}, \mathbf{t}_{n\$32082[3]} \rightarrow \mathbf{t}_{n\$32082[3]}, \alpha_{n\$32082[3]} \rightarrow \alpha_{n\$32082[3]}, \\
& \mathbf{b}_{n\$32082[4]} \rightarrow \mathbf{b}_2 + \mathbf{b}_{n\$32082[4]}, \mathbf{t}_{n\$32082[4]} \rightarrow \mathbf{t}_{n\$32082[4]}, \alpha_{n\$32082[4]} \rightarrow \hbar (\mathbf{b}_1 + \mathbf{b}_{n\$32082[3]}) + \alpha_{n\$32082[4]} \} \\
& \{ \mathbf{B}_{n\$32082[1]} \rightarrow \mathbf{B}_1 \mathbf{B}_{n\$32082[1]}, \mathbf{T}_{n\$32082[1]} \rightarrow \mathbf{T}_{n\$32082[1]}, \mathcal{A}_{n\$32082[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$32082[1]}, \mathbf{B}_{n\$32082[2]} \rightarrow \mathbf{B}_2 \mathbf{B}_{n\$32082[2]}, \\
& \mathbf{T}_{n\$32082[2]} \rightarrow \mathbf{T}_{n\$32082[2]}, \mathcal{A}_{n\$32082[2]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$32082[2]}, \mathbf{B}_{n\$32082[3]} \rightarrow \mathbf{B}_1 \mathbf{B}_{n\$32082[3]}, \mathbf{T}_{n\$32082[3]} \rightarrow \mathbf{T}_{n\$32082[3]}, \\
& \mathcal{A}_{n\$32082[3]} \rightarrow \mathcal{A}_{n\$32082[3]}, \mathbf{B}_{n\$32082[4]} \rightarrow \mathbf{B}_2 \mathbf{B}_{n\$32082[4]}, \mathbf{T}_{n\$32082[4]} \rightarrow \mathbf{T}_{n\$32082[4]}, \mathcal{A}_{n\$32082[4]} \rightarrow \frac{\mathcal{A}_{n\$32082[4]}}{\mathbf{B}_1 \mathbf{B}_{n\$32082[3]}} \} \\
& \{ \mathbf{b}_{n\$33120[1]} \rightarrow \mathbf{b}_1 + \mathbf{b}_{n\$33120[1]}, \mathbf{t}_{n\$33120[1]} \rightarrow \mathbf{t}_{n\$33120[1]}, \alpha_{n\$33120[1]} \rightarrow \alpha_1 + \alpha_{n\$33120[1]}, \\
& \mathbf{b}_{n\$33120[2]} \rightarrow \mathbf{b}_2 + \mathbf{b}_{n\$33120[2]}, \mathbf{t}_{n\$33120[2]} \rightarrow \mathbf{t}_{n\$33120[2]}, \alpha_{n\$33120[2]} \rightarrow \alpha_1 + \alpha_{n\$33120[2]}, \\
& \mathbf{b}_{n\$33120[3]} \rightarrow \mathbf{b}_1 + \mathbf{b}_{n\$33120[3]}, \mathbf{t}_{n\$33120[3]} \rightarrow \mathbf{t}_{n\$33120[3]}, \alpha_{n\$33120[3]} \rightarrow \alpha_{n\$33120[3]}, \\
& \mathbf{b}_{n\$33120[4]} \rightarrow \mathbf{b}_2 + \mathbf{b}_{n\$33120[4]}, \mathbf{t}_{n\$33120[4]} \rightarrow \mathbf{t}_{n\$33120[4]}, \alpha_{n\$33120[4]} \rightarrow \hbar (\mathbf{b}_1 + \mathbf{b}_{n\$33120[3]}) + \alpha_{n\$33120[4]} \} \\
& \{ \mathbf{B}_{n\$33120[1]} \rightarrow \mathbf{B}_1 \mathbf{B}_{n\$33120[1]}, \mathbf{T}_{n\$33120[1]} \rightarrow \mathbf{T}_{n\$33120[1]}, \mathcal{A}_{n\$33120[1]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$33120[1]}, \mathbf{B}_{n\$33120[2]} \rightarrow \mathbf{B}_2 \mathbf{B}_{n\$33120[2]}, \\
& \mathbf{T}_{n\$33120[2]} \rightarrow \mathbf{T}_{n\$33120[2]}, \mathcal{A}_{n\$33120[2]} \rightarrow \mathcal{A}_1 \mathcal{A}_{n\$33120[2]}, \mathbf{B}_{n\$33120[3]} \rightarrow \mathbf{B}_1 \mathbf{B}_{n\$33120[3]}, \mathbf{T}_{n\$33120[3]} \rightarrow \mathbf{T}_{n\$33120[3]}, \\
& \mathcal{A}_{n\$33120[3]} \rightarrow \mathcal{A}_{n\$33120[3]}, \mathbf{B}_{n\$33120[4]} \rightarrow \mathbf{B}_2 \mathbf{B}_{n\$33120[4]}, \mathbf{T}_{n\$33120[4]} \rightarrow \mathbf{T}_{n\$33120[4]}, \mathcal{A}_{n\$33120[4]} \rightarrow \frac{\mathcal{A}_{n\$33120[4]}}{\mathbf{B}_1 \mathbf{B}_{n\$33120[3]}} \}
\end{aligned}$$

Out[]:= { 5.9375, True }

The Drinfel'd element inverse property, $(u_1 \bar{u}_2) \sim B_{1,2} \sim dm_{1,2 \rightarrow 1} \equiv \mathbb{E}[0, 0, 1]$:

In[]:= **Timing@**

$$\mathbf{HL} \left[\left(\left(\mathbf{R}_{1,2} \sim \mathbf{B}_1 \sim \mathbf{dS}_1 \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{2,1 \rightarrow i} \right) \left(\mathbf{R}_{1,2} \sim \mathbf{B}_2 \sim \mathbf{dS}_2 \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{2,1 \rightarrow j} \right) \right) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow i} \equiv \mathbb{E} [\mathbf{0}, \mathbf{0}, \mathbf{1}] \right]$$

```

" {bn$34131[1] -> -b1 + bn$34131[1], tn$34131[1] -> tn$34131[1], an$34131[1] -> an$34131[1]}
» {Bn$34131[1] -> Bn$34131[1]/B1, Tn$34131[1] -> Tn$34131[1], An$34131[1] -> An$34131[1]}
" {bn$34478[1] -> bi + bn$34478[1], tn$34478[1] -> tn$34478[1], an$34478[1] -> an$34478[1],
bn$34478[2] -> bi + bn$34478[2], tn$34478[2] -> tn$34478[2], an$34478[2] -> -h (bi + bn$34478[1]) + an$34478[2]}
" {Bn$34478[1] -> Bi Bn$34478[1], Tn$34478[1] -> Tn$34478[1], An$34478[1] -> An$34478[1],
Bn$34478[2] -> Bi Bn$34478[2], Tn$34478[2] -> Tn$34478[2], An$34478[2] -> Bi Bn$34478[1] An$34478[2]}
" {bn$34827[2] -> -b2 + bn$34827[2], tn$34827[2] -> tn$34827[2], an$34827[2] -> h b1 + an$34827[2]}
» {Bn$34827[2] -> Bn$34827[2]/B2, Tn$34827[2] -> Tn$34827[2], An$34827[2] -> An$34827[2]/B1}
" {bn$35118[2] -> -b2 + bn$35118[2], tn$35118[2] -> tn$35118[2], an$35118[2] -> -h b1 + an$35118[2]}
» {Bn$35118[2] -> Bn$35118[2]/B2, Tn$35118[2] -> Tn$35118[2], An$35118[2] -> B1 An$35118[2]}
" {bn$35542[1] -> bj + bn$35542[1], tn$35542[1] -> tn$35542[1], an$35542[1] -> an$35542[1],
bn$35542[2] -> bj + bn$35542[2], tn$35542[2] -> tn$35542[2], an$35542[2] -> h (bj + bn$35542[1]) + an$35542[2]}
» {Bn$35542[1] -> Bj Bn$35542[1], Tn$35542[1] -> Tn$35542[1], An$35542[1] -> An$35542[1],
Bn$35542[2] -> Bj Bn$35542[2], Tn$35542[2] -> Tn$35542[2], An$35542[2] -> An$35542[2]/Bj Bn$35542[1]}
" {bn$35828[i] -> bi + bn$35828[i], tn$35828[i] -> tn$35828[i], an$35828[i] -> -h (bi + bn$35828[i]) + an$35828[i],
bn$35828[j] -> bi + bn$35828[j], tn$35828[j] -> tn$35828[j], an$35828[j] -> h (bi + bn$35828[j]) + an$35828[j]}
» {Bn$35828[i] -> Bi Bn$35828[i], Tn$35828[i] -> Tn$35828[i], An$35828[i] -> Bi Bn$35828[i] An$35828[i],
Bn$35828[j] -> Bi Bn$35828[j], Tn$35828[j] -> Tn$35828[j], An$35828[j] -> An$35828[j]/Bi Bn$35828[j]}

```

Out[]:= {2.96875, True}

The ribbon element v satisfies $v^2 = S(u)u$. The spinner $C=uv^{-1}$. It is convenient to compute $z = S(u)u^{-1}$ which is something easy.

```

In[ ]:= Timing@Block[{k = 2},
((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1-i) ~ Bi ~ dSi) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1-j) ~ Bi,j ~ dm_i,j-i]

```

" $\{b_n\$36291[1] \rightarrow -b_1 + b_n\$36291[1], t_n\$36291[1] \rightarrow t_n\$36291[1], \alpha_n\$36291[1] \rightarrow \alpha_n\$36291[1]\}$
 » $\left\{B_n\$36291[1] \rightarrow \frac{B_n\$36291[1]}{B_1}, T_n\$36291[1] \rightarrow T_n\$36291[1], \mathcal{A}_n\$36291[1] \rightarrow \mathcal{A}_n\$36291[1]\right\}$
 " $\{b_n\$36637[1] \rightarrow b_i + b_n\$36637[1], t_n\$36637[1] \rightarrow t_n\$36637[1], \alpha_n\$36637[1] \rightarrow \alpha_n\$36637[1],$
 $b_n\$36637[2] \rightarrow b_i + b_n\$36637[2], t_n\$36637[2] \rightarrow t_n\$36637[2], \alpha_n\$36637[2] \rightarrow -\hbar (b_i + b_n\$36637[1]) + \alpha_n\$36637[2]\}$
 " $\{B_n\$36637[1] \rightarrow B_i B_n\$36637[1], T_n\$36637[1] \rightarrow T_n\$36637[1], \mathcal{A}_n\$36637[1] \rightarrow \mathcal{A}_n\$36637[1],$
 $B_n\$36637[2] \rightarrow B_i B_n\$36637[2], T_n\$36637[2] \rightarrow T_n\$36637[2], \mathcal{A}_n\$36637[2] \rightarrow B_i B_n\$36637[1] \mathcal{A}_n\$36637[2]\}$
 " $\{b_n\$36990[i] \rightarrow -b_i + b_n\$36990[i], t_n\$36990[i] \rightarrow t_n\$36990[i], \alpha_n\$36990[i] \rightarrow -\hbar (-b_i + b_n\$36990[i]) + \alpha_n\$36990[i]\}$
 » $\left\{B_n\$36990[i] \rightarrow \frac{B_n\$36990[i]}{B_i}, T_n\$36990[i] \rightarrow T_n\$36990[i], \mathcal{A}_n\$36990[i] \rightarrow \frac{B_n\$36990[i] \mathcal{A}_n\$36990[i]}{B_i}\right\}$
 " $\{b_n\$37782[2] \rightarrow -b_2 + b_n\$37782[2], t_n\$37782[2] \rightarrow t_n\$37782[2], \alpha_n\$37782[2] \rightarrow \hbar b_1 + \alpha_n\$37782[2]\}$
 » $\left\{B_n\$37782[2] \rightarrow \frac{B_n\$37782[2]}{B_2}, T_n\$37782[2] \rightarrow T_n\$37782[2], \mathcal{A}_n\$37782[2] \rightarrow \frac{\mathcal{A}_n\$37782[2]}{B_1}\right\}$
 " $\{b_n\$38073[2] \rightarrow -b_2 + b_n\$38073[2], t_n\$38073[2] \rightarrow t_n\$38073[2], \alpha_n\$38073[2] \rightarrow -\hbar b_1 + \alpha_n\$38073[2]\}$
 » $\left\{B_n\$38073[2] \rightarrow \frac{B_n\$38073[2]}{B_2}, T_n\$38073[2] \rightarrow T_n\$38073[2], \mathcal{A}_n\$38073[2] \rightarrow B_1 \mathcal{A}_n\$38073[2]\right\}$
 " $\{b_n\$38497[1] \rightarrow b_j + b_n\$38497[1], t_n\$38497[1] \rightarrow t_n\$38497[1], \alpha_n\$38497[1] \rightarrow \alpha_n\$38497[1],$
 $b_n\$38497[2] \rightarrow b_j + b_n\$38497[2], t_n\$38497[2] \rightarrow t_n\$38497[2], \alpha_n\$38497[2] \rightarrow \hbar (b_j + b_n\$38497[1]) + \alpha_n\$38497[2]\}$
 » $\left\{B_n\$38497[1] \rightarrow B_j B_n\$38497[1], T_n\$38497[1] \rightarrow T_n\$38497[1], \mathcal{A}_n\$38497[1] \rightarrow \mathcal{A}_n\$38497[1],$
 $B_n\$38497[2] \rightarrow B_j B_n\$38497[2], T_n\$38497[2] \rightarrow T_n\$38497[2], \mathcal{A}_n\$38497[2] \rightarrow \frac{\mathcal{A}_n\$38497[2]}{B_j B_n\$38497[1]}\right\}$
 " $\{b_n\$38781[i] \rightarrow b_i + b_n\$38781[i], t_n\$38781[i] \rightarrow t_n\$38781[i], \alpha_n\$38781[i] \rightarrow -\hbar (b_i + b_n\$38781[i]) + \alpha_n\$38781[i],$
 $b_n\$38781[j] \rightarrow b_i + b_n\$38781[j], t_n\$38781[j] \rightarrow t_n\$38781[j], \alpha_n\$38781[j] \rightarrow \hbar (b_i + b_n\$38781[j]) + \alpha_n\$38781[j]\}$
 » $\left\{B_n\$38781[i] \rightarrow B_i B_n\$38781[i], T_n\$38781[i] \rightarrow T_n\$38781[i], \mathcal{A}_n\$38781[i] \rightarrow B_i B_n\$38781[i] \mathcal{A}_n\$38781[i],$
 $B_n\$38781[j] \rightarrow B_i B_n\$38781[j], T_n\$38781[j] \rightarrow T_n\$38781[j], \mathcal{A}_n\$38781[j] \rightarrow \frac{\mathcal{A}_n\$38781[j]}{B_i B_n\$38781[j]}\right\}$

$$\text{Out[*]} = \left\{3.64063, \mathbb{E} \left[0, 0, \frac{1}{B_i} + \frac{\hbar a_i}{B_i} + \frac{\hbar^2 a_i^2}{2 B_i} + O[\epsilon]^3\right]\right\}$$

$$\text{In[*]} = \text{Timing@Block}[\{\$k = 2\}, \text{HL} / @ \left\{ \left(C_i \bar{C}_j \right) \sim B_{i,j} \sim dm_{i,j \rightarrow i} \equiv \mathbb{E} [0, 0, 1], \left(\bar{C}_i \bar{C}_j \right) \sim B_{i,j} \sim dm_{i,j \rightarrow i} \equiv \right. \\ \left. \left(\left(R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1 \rightarrow i} \right) \sim B_i \sim dS_i \right) \left(R_{1,2} \sim B_2 \sim dS_2 \sim B_{2,2} \sim dS_2 \sim B_{1,2} \sim dm_{2,1 \rightarrow j} \right) \sim B_{i,j} \sim dm_{i,j \rightarrow i} \right\}$$

```

" { b_n$39251[i] -> b_i + b_n$39251[i], t_n$39251[i] -> t_n$39251[i], alpha_n$39251[i] -> alpha_n$39251[i],
  b_n$39251[j] -> b_i + b_n$39251[j], t_n$39251[j] -> t_n$39251[j], alpha_n$39251[j] -> alpha_n$39251[j] }
" { B_n$39251[i] -> B_i B_n$39251[i], T_n$39251[i] -> T_n$39251[i], A_n$39251[i] -> A_n$39251[i],
  B_n$39251[j] -> B_i B_n$39251[j], T_n$39251[j] -> T_n$39251[j], A_n$39251[j] -> A_n$39251[j] }
" { b_n$39380[i] -> b_i + b_n$39380[i], t_n$39380[i] -> t_n$39380[i], alpha_n$39380[i] -> alpha_n$39380[i],
  b_n$39380[j] -> b_i + b_n$39380[j], t_n$39380[j] -> t_n$39380[j], alpha_n$39380[j] -> alpha_n$39380[j] }
" { B_n$39380[i] -> B_i B_n$39380[i], T_n$39380[i] -> T_n$39380[i], A_n$39380[i] -> A_n$39380[i],
  B_n$39380[j] -> B_i B_n$39380[j], T_n$39380[j] -> T_n$39380[j], A_n$39380[j] -> A_n$39380[j] }
" { b_n$39529[1] -> -b_1 + b_n$39529[1], t_n$39529[1] -> t_n$39529[1], alpha_n$39529[1] -> alpha_n$39529[1] }
" { B_n$39529[1] -> B_n$39529[1] / B_1, T_n$39529[1] -> T_n$39529[1], A_n$39529[1] -> A_n$39529[1] }
" { b_n$39875[1] -> b_i + b_n$39875[1], t_n$39875[1] -> t_n$39875[1], alpha_n$39875[1] -> alpha_n$39875[1],
  b_n$39875[2] -> b_i + b_n$39875[2], t_n$39875[2] -> t_n$39875[2], alpha_n$39875[2] -> -hbar (b_i + b_n$39875[1]) + alpha_n$39875[2] }
" { B_n$39875[1] -> B_i B_n$39875[1], T_n$39875[1] -> T_n$39875[1], A_n$39875[1] -> A_n$39875[1],
  B_n$39875[2] -> B_i B_n$39875[2], T_n$39875[2] -> T_n$39875[2], A_n$39875[2] -> B_i B_n$39875[1] A_n$39875[2] }
" { b_n$40229[i] -> -b_i + b_n$40229[i], t_n$40229[i] -> t_n$40229[i], alpha_n$40229[i] -> -hbar (-b_i + b_n$40229[i]) + alpha_n$40229[i] }
" { B_n$40229[i] -> B_n$40229[i] / B_i, T_n$40229[i] -> T_n$40229[i], A_n$40229[i] -> B_n$40229[i] A_n$40229[i] / B_i }
" { b_n$41013[2] -> -b_2 + b_n$41013[2], t_n$41013[2] -> t_n$41013[2], alpha_n$41013[2] -> hbar b_1 + alpha_n$41013[2] }
" { B_n$41013[2] -> B_n$41013[2] / B_2, T_n$41013[2] -> T_n$41013[2], A_n$41013[2] -> A_n$41013[2] / B_1 }
" { b_n$41304[2] -> -b_2 + b_n$41304[2], t_n$41304[2] -> t_n$41304[2], alpha_n$41304[2] -> -hbar b_1 + alpha_n$41304[2] }
" { B_n$41304[2] -> B_n$41304[2] / B_2, T_n$41304[2] -> T_n$41304[2], A_n$41304[2] -> B_1 A_n$41304[2] }
" { b_n$41728[1] -> b_j + b_n$41728[1], t_n$41728[1] -> t_n$41728[1], alpha_n$41728[1] -> alpha_n$41728[1],
  b_n$41728[2] -> b_j + b_n$41728[2], t_n$41728[2] -> t_n$41728[2], alpha_n$41728[2] -> hbar (b_j + b_n$41728[1]) + alpha_n$41728[2] }
" { B_n$41728[1] -> B_j B_n$41728[1], T_n$41728[1] -> T_n$41728[1], A_n$41728[1] -> A_n$41728[1],
  B_n$41728[2] -> B_j B_n$41728[2], T_n$41728[2] -> T_n$41728[2], A_n$41728[2] -> A_n$41728[2] / (B_j B_n$41728[1]) }
" { b_n$42016[i] -> b_i + b_n$42016[i], t_n$42016[i] -> t_n$42016[i], alpha_n$42016[i] -> -hbar (b_i + b_n$42016[i]) + alpha_n$42016[i],
  b_n$42016[j] -> b_i + b_n$42016[j], t_n$42016[j] -> t_n$42016[j], alpha_n$42016[j] -> hbar (b_i + b_n$42016[j]) + alpha_n$42016[j] }
" { B_n$42016[i] -> B_i B_n$42016[i], T_n$42016[i] -> T_n$42016[i], A_n$42016[i] -> B_i B_n$42016[i] A_n$42016[i],
  B_n$42016[j] -> B_i B_n$42016[j], T_n$42016[j] -> T_n$42016[j], A_n$42016[j] -> A_n$42016[j] / (B_i B_n$42016[j]) }

```

Out[]:= { 4.1875, { True, True } }

Reidemeister 2:

In[]:= Timing[HL[# == E[0, 0, 1]] & /@
 { (R1,2 R3,4) ~ B1,2,3,4 ~ (dm1,3 ->1 dm2,4 ->2), (R1,2 R3,4) ~ B1,2,3,4 ~ (dm1,3 ->1 dm2,4 ->2) }]

$$\{ b_{n\$42490[1]} \rightarrow b_1 + b_{n\$42490[1]}, t_{n\$42490[1]} \rightarrow t_{n\$42490[1]}, \alpha_{n\$42490[1]} \rightarrow \alpha_{n\$42490[1]},$$

$$b_{n\$42490[2]} \rightarrow b_2 + b_{n\$42490[2]}, t_{n\$42490[2]} \rightarrow t_{n\$42490[2]}, \alpha_{n\$42490[2]} \rightarrow -\hbar (b_1 + b_{n\$42490[1]}) + \alpha_{n\$42490[2]},$$

$$b_{n\$42490[3]} \rightarrow b_1 + b_{n\$42490[3]}, t_{n\$42490[3]} \rightarrow t_{n\$42490[3]}, \alpha_{n\$42490[3]} \rightarrow \alpha_{n\$42490[3]},$$

$$b_{n\$42490[4]} \rightarrow b_2 + b_{n\$42490[4]}, t_{n\$42490[4]} \rightarrow t_{n\$42490[4]}, \alpha_{n\$42490[4]} \rightarrow \hbar (b_1 + b_{n\$42490[3]}) + \alpha_{n\$42490[4]} \}$$

$$\{ B_{n\$42490[1]} \rightarrow B_1 B_{n\$42490[1]}, T_{n\$42490[1]} \rightarrow T_{n\$42490[1]}, \mathcal{A}_{n\$42490[1]} \rightarrow \mathcal{A}_{n\$42490[1]}, B_{n\$42490[2]} \rightarrow B_2 B_{n\$42490[2]},$$

$$T_{n\$42490[2]} \rightarrow T_{n\$42490[2]}, \mathcal{A}_{n\$42490[2]} \rightarrow B_1 B_{n\$42490[1]} \mathcal{A}_{n\$42490[2]}, B_{n\$42490[3]} \rightarrow B_1 B_{n\$42490[3]}, T_{n\$42490[3]} \rightarrow T_{n\$42490[3]},$$

$$\mathcal{A}_{n\$42490[3]} \rightarrow \mathcal{A}_{n\$42490[3]}, B_{n\$42490[4]} \rightarrow B_2 B_{n\$42490[4]}, T_{n\$42490[4]} \rightarrow T_{n\$42490[4]}, \mathcal{A}_{n\$42490[4]} \rightarrow \frac{\mathcal{A}_{n\$42490[4]}}{B_1 B_{n\$42490[3]}} \}$$

$$\{ b_{n\$43321[1]} \rightarrow b_1 + b_{n\$43321[1]}, t_{n\$43321[1]} \rightarrow t_{n\$43321[1]}, \alpha_{n\$43321[1]} \rightarrow \alpha_{n\$43321[1]},$$

$$b_{n\$43321[2]} \rightarrow b_2 + b_{n\$43321[2]}, t_{n\$43321[2]} \rightarrow t_{n\$43321[2]}, \alpha_{n\$43321[2]} \rightarrow \hbar (b_1 + b_{n\$43321[1]}) + \alpha_{n\$43321[2]},$$

$$b_{n\$43321[3]} \rightarrow b_1 + b_{n\$43321[3]}, t_{n\$43321[3]} \rightarrow t_{n\$43321[3]}, \alpha_{n\$43321[3]} \rightarrow \alpha_{n\$43321[3]},$$

$$b_{n\$43321[4]} \rightarrow b_2 + b_{n\$43321[4]}, t_{n\$43321[4]} \rightarrow t_{n\$43321[4]}, \alpha_{n\$43321[4]} \rightarrow -\hbar (b_1 + b_{n\$43321[3]}) + \alpha_{n\$43321[4]} \}$$

$$\{ B_{n\$43321[1]} \rightarrow B_1 B_{n\$43321[1]}, T_{n\$43321[1]} \rightarrow T_{n\$43321[1]}, \mathcal{A}_{n\$43321[1]} \rightarrow \mathcal{A}_{n\$43321[1]}, B_{n\$43321[2]} \rightarrow B_2 B_{n\$43321[2]},$$

$$T_{n\$43321[2]} \rightarrow T_{n\$43321[2]}, \mathcal{A}_{n\$43321[2]} \rightarrow \frac{\mathcal{A}_{n\$43321[2]}}{B_1 B_{n\$43321[1]}}, B_{n\$43321[3]} \rightarrow B_1 B_{n\$43321[3]}, T_{n\$43321[3]} \rightarrow T_{n\$43321[3]},$$

$$\mathcal{A}_{n\$43321[3]} \rightarrow \mathcal{A}_{n\$43321[3]}, B_{n\$43321[4]} \rightarrow B_2 B_{n\$43321[4]}, T_{n\$43321[4]} \rightarrow T_{n\$43321[4]}, \mathcal{A}_{n\$43321[4]} \rightarrow B_1 B_{n\$43321[3]} \mathcal{A}_{n\$43321[4]} \}$$

Out[]:= {4.78125, {True, True}}

Cyclic Reidemeister 2:

ln[]:= Timing@HL [(R_{1,4} R̄_{5,2} C̄₃) ~ B_{2,4} ~ dm_{2,4→2} ~ B_{1,3} ~ dm_{1,3→1} ~ B_{1,5} ~ dm_{1,5→1} ≡ C̄₁]

$$\{ b_{n\$44162[2]} \rightarrow b_2 + b_{n\$44162[2]}, t_{n\$44162[2]} \rightarrow t_{n\$44162[2]}, \alpha_{n\$44162[2]} \rightarrow -\hbar b_5 + \alpha_{n\$44162[2]},$$

$$b_{n\$44162[4]} \rightarrow b_2 + b_{n\$44162[4]}, t_{n\$44162[4]} \rightarrow t_{n\$44162[4]}, \alpha_{n\$44162[4]} \rightarrow \hbar b_1 + \alpha_{n\$44162[4]} \}$$

$$\{ B_{n\$44162[2]} \rightarrow B_2 B_{n\$44162[2]}, T_{n\$44162[2]} \rightarrow T_{n\$44162[2]}, \mathcal{A}_{n\$44162[2]} \rightarrow B_5 \mathcal{A}_{n\$44162[2]},$$

$$B_{n\$44162[4]} \rightarrow B_2 B_{n\$44162[4]}, T_{n\$44162[4]} \rightarrow T_{n\$44162[4]}, \mathcal{A}_{n\$44162[4]} \rightarrow \frac{\mathcal{A}_{n\$44162[4]}}{B_1} \}$$

$$\{ b_{n\$44499[1]} \rightarrow b_1 + b_{n\$44499[1]}, t_{n\$44499[1]} \rightarrow t_{n\$44499[1]}, \alpha_{n\$44499[1]} \rightarrow \alpha_{n\$44499[1]},$$

$$b_{n\$44499[3]} \rightarrow b_1 + b_{n\$44499[3]}, t_{n\$44499[3]} \rightarrow t_{n\$44499[3]}, \alpha_{n\$44499[3]} \rightarrow \alpha_{n\$44499[3]} \}$$

$$\{ B_{n\$44499[1]} \rightarrow B_1 B_{n\$44499[1]}, T_{n\$44499[1]} \rightarrow T_{n\$44499[1]}, \mathcal{A}_{n\$44499[1]} \rightarrow \mathcal{A}_{n\$44499[1]},$$

$$B_{n\$44499[3]} \rightarrow B_1 B_{n\$44499[3]}, T_{n\$44499[3]} \rightarrow T_{n\$44499[3]}, \mathcal{A}_{n\$44499[3]} \rightarrow \mathcal{A}_{n\$44499[3]} \}$$

$$\{ b_{n\$44868[1]} \rightarrow b_1 + b_{n\$44868[1]}, t_{n\$44868[1]} \rightarrow t_{n\$44868[1]}, \alpha_{n\$44868[1]} \rightarrow \alpha_{n\$44868[1]},$$

$$b_{n\$44868[5]} \rightarrow b_1 + b_{n\$44868[5]}, t_{n\$44868[5]} \rightarrow t_{n\$44868[5]}, \alpha_{n\$44868[5]} \rightarrow \alpha_{n\$44868[5]} \}$$

$$\{ B_{n\$44868[1]} \rightarrow B_1 B_{n\$44868[1]}, T_{n\$44868[1]} \rightarrow T_{n\$44868[1]}, \mathcal{A}_{n\$44868[1]} \rightarrow \mathcal{A}_{n\$44868[1]},$$

$$B_{n\$44868[5]} \rightarrow B_1 B_{n\$44868[5]}, T_{n\$44868[5]} \rightarrow T_{n\$44868[5]}, \mathcal{A}_{n\$44868[5]} \rightarrow \mathcal{A}_{n\$44868[5]} \}$$

Out[]:= {2.09375, True}

Reidemeister 3:

ln[]:= Timing@HL [((R_{1,2} R_{4,3} R_{5,6}) ~ B_{1,4} ~ dm_{1,4→1} ~ B_{2,5} ~ dm_{2,5→2} ~ B_{3,6} ~ dm_{3,6→3}) ≡ (R_{1,6} R_{2,3} R_{4,5}) ~ B_{1,4} ~ dm_{1,4→1} ~ B_{2,5} ~ dm_{2,5→2} ~ B_{3,6} ~ dm_{3,6→3})]

```

" { b_n$45346[1] -> b_1 + b_n$45346[1], t_n$45346[1] -> t_n$45346[1], alpha_n$45346[1] -> alpha_n$45346[1],
  b_n$45346[4] -> b_1 + b_n$45346[4], t_n$45346[4] -> t_n$45346[4], alpha_n$45346[4] -> alpha_n$45346[4] }
" { B_n$45346[1] -> B_1 B_n$45346[1], T_n$45346[1] -> T_n$45346[1], A_n$45346[1] -> A_n$45346[1],
  B_n$45346[4] -> B_1 B_n$45346[4], T_n$45346[4] -> T_n$45346[4], A_n$45346[4] -> A_n$45346[4] }
" { b_n$45553[2] -> b_2 + b_n$45553[2], t_n$45553[2] -> t_n$45553[2], alpha_n$45553[2] -> hbar b_1 + alpha_n$45553[2],
  b_n$45553[5] -> b_2 + b_n$45553[5], t_n$45553[5] -> t_n$45553[5], alpha_n$45553[5] -> alpha_n$45553[5] }
" { B_n$45553[2] -> B_2 B_n$45553[2], T_n$45553[2] -> T_n$45553[2], A_n$45553[2] -> A_n$45553[2] / B_1,
  B_n$45553[5] -> B_2 B_n$45553[5], T_n$45553[5] -> T_n$45553[5], A_n$45553[5] -> A_n$45553[5] }
" { b_n$45865[3] -> b_3 + b_n$45865[3], t_n$45865[3] -> t_n$45865[3], alpha_n$45865[3] -> hbar b_1 + alpha_n$45865[3],
  b_n$45865[6] -> b_3 + b_n$45865[6], t_n$45865[6] -> t_n$45865[6], alpha_n$45865[6] -> hbar b_2 + alpha_n$45865[6] }
" { B_n$45865[3] -> B_3 B_n$45865[3], T_n$45865[3] -> T_n$45865[3], A_n$45865[3] -> A_n$45865[3] / B_1,
  B_n$45865[6] -> B_3 B_n$45865[6], T_n$45865[6] -> T_n$45865[6], A_n$45865[6] -> A_n$45865[6] / B_2 }
" { b_n$46468[1] -> b_1 + b_n$46468[1], t_n$46468[1] -> t_n$46468[1], alpha_n$46468[1] -> alpha_n$46468[1],
  b_n$46468[4] -> b_1 + b_n$46468[4], t_n$46468[4] -> t_n$46468[4], alpha_n$46468[4] -> alpha_n$46468[4] }
" { B_n$46468[1] -> B_1 B_n$46468[1], T_n$46468[1] -> T_n$46468[1], A_n$46468[1] -> A_n$46468[1],
  B_n$46468[4] -> B_1 B_n$46468[4], T_n$46468[4] -> T_n$46468[4], A_n$46468[4] -> A_n$46468[4] }
" { b_n$46675[2] -> b_2 + b_n$46675[2], t_n$46675[2] -> t_n$46675[2], alpha_n$46675[2] -> alpha_n$46675[2],
  b_n$46675[5] -> b_2 + b_n$46675[5], t_n$46675[5] -> t_n$46675[5], alpha_n$46675[5] -> hbar b_1 + alpha_n$46675[5] }
" { B_n$46675[2] -> B_2 B_n$46675[2], T_n$46675[2] -> T_n$46675[2], A_n$46675[2] -> A_n$46675[2],
  B_n$46675[5] -> B_2 B_n$46675[5], T_n$46675[5] -> T_n$46675[5], A_n$46675[5] -> A_n$46675[5] / B_1 }
" { b_n$46908[3] -> b_3 + b_n$46908[3], t_n$46908[3] -> t_n$46908[3], alpha_n$46908[3] -> hbar b_2 + alpha_n$46908[3],
  b_n$46908[6] -> b_3 + b_n$46908[6], t_n$46908[6] -> t_n$46908[6], alpha_n$46908[6] -> hbar b_1 + alpha_n$46908[6] }
" { B_n$46908[3] -> B_3 B_n$46908[3], T_n$46908[3] -> T_n$46908[3], A_n$46908[3] -> A_n$46908[3] / B_2,
  B_n$46908[6] -> B_3 B_n$46908[6], T_n$46908[6] -> T_n$46908[6], A_n$46908[6] -> A_n$46908[6] / B_1 }

```

Out[*]= {4.3125, True}

Relations between the four kinks:

```

In[*]:= Timing[HL /@ {Kink_i == (R_{3,1} C_2) ~ B_{1,2} ~ dm_{1,2->1} ~ B_{1,3} ~ dm_{1,3->i},
  Kink_j == (R_{3,1} C_2) ~ B_{1,2} ~ dm_{1,2->1} ~ B_{1,3} ~ dm_{1,3->j}, (Kink_i Kink_j) ~ B_{i,j} ~ dm_{i,j->1} == E[0, 0, 1] } ]

```


$$\{ b_{n\$47192[1]} \rightarrow b_1 + b_{n\$47192[1]}, t_{n\$47192[1]} \rightarrow t_{n\$47192[1]}, \alpha_{n\$47192[1]} \rightarrow \alpha_{n\$47192[1]}, \\ b_{n\$47192[2]} \rightarrow b_1 + b_{n\$47192[2]}, t_{n\$47192[2]} \rightarrow t_{n\$47192[2]}, \alpha_{n\$47192[2]} \rightarrow \alpha_{n\$47192[2]} \}$$

$$\{ B_{n\$47192[1]} \rightarrow B_1 B_{n\$47192[1]}, T_{n\$47192[1]} \rightarrow T_{n\$47192[1]}, \mathcal{A}_{n\$47192[1]} \rightarrow \mathcal{A}_{n\$47192[1]}, \\ B_{n\$47192[2]} \rightarrow B_1 B_{n\$47192[2]}, T_{n\$47192[2]} \rightarrow T_{n\$47192[2]}, \mathcal{A}_{n\$47192[2]} \rightarrow \mathcal{A}_{n\$47192[2]} \}$$

$$\{ b_{n\$47449[1]} \rightarrow b_i + b_{n\$47449[1]}, t_{n\$47449[1]} \rightarrow t_{n\$47449[1]}, \alpha_{n\$47449[1]} \rightarrow \alpha_{n\$47449[1]}, \\ b_{n\$47449[3]} \rightarrow b_i + b_{n\$47449[3]}, t_{n\$47449[3]} \rightarrow t_{n\$47449[3]}, \alpha_{n\$47449[3]} \rightarrow \hbar (b_i + b_{n\$47449[1]}) + \alpha_{n\$47449[3]} \}$$

$$\left\{ B_{n\$47449[1]} \rightarrow B_i B_{n\$47449[1]}, T_{n\$47449[1]} \rightarrow T_{n\$47449[1]}, \mathcal{A}_{n\$47449[1]} \rightarrow \mathcal{A}_{n\$47449[1]}, \right. \\ \left. B_{n\$47449[3]} \rightarrow B_i B_{n\$47449[3]}, T_{n\$47449[3]} \rightarrow T_{n\$47449[3]}, \mathcal{A}_{n\$47449[3]} \rightarrow \frac{\mathcal{A}_{n\$47449[3]}}{B_i B_{n\$47449[1]}} \right\}$$

$$\{ b_{n\$47692[1]} \rightarrow b_1 + b_{n\$47692[1]}, t_{n\$47692[1]} \rightarrow t_{n\$47692[1]}, \alpha_{n\$47692[1]} \rightarrow \hbar b_3 + \alpha_{n\$47692[1]}, \\ b_{n\$47692[2]} \rightarrow b_1 + b_{n\$47692[2]}, t_{n\$47692[2]} \rightarrow t_{n\$47692[2]}, \alpha_{n\$47692[2]} \rightarrow \alpha_{n\$47692[2]} \}$$

$$\left\{ B_{n\$47692[1]} \rightarrow B_1 B_{n\$47692[1]}, T_{n\$47692[1]} \rightarrow T_{n\$47692[1]}, \mathcal{A}_{n\$47692[1]} \rightarrow \frac{\mathcal{A}_{n\$47692[1]}}{B_3}, \right. \\ \left. B_{n\$47692[2]} \rightarrow B_1 B_{n\$47692[2]}, T_{n\$47692[2]} \rightarrow T_{n\$47692[2]}, \mathcal{A}_{n\$47692[2]} \rightarrow \mathcal{A}_{n\$47692[2]} \right\}$$

$$\{ b_{n\$47921[1]} \rightarrow b_i + b_{n\$47921[1]}, t_{n\$47921[1]} \rightarrow t_{n\$47921[1]}, \alpha_{n\$47921[1]} \rightarrow \hbar (b_i + b_{n\$47921[3]}) + \alpha_{n\$47921[1]}, \\ b_{n\$47921[3]} \rightarrow b_i + b_{n\$47921[3]}, t_{n\$47921[3]} \rightarrow t_{n\$47921[3]}, \alpha_{n\$47921[3]} \rightarrow \alpha_{n\$47921[3]} \}$$

$$\left\{ B_{n\$47921[1]} \rightarrow B_i B_{n\$47921[1]}, T_{n\$47921[1]} \rightarrow T_{n\$47921[1]}, \mathcal{A}_{n\$47921[1]} \rightarrow \frac{\mathcal{A}_{n\$47921[1]}}{B_i B_{n\$47921[3]}}, \right. \\ \left. B_{n\$47921[3]} \rightarrow B_i B_{n\$47921[3]}, T_{n\$47921[3]} \rightarrow T_{n\$47921[3]}, \mathcal{A}_{n\$47921[3]} \rightarrow \mathcal{A}_{n\$47921[3]} \right\}$$

$$\{ b_{n\$48431[1]} \rightarrow b_1 + b_{n\$48431[1]}, t_{n\$48431[1]} \rightarrow t_{n\$48431[1]}, \alpha_{n\$48431[1]} \rightarrow \alpha_{n\$48431[1]}, \\ b_{n\$48431[2]} \rightarrow b_1 + b_{n\$48431[2]}, t_{n\$48431[2]} \rightarrow t_{n\$48431[2]}, \alpha_{n\$48431[2]} \rightarrow \alpha_{n\$48431[2]} \}$$

$$\{ B_{n\$48431[1]} \rightarrow B_1 B_{n\$48431[1]}, T_{n\$48431[1]} \rightarrow T_{n\$48431[1]}, \mathcal{A}_{n\$48431[1]} \rightarrow \mathcal{A}_{n\$48431[1]}, \\ B_{n\$48431[2]} \rightarrow B_1 B_{n\$48431[2]}, T_{n\$48431[2]} \rightarrow T_{n\$48431[2]}, \mathcal{A}_{n\$48431[2]} \rightarrow \mathcal{A}_{n\$48431[2]} \}$$

$$\{ b_{n\$48772[1]} \rightarrow b_i + b_{n\$48772[1]}, t_{n\$48772[1]} \rightarrow t_{n\$48772[1]}, \alpha_{n\$48772[1]} \rightarrow \alpha_{n\$48772[1]}, \\ b_{n\$48772[3]} \rightarrow b_i + b_{n\$48772[3]}, t_{n\$48772[3]} \rightarrow t_{n\$48772[3]}, \alpha_{n\$48772[3]} \rightarrow -\hbar (b_i + b_{n\$48772[1]}) + \alpha_{n\$48772[3]} \}$$

$$\{ B_{n\$48772[1]} \rightarrow B_i B_{n\$48772[1]}, T_{n\$48772[1]} \rightarrow T_{n\$48772[1]}, \mathcal{A}_{n\$48772[1]} \rightarrow \mathcal{A}_{n\$48772[1]}, \\ B_{n\$48772[3]} \rightarrow B_i B_{n\$48772[3]}, T_{n\$48772[3]} \rightarrow T_{n\$48772[3]}, \mathcal{A}_{n\$48772[3]} \rightarrow B_i B_{n\$48772[1]} \mathcal{A}_{n\$48772[3]} \}$$

$$\{ b_{n\$49091[1]} \rightarrow b_1 + b_{n\$49091[1]}, t_{n\$49091[1]} \rightarrow t_{n\$49091[1]}, \alpha_{n\$49091[1]} \rightarrow -\hbar b_3 + \alpha_{n\$49091[1]}, \\ b_{n\$49091[2]} \rightarrow b_1 + b_{n\$49091[2]}, t_{n\$49091[2]} \rightarrow t_{n\$49091[2]}, \alpha_{n\$49091[2]} \rightarrow \alpha_{n\$49091[2]} \}$$

$$\{ B_{n\$49091[1]} \rightarrow B_1 B_{n\$49091[1]}, T_{n\$49091[1]} \rightarrow T_{n\$49091[1]}, \mathcal{A}_{n\$49091[1]} \rightarrow B_3 \mathcal{A}_{n\$49091[1]}, \\ B_{n\$49091[2]} \rightarrow B_1 B_{n\$49091[2]}, T_{n\$49091[2]} \rightarrow T_{n\$49091[2]}, \mathcal{A}_{n\$49091[2]} \rightarrow \mathcal{A}_{n\$49091[2]} \}$$

$$\{ b_{n\$49386[1]} \rightarrow b_j + b_{n\$49386[1]}, t_{n\$49386[1]} \rightarrow t_{n\$49386[1]}, \alpha_{n\$49386[1]} \rightarrow -\hbar (b_j + b_{n\$49386[3]}) + \alpha_{n\$49386[1]}, \\ b_{n\$49386[3]} \rightarrow b_j + b_{n\$49386[3]}, t_{n\$49386[3]} \rightarrow t_{n\$49386[3]}, \alpha_{n\$49386[3]} \rightarrow \alpha_{n\$49386[3]} \}$$

$$\{ B_{n\$49386[1]} \rightarrow B_j B_{n\$49386[1]}, T_{n\$49386[1]} \rightarrow T_{n\$49386[1]}, \mathcal{A}_{n\$49386[1]} \rightarrow B_j B_{n\$49386[3]} \mathcal{A}_{n\$49386[1]}, \\ B_{n\$49386[3]} \rightarrow B_j B_{n\$49386[3]}, T_{n\$49386[3]} \rightarrow T_{n\$49386[3]}, \mathcal{A}_{n\$49386[3]} \rightarrow \mathcal{A}_{n\$49386[3]} \}$$

$$\{ b_{n\$49994[i]} \rightarrow b_1 + b_{n\$49994[i]}, t_{n\$49994[i]} \rightarrow t_{n\$49994[i]}, \alpha_{n\$49994[i]} \rightarrow \hbar (b_1 + b_{n\$49994[i]}) + \alpha_{n\$49994[i]}, \\ b_{n\$49994[j]} \rightarrow b_1 + b_{n\$49994[j]}, t_{n\$49994[j]} \rightarrow t_{n\$49994[j]}, \alpha_{n\$49994[j]} \rightarrow -\hbar (b_1 + b_{n\$49994[j]}) + \alpha_{n\$49994[j]} \}$$

$$\left\{ B_{n\$49994[i]} \rightarrow B_1 B_{n\$49994[i]}, T_{n\$49994[i]} \rightarrow T_{n\$49994[i]}, \mathcal{A}_{n\$49994[i]} \rightarrow \frac{\mathcal{A}_{n\$49994[i]}}{B_1 B_{n\$49994[i]}}, \right. \\ \left. B_{n\$49994[j]} \rightarrow B_1 B_{n\$49994[j]}, T_{n\$49994[j]} \rightarrow T_{n\$49994[j]}, \mathcal{A}_{n\$49994[j]} \rightarrow B_1 B_{n\$49994[j]} \mathcal{A}_{n\$49994[j]} \right\}$$

Out[*]= {4.53125, {True, True, True}}

The Trefoil

```

In[ ]:= Timing@Block[{ $k = 1,
  Z = R1,5 R6,2 R3,7 C4 Kink8 Kink9 Kink10;
  Do[Z = Z ~ B1,r ~ dm1,r→1, {r, 2, 10}];
  {Simplify /@ Z, Simplify /@ (Z ~ B1 ~ b2t1 /. T1 → T)} } ]

"
{ bn$50368[1] → b1 + bn$50368[1], tn$50368[1] → tn$50368[1], αn$50368[1] → αn$50368[1],
  bn$50368[2] → b1 + bn$50368[2], tn$50368[2] → tn$50368[2], αn$50368[2] → ħ b6 + αn$50368[2] }

"
{ Bn$50368[1] → B1 Bn$50368[1], Tn$50368[1] → Tn$50368[1], An$50368[1] → An$50368[1],
  Bn$50368[2] → B1 Bn$50368[2], Tn$50368[2] → Tn$50368[2], An$50368[2] →  $\frac{A_{n$50368[2]}}{B_6}$  }

"
{ bn$50445[1] → b1 + bn$50445[1], tn$50445[1] → tn$50445[1], αn$50445[1] → ħ b6 + αn$50445[1],
  bn$50445[3] → b1 + bn$50445[3], tn$50445[3] → tn$50445[3], αn$50445[3] → αn$50445[3] }

"
{ Bn$50445[1] → B1 Bn$50445[1], Tn$50445[1] → Tn$50445[1], An$50445[1] →  $\frac{A_{n$50445[1]}}{B_6}$ ,
  Bn$50445[3] → B1 Bn$50445[3], Tn$50445[3] → Tn$50445[3], An$50445[3] → An$50445[3] }

"
{ bn$50524[1] → b1 + bn$50524[1], tn$50524[1] → tn$50524[1], αn$50524[1] → ħ b6 + αn$50524[1],
  bn$50524[4] → b1 + bn$50524[4], tn$50524[4] → tn$50524[4], αn$50524[4] → αn$50524[4] }

"
{ Bn$50524[1] → B1 Bn$50524[1], Tn$50524[1] → Tn$50524[1], An$50524[1] →  $\frac{A_{n$50524[1]}}{B_6}$ ,
  Bn$50524[4] → B1 Bn$50524[4], Tn$50524[4] → Tn$50524[4], An$50524[4] → An$50524[4] }

"
{ bn$50627[1] → b1 + bn$50627[1], tn$50627[1] → tn$50627[1], αn$50627[1] → ħ b6 + αn$50627[1],
  bn$50627[5] → b1 + bn$50627[5], tn$50627[5] → tn$50627[5], αn$50627[5] → ħ (b1 + bn$50627[1]) + αn$50627[5] }

"
{ Bn$50627[1] → B1 Bn$50627[1], Tn$50627[1] → Tn$50627[1], An$50627[1] →  $\frac{A_{n$50627[1]}}{B_6}$ ,
  Bn$50627[5] → B1 Bn$50627[5], Tn$50627[5] → Tn$50627[5], An$50627[5] →  $\frac{A_{n$50627[5]}}{B_1 B_{n$50627[1]}}$  }

"
{ bn$50736[1] → b1 + bn$50736[1], tn$50736[1] → tn$50736[1],
  αn$50736[1] → ħ (b1 + bn$50736[1]) + ħ (b1 + bn$50736[6]) + αn$50736[1],
  bn$50736[6] → b1 + bn$50736[6], tn$50736[6] → tn$50736[6], αn$50736[6] → αn$50736[6] }

"
{ Bn$50736[1] → B1 Bn$50736[1], Tn$50736[1] → Tn$50736[1], An$50736[1] →  $\frac{A_{n$50736[1]}}{B_1^2 B_{n$50736[1]} B_{n$50736[6]}}$ ,
  Bn$50736[6] → B1 Bn$50736[6], Tn$50736[6] → Tn$50736[6], An$50736[6] → An$50736[6] }

"
{ bn$50867[1] → b1 + bn$50867[1], tn$50867[1] → tn$50867[1], αn$50867[1] → 2 ħ (b1 + bn$50867[1]) + αn$50867[1],
  bn$50867[7] → b1 + bn$50867[7], tn$50867[7] → tn$50867[7], αn$50867[7] → ħ (b1 + bn$50867[1]) + αn$50867[7] }

"
{ Bn$50867[1] → B1 Bn$50867[1], Tn$50867[1] → Tn$50867[1], An$50867[1] →  $\frac{A_{n$50867[1]}}{B_1^2 B_{n$50867[1]}}$ ,
  Bn$50867[7] → B1 Bn$50867[7], Tn$50867[7] → Tn$50867[7], An$50867[7] →  $\frac{A_{n$50867[7]}}{B_1 B_{n$50867[1]}}$  }

"
{ bn$52748[1] → b1 + bn$52748[1], tn$52748[1] → tn$52748[1], αn$52748[1] → 3 ħ (b1 + bn$52748[1]) + αn$52748[1],
  bn$52748[8] → b1 + bn$52748[8], tn$52748[8] → tn$52748[8], αn$52748[8] → -ħ (b1 + bn$52748[8]) + αn$52748[8] }

```

- » $\left\{ B_{n\$52748[1]} \rightarrow B_1 B_{n\$52748[1]}, T_{n\$52748[1]} \rightarrow T_{n\$52748[1]}, \mathcal{A}_{n\$52748[1]} \rightarrow \frac{\mathcal{A}_{n\$52748[1]}}{B_1^3 B_{n\$52748[1]}^3}, \right.$
 $B_{n\$52748[8]} \rightarrow B_1 B_{n\$52748[8]}, T_{n\$52748[8]} \rightarrow T_{n\$52748[8]}, \mathcal{A}_{n\$52748[8]} \rightarrow B_1 B_{n\$52748[8]} \mathcal{A}_{n\$52748[8]} \left. \right\}$
- “ $\{ b_{n\$54287[1]} \rightarrow b_1 + b_{n\$54287[1]}, t_{n\$54287[1]} \rightarrow t_{n\$54287[1]}, \alpha_{n\$54287[1]} \rightarrow 2 \hbar (b_1 + b_{n\$54287[1]}) + \alpha_{n\$54287[1]},$
 $b_{n\$54287[9]} \rightarrow b_1 + b_{n\$54287[9]}, t_{n\$54287[9]} \rightarrow t_{n\$54287[9]}, \alpha_{n\$54287[9]} \rightarrow -\hbar (b_1 + b_{n\$54287[9]}) + \alpha_{n\$54287[9]} \}$
- » $\left\{ B_{n\$54287[1]} \rightarrow B_1 B_{n\$54287[1]}, T_{n\$54287[1]} \rightarrow T_{n\$54287[1]}, \mathcal{A}_{n\$54287[1]} \rightarrow \frac{\mathcal{A}_{n\$54287[1]}}{B_1^2 B_{n\$54287[1]}^2}, \right.$
 $B_{n\$54287[9]} \rightarrow B_1 B_{n\$54287[9]}, T_{n\$54287[9]} \rightarrow T_{n\$54287[9]}, \mathcal{A}_{n\$54287[9]} \rightarrow B_1 B_{n\$54287[9]} \mathcal{A}_{n\$54287[9]} \left. \right\}$
- “ $\{ b_{n\$55478[1]} \rightarrow b_1 + b_{n\$55478[1]}, t_{n\$55478[1]} \rightarrow t_{n\$55478[1]}, \alpha_{n\$55478[1]} \rightarrow \hbar (b_1 + b_{n\$55478[1]}) + \alpha_{n\$55478[1]},$
 $b_{n\$55478[10]} \rightarrow b_1 + b_{n\$55478[10]}, t_{n\$55478[10]} \rightarrow t_{n\$55478[10]}, \alpha_{n\$55478[10]} \rightarrow -\hbar (b_1 + b_{n\$55478[10]}) + \alpha_{n\$55478[10]} \}$
- » $\left\{ B_{n\$55478[1]} \rightarrow B_1 B_{n\$55478[1]}, T_{n\$55478[1]} \rightarrow T_{n\$55478[1]}, \mathcal{A}_{n\$55478[1]} \rightarrow \frac{\mathcal{A}_{n\$55478[1]}}{B_1 B_{n\$55478[1]}}, \right.$
 $B_{n\$55478[10]} \rightarrow B_1 B_{n\$55478[10]}, T_{n\$55478[10]} \rightarrow T_{n\$55478[10]}, \mathcal{A}_{n\$55478[10]} \rightarrow B_1 B_{n\$55478[10]} \mathcal{A}_{n\$55478[10]} \left. \right\}$
- » $\left\{ b_{n\$56237[1]} \rightarrow b_{n\$56237[1]} - \frac{t_1}{\gamma}, t_{n\$56237[1]} \rightarrow t_{n\$56237[1]}, \alpha_{n\$56237[1]} \rightarrow \alpha_{n\$56237[1]} \right\}$
- “ $\{ B_{n\$56237[1]} \rightarrow B_{n\$56237[1]} T_1, T_{n\$56237[1]} \rightarrow T_{n\$56237[1]}, \mathcal{A}_{n\$56237[1]} \rightarrow \mathcal{A}_{n\$56237[1]} \}$

Out[*]= $\left\{ 12.7656, \left\{ \mathbb{E} \left[\theta, \theta, \frac{B_1}{1 - B_1 + B_1^2} - \left(\hbar B_1 (-a_1 (-1 + B_1 - B_1^3 + B_1^4)) + \gamma (B_1 - 2 B_1^2 - 2 B_1^4 + 2 \hbar x_1 y_1 + B_1^3 (3 + 2 \hbar x_1 y_1)) \right) \right] \right\} / \right.$
 $\left. \left(1 - B_1 + B_1^2 \right)^3 + O[\epsilon]^2 \right\},$
 $\mathbb{E} \left[\theta, \theta, \frac{T}{1 - T + T^2} + \left(T \hbar (T (-1 + 2 T - 3 T^2 + 2 T^3)) \gamma + 2 (-1 + T - T^3 + T^4) a_1 - 2 (1 + T^3) \gamma \hbar x_1 y_1 \right) \right] \right\} /$
 $\left. \left(1 - T + T^2 \right)^3 + O[\epsilon]^2 \right\} \}$

Program

```
In[*]:= Define [kRi,j = Ri,j ~ Bi,j ~ (b2ti b2tj) /. {ti|j → t,
kRi,j = Ri,j ~ Bi,j ~ (b2ti b2tj) /. {ti|j → t, Ti|j → T},
kmi,j→k = (t2bi t2bj) ~ Bi,j ~ dmi,j→k ~ Bk ~ b2tk /. {tk → t, Tk → T, vi|j → 0},
kCi = Ci ~ Bi ~ b2ti /. Ti → T,
kCi = Ci ~ Bi ~ b2ti /. Ti → T,
kKinki = Kinki ~ Bi ~ b2ti /. {ti → t, Ti → T},
kKinki = Kinki ~ Bi ~ b2ti /. {ti → t, Ti → T}]
```

```
In[*]:= Timing@Block[{ $k = 1},
Z = kR1,5 kR6,2 kR3,7 kC4 kKink8 kKink9 kKink10;
Do[Z = Z ~ B1,r ~ km1,r→1, {r, 2, 10}];
Simplify /@ Z]
```

- » $\left\{ b_{n\$56432[i]} \rightarrow b_{n\$56432[i]} - \frac{t_i}{\gamma}, t_{n\$56432[i]} \rightarrow t_{n\$56432[i]}, \alpha_{n\$56432[i]} \rightarrow \alpha_{n\$56432[i]}, \right.$
 $b_{n\$56432[j]} \rightarrow b_{n\$56432[j]} - \frac{t_j}{\gamma}, t_{n\$56432[j]} \rightarrow t_{n\$56432[j]}, \alpha_{n\$56432[j]} \rightarrow \hbar \left(b_{n\$56432[i]} - \frac{t_i}{\gamma} \right) + \alpha_{n\$56432[j]} \left. \right\}$
- » $\left\{ B_{n\$56432[i]} \rightarrow B_{n\$56432[i]} T_i, T_{n\$56432[i]} \rightarrow T_{n\$56432[i]}, \mathcal{A}_{n\$56432[i]} \rightarrow \mathcal{A}_{n\$56432[i]}, \right.$
 $B_{n\$56432[j]} \rightarrow B_{n\$56432[j]} T_j, T_{n\$56432[j]} \rightarrow T_{n\$56432[j]}, \mathcal{A}_{n\$56432[j]} \rightarrow \frac{\mathcal{A}_{n\$56432[j]}}{B_{n\$56432[i]} T_i} \left. \right\}$

- » $\left\{ b_{n\$56451[i]} \rightarrow b_{n\$56451[i]} - \frac{t_i}{\gamma}, t_{n\$56451[i]} \rightarrow t_{n\$56451[i]}, \alpha_{n\$56451[i]} \rightarrow \alpha_{n\$56451[i]} \right\}$
- “ $\{ B_{n\$56451[i]} \rightarrow B_{n\$56451[i]} T_i, T_{n\$56451[i]} \rightarrow T_{n\$56451[i]}, \mathcal{A}_{n\$56451[i]} \rightarrow \mathcal{A}_{n\$56451[i]} \}$
- » $\left\{ b_{n\$56474[i]} \rightarrow b_{n\$56474[i]} - \frac{t_i}{\gamma}, t_{n\$56474[i]} \rightarrow t_{n\$56474[i]}, \alpha_{n\$56474[i]} \rightarrow -\hbar \left(b_{n\$56474[i]} - \frac{t_i}{\gamma} \right) + \alpha_{n\$56474[i]} \right\}$
- “ $\{ B_{n\$56474[i]} \rightarrow B_{n\$56474[i]} T_i, T_{n\$56474[i]} \rightarrow T_{n\$56474[i]}, \mathcal{A}_{n\$56474[i]} \rightarrow B_{n\$56474[i]} T_i \mathcal{A}_{n\$56474[i]} \}$
- “ $\{ b_{n\$56505[i]} \rightarrow b_k + b_{n\$56505[i]}, t_{n\$56505[i]} \rightarrow t_{n\$56505[i]}, \alpha_{n\$56505[i]} \rightarrow \alpha_i + \alpha_{n\$56505[i]}, b_{n\$56505[j]} \rightarrow b_k + b_{n\$56505[j]}, t_{n\$56505[j]} \rightarrow t_{n\$56505[j]}, \alpha_{n\$56505[j]} \rightarrow \alpha_j + \alpha_{n\$56505[j]} \}$
- “ $\{ B_{n\$56505[i]} \rightarrow B_k B_{n\$56505[i]}, T_{n\$56505[i]} \rightarrow T_{n\$56505[i]}, \mathcal{A}_{n\$56505[i]} \rightarrow \mathcal{A}_i \mathcal{A}_{n\$56505[i]}, B_{n\$56505[j]} \rightarrow B_k B_{n\$56505[j]}, T_{n\$56505[j]} \rightarrow T_{n\$56505[j]}, \mathcal{A}_{n\$56505[j]} \rightarrow \mathcal{A}_j \mathcal{A}_{n\$56505[j]} \}$
- » $\left\{ b_{n\$56546[k]} \rightarrow b_{n\$56546[k]} - \frac{t_k}{\gamma}, t_{n\$56546[k]} \rightarrow t_{n\$56546[k]}, \alpha_{n\$56546[k]} \rightarrow \alpha_i + \alpha_j + \alpha_{n\$56546[k]} \right\}$
- “ $\{ B_{n\$56546[k]} \rightarrow B_{n\$56546[k]} T_k, T_{n\$56546[k]} \rightarrow T_{n\$56546[k]}, \mathcal{A}_{n\$56546[k]} \rightarrow \mathcal{A}_i \mathcal{A}_j \mathcal{A}_{n\$56546[k]} \}$
- » $\left\{ b_{n\$56587[1]} \rightarrow b_{n\$56587[1]}, t_{n\$56587[1]} \rightarrow t_{n\$56587[1]}, \alpha_{n\$56587[1]} \rightarrow \alpha_{n\$56587[1]}, b_{n\$56587[2]} \rightarrow b_{n\$56587[2]}, t_{n\$56587[2]} \rightarrow t_{n\$56587[2]}, \alpha_{n\$56587[2]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$56587[2]} \right\}$
- » $\left\{ B_{n\$56587[1]} \rightarrow B_{n\$56587[1]}, T_{n\$56587[1]} \rightarrow T_{n\$56587[1]}, \mathcal{A}_{n\$56587[1]} \rightarrow \mathcal{A}_{n\$56587[1]}, B_{n\$56587[2]} \rightarrow B_{n\$56587[2]}, T_{n\$56587[2]} \rightarrow T_{n\$56587[2]}, \mathcal{A}_{n\$56587[2]} \rightarrow \frac{\mathcal{A}_{n\$56587[2]}}{T} \right\}$
- » $\left\{ b_{n\$56672[1]} \rightarrow b_{n\$56672[1]}, t_{n\$56672[1]} \rightarrow t_{n\$56672[1]}, \alpha_{n\$56672[1]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$56672[1]}, b_{n\$56672[3]} \rightarrow b_{n\$56672[3]}, t_{n\$56672[3]} \rightarrow t_{n\$56672[3]}, \alpha_{n\$56672[3]} \rightarrow \alpha_{n\$56672[3]} \right\}$
- » $\left\{ B_{n\$56672[1]} \rightarrow B_{n\$56672[1]}, T_{n\$56672[1]} \rightarrow T_{n\$56672[1]}, \mathcal{A}_{n\$56672[1]} \rightarrow \frac{\mathcal{A}_{n\$56672[1]}}{T}, B_{n\$56672[3]} \rightarrow B_{n\$56672[3]}, T_{n\$56672[3]} \rightarrow T_{n\$56672[3]}, \mathcal{A}_{n\$56672[3]} \rightarrow \mathcal{A}_{n\$56672[3]} \right\}$
- » $\left\{ b_{n\$56759[1]} \rightarrow b_{n\$56759[1]}, t_{n\$56759[1]} \rightarrow t_{n\$56759[1]}, \alpha_{n\$56759[1]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$56759[1]}, b_{n\$56759[4]} \rightarrow b_{n\$56759[4]}, t_{n\$56759[4]} \rightarrow t_{n\$56759[4]}, \alpha_{n\$56759[4]} \rightarrow \alpha_{n\$56759[4]} \right\}$
- » $\left\{ B_{n\$56759[1]} \rightarrow B_{n\$56759[1]}, T_{n\$56759[1]} \rightarrow T_{n\$56759[1]}, \mathcal{A}_{n\$56759[1]} \rightarrow \frac{\mathcal{A}_{n\$56759[1]}}{T}, B_{n\$56759[4]} \rightarrow B_{n\$56759[4]}, T_{n\$56759[4]} \rightarrow T_{n\$56759[4]}, \mathcal{A}_{n\$56759[4]} \rightarrow \mathcal{A}_{n\$56759[4]} \right\}$
- » $\left\{ b_{n\$56872[1]} \rightarrow b_{n\$56872[1]}, t_{n\$56872[1]} \rightarrow t_{n\$56872[1]}, \alpha_{n\$56872[1]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$56872[1]}, b_{n\$56872[5]} \rightarrow b_{n\$56872[5]}, t_{n\$56872[5]} \rightarrow t_{n\$56872[5]}, \alpha_{n\$56872[5]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$56872[5]} \right\}$
- » $\left\{ B_{n\$56872[1]} \rightarrow B_{n\$56872[1]}, T_{n\$56872[1]} \rightarrow T_{n\$56872[1]}, \mathcal{A}_{n\$56872[1]} \rightarrow \frac{\mathcal{A}_{n\$56872[1]}}{T}, B_{n\$56872[5]} \rightarrow B_{n\$56872[5]}, T_{n\$56872[5]} \rightarrow T_{n\$56872[5]}, \mathcal{A}_{n\$56872[5]} \rightarrow \frac{\mathcal{A}_{n\$56872[5]}}{T} \right\}$
- » $\left\{ b_{n\$56993[1]} \rightarrow b_{n\$56993[1]}, t_{n\$56993[1]} \rightarrow t_{n\$56993[1]}, \alpha_{n\$56993[1]} \rightarrow -\frac{2 t \hbar}{\gamma} + \alpha_{n\$56993[1]}, b_{n\$56993[6]} \rightarrow b_{n\$56993[6]}, t_{n\$56993[6]} \rightarrow t_{n\$56993[6]}, \alpha_{n\$56993[6]} \rightarrow \alpha_{n\$56993[6]} \right\}$

- » $\left\{ \begin{aligned} &B_n\$56993[1] \rightarrow B_n\$56993[1], T_n\$56993[1] \rightarrow T_n\$56993[1], \mathcal{A}_n\$56993[1] \rightarrow \frac{\mathcal{A}_n\$56993[1]}{T^2}, \\ &B_n\$56993[6] \rightarrow B_n\$56993[6], T_n\$56993[6] \rightarrow T_n\$56993[6], \mathcal{A}_n\$56993[6] \rightarrow \mathcal{A}_n\$56993[6] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &b_n\$57128[1] \rightarrow b_n\$57128[1], t_n\$57128[1] \rightarrow t_n\$57128[1], \alpha_n\$57128[1] \rightarrow -\frac{2 t \hbar}{\gamma} + \alpha_n\$57128[1], \\ &b_n\$57128[7] \rightarrow b_n\$57128[7], t_n\$57128[7] \rightarrow t_n\$57128[7], \alpha_n\$57128[7] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$57128[7] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &B_n\$57128[1] \rightarrow B_n\$57128[1], T_n\$57128[1] \rightarrow T_n\$57128[1], \mathcal{A}_n\$57128[1] \rightarrow \frac{\mathcal{A}_n\$57128[1]}{T^2}, \\ &B_n\$57128[7] \rightarrow B_n\$57128[7], T_n\$57128[7] \rightarrow T_n\$57128[7], \mathcal{A}_n\$57128[7] \rightarrow \frac{\mathcal{A}_n\$57128[7]}{T} \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &b_n\$59583[1] \rightarrow b_n\$59583[1], t_n\$59583[1] \rightarrow t_n\$59583[1], \alpha_n\$59583[1] \rightarrow -\frac{3 t \hbar}{\gamma} + \alpha_n\$59583[1], \\ &b_n\$59583[8] \rightarrow b_n\$59583[8], t_n\$59583[8] \rightarrow t_n\$59583[8], \alpha_n\$59583[8] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$59583[8] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &B_n\$59583[1] \rightarrow B_n\$59583[1], T_n\$59583[1] \rightarrow T_n\$59583[1], \mathcal{A}_n\$59583[1] \rightarrow \frac{\mathcal{A}_n\$59583[1]}{T^3}, \\ &B_n\$59583[8] \rightarrow B_n\$59583[8], T_n\$59583[8] \rightarrow T_n\$59583[8], \mathcal{A}_n\$59583[8] \rightarrow T \mathcal{A}_n\$59583[8] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &b_n\$61458[1] \rightarrow b_n\$61458[1], t_n\$61458[1] \rightarrow t_n\$61458[1], \alpha_n\$61458[1] \rightarrow -\frac{2 t \hbar}{\gamma} + \alpha_n\$61458[1], \\ &b_n\$61458[9] \rightarrow b_n\$61458[9], t_n\$61458[9] \rightarrow t_n\$61458[9], \alpha_n\$61458[9] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$61458[9] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &B_n\$61458[1] \rightarrow B_n\$61458[1], T_n\$61458[1] \rightarrow T_n\$61458[1], \mathcal{A}_n\$61458[1] \rightarrow \frac{\mathcal{A}_n\$61458[1]}{T^2}, \\ &B_n\$61458[9] \rightarrow B_n\$61458[9], T_n\$61458[9] \rightarrow T_n\$61458[9], \mathcal{A}_n\$61458[9] \rightarrow T \mathcal{A}_n\$61458[9] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &b_n\$62979[1] \rightarrow b_n\$62979[1], t_n\$62979[1] \rightarrow t_n\$62979[1], \alpha_n\$62979[1] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$62979[1], \\ &b_n\$62979[10] \rightarrow b_n\$62979[10], t_n\$62979[10] \rightarrow t_n\$62979[10], \alpha_n\$62979[10] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$62979[10] \end{aligned} \right\}$
- » $\left\{ \begin{aligned} &B_n\$62979[1] \rightarrow B_n\$62979[1], T_n\$62979[1] \rightarrow T_n\$62979[1], \mathcal{A}_n\$62979[1] \rightarrow \frac{\mathcal{A}_n\$62979[1]}{T}, \\ &B_n\$62979[10] \rightarrow B_n\$62979[10], T_n\$62979[10] \rightarrow T_n\$62979[10], \mathcal{A}_n\$62979[10] \rightarrow T \mathcal{A}_n\$62979[10] \end{aligned} \right\}$

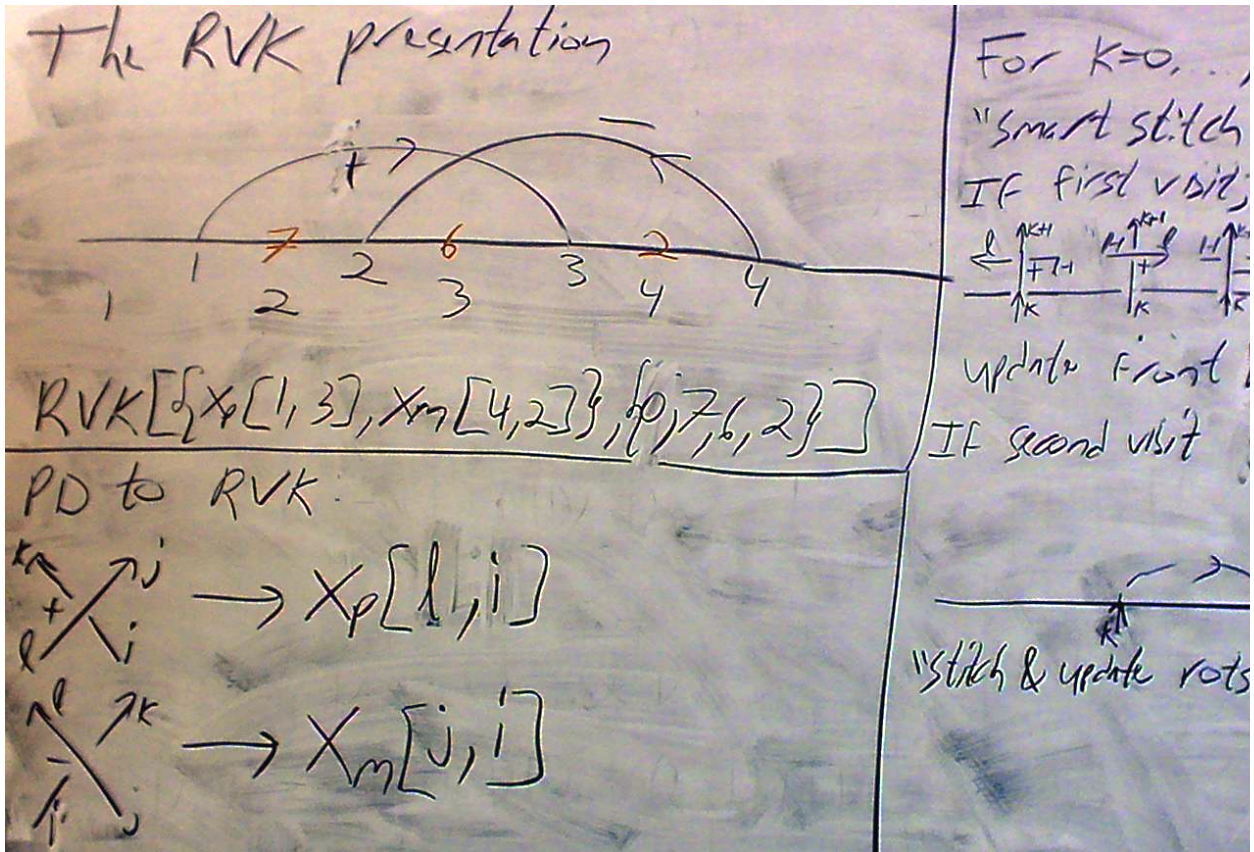
Out[*]= {5.45313,

$$\mathbb{E} \left[\mathbf{0}, \mathbf{0}, \frac{T}{1 - T + T^2} + (T \hbar (T (-1 + 2 T - 3 T^2 + 2 T^3) \gamma + 2 (-1 + T - T^3 + T^4) \mathbf{a}_1 - 2 (1 + T^3) \gamma \hbar \mathbf{x}_1 \mathbf{y}_1) \epsilon) / (1 - T + T^2)^3 + \mathbf{0}[\epsilon]^2 \right]$$

RVK, rot, Z from 2016-09/OneSmidgen.nb. See also local version in this folder.

Some details of the code below are at

<http://drorbn.net/bbs/show?shot=Dror-160920-151350.jpg>



Program

In[]:=

RVK::usage =

"RVK[*xs*, *rots*] represents a Rotational Virtual Knot with a list of *n* X_p/X_m crossings *xs* and a length $2n$ list of rotation numbers *rots*. Crossing sites are indexed 1 through $2n$, and *rots*[[*k*]] is the rotation between site *k*-1 and site *k*. RVK is also a casting operator converting to the RVK presentation from other knot presentations.";

Program

```

In[ ]:= RVK[pd_PD] := PPRVK@Module[{n, xs, x, rots, front = {0}, k},
  n = Length@pd; rots = Table[0, {2 n}];
  xs = List@@pd /. x_X => If[PositiveQ@x, Xp[x[[4]], x[[1]], Xm[x[[2]], x[[1]]];
  For[k = 0, k < 2 n, ++k,
    If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → Catch[xs /. {
        Xp[k + 1, L_] | Xm[L_, k + 1] => Throw[{L, k + 1, 1 - L}],
        Xp[L_, k + 1] | Xm[k + 1, L_] => (++rots[[L]];
          Throw[{1 - L, k + 1, L})
      }]],
      If[MatchQ[front, {___, k, ___, -k, ___}], --rots[[k + 1]]
    ]
  ];
  RVK[xs, rots]
];
RVK[K_] := RVK[PD[K]];

```

In[]:= RVK[Knot[3, 1]]

 KnotTheory: Loading precomputed data in PD4Knots` 

Out[]:= RVK[{Xm[4, 1], Xm[6, 3], Xm[2, 5]}, {0, 0, 0, -1, 0, 0}]

Program

```

In[ ]:= rot[_ , 0] = E[0, 0, 1];
  rot[i_, n_] := Module[{j},
    rot[i, n] = If[n > 0, rot[i, n - 1] kCj, rot[i, n + 1] kCj] ~Bi,j ~kmi,j→i;

```

In[]:= {rot[*i*, 3], rot[*i*, -3]}

- » $\left\{ b_{n\$63944[i]} \rightarrow b_{n\$63944[i]} - \frac{t_i}{\gamma}, t_{n\$63944[i]} \rightarrow t_{n\$63944[i]}, \alpha_{n\$63944[i]} \rightarrow \alpha_{n\$63944[i]} \right\}$
- “ $\{ B_{n\$63944[i]} \rightarrow B_{n\$63944[i]} T_i, T_{n\$63944[i]} \rightarrow T_{n\$63944[i]}, \mathcal{A}_{n\$63944[i]} \rightarrow \mathcal{A}_{n\$63944[i]} \}$
- ” $\{ b_{n\$63976[i]} \rightarrow b_k + b_{n\$63976[i]}, t_{n\$63976[i]} \rightarrow t_{n\$63976[i]}, \alpha_{n\$63976[i]} \rightarrow \alpha_i + \alpha_{n\$63976[i]},$
 $b_{n\$63976[j]} \rightarrow b_k + b_{n\$63976[j]}, t_{n\$63976[j]} \rightarrow t_{n\$63976[j]}, \alpha_{n\$63976[j]} \rightarrow \alpha_j + \alpha_{n\$63976[j]} \}$
- ” $\{ B_{n\$63976[i]} \rightarrow B_k B_{n\$63976[i]}, T_{n\$63976[i]} \rightarrow T_{n\$63976[i]}, \mathcal{A}_{n\$63976[i]} \rightarrow \mathcal{A}_i \mathcal{A}_{n\$63976[i]},$
 $B_{n\$63976[j]} \rightarrow B_k B_{n\$63976[j]}, T_{n\$63976[j]} \rightarrow T_{n\$63976[j]}, \mathcal{A}_{n\$63976[j]} \rightarrow \mathcal{A}_j \mathcal{A}_{n\$63976[j]} \}$
- » $\left\{ b_{n\$64181[k]} \rightarrow b_{n\$64181[k]} - \frac{t_k}{\gamma}, t_{n\$64181[k]} \rightarrow t_{n\$64181[k]}, \alpha_{n\$64181[k]} \rightarrow \alpha_i + \alpha_j + \alpha_{n\$64181[k]} \right\}$
- “ $\{ B_{n\$64181[k]} \rightarrow B_{n\$64181[k]} T_k, T_{n\$64181[k]} \rightarrow T_{n\$64181[k]}, \mathcal{A}_{n\$64181[k]} \rightarrow \mathcal{A}_i \mathcal{A}_j \mathcal{A}_{n\$64181[k]} \}$
- ” $\{ b_{n\$64354[i]} \rightarrow b_{n\$64354[i]}, t_{n\$64354[i]} \rightarrow t_{n\$64354[i]}, \alpha_{n\$64354[i]} \rightarrow \alpha_{n\$64354[i]},$
 $b_{n\$64354[j\$63943]} \rightarrow b_{n\$64354[j\$63943]}, t_{n\$64354[j\$63943]} \rightarrow t_{n\$64354[j\$63943]}, \alpha_{n\$64354[j\$63943]} \rightarrow \alpha_{n\$64354[j\$63943]} \}$
- ” $\{ B_{n\$64354[i]} \rightarrow B_{n\$64354[i]}, T_{n\$64354[i]} \rightarrow T_{n\$64354[i]}, \mathcal{A}_{n\$64354[i]} \rightarrow \mathcal{A}_{n\$64354[i]},$
 $B_{n\$64354[j\$63943]} \rightarrow B_{n\$64354[j\$63943]}, T_{n\$64354[j\$63943]} \rightarrow T_{n\$64354[j\$63943]}, \mathcal{A}_{n\$64354[j\$63943]} \rightarrow \mathcal{A}_{n\$64354[j\$63943]} \}$
- ” $\{ b_{n\$64553[i]} \rightarrow b_{n\$64553[i]}, t_{n\$64553[i]} \rightarrow t_{n\$64553[i]}, \alpha_{n\$64553[i]} \rightarrow \alpha_{n\$64553[i]},$
 $b_{n\$64553[j\$63942]} \rightarrow b_{n\$64553[j\$63942]}, t_{n\$64553[j\$63942]} \rightarrow t_{n\$64553[j\$63942]}, \alpha_{n\$64553[j\$63942]} \rightarrow \alpha_{n\$64553[j\$63942]} \}$
- ” $\{ B_{n\$64553[i]} \rightarrow B_{n\$64553[i]}, T_{n\$64553[i]} \rightarrow T_{n\$64553[i]}, \mathcal{A}_{n\$64553[i]} \rightarrow \mathcal{A}_{n\$64553[i]},$
 $B_{n\$64553[j\$63942]} \rightarrow B_{n\$64553[j\$63942]}, T_{n\$64553[j\$63942]} \rightarrow T_{n\$64553[j\$63942]}, \mathcal{A}_{n\$64553[j\$63942]} \rightarrow \mathcal{A}_{n\$64553[j\$63942]} \}$
- ” $\{ b_{n\$64760[i]} \rightarrow b_{n\$64760[i]}, t_{n\$64760[i]} \rightarrow t_{n\$64760[i]}, \alpha_{n\$64760[i]} \rightarrow \alpha_{n\$64760[i]},$
 $b_{n\$64760[j\$63941]} \rightarrow b_{n\$64760[j\$63941]}, t_{n\$64760[j\$63941]} \rightarrow t_{n\$64760[j\$63941]}, \alpha_{n\$64760[j\$63941]} \rightarrow \alpha_{n\$64760[j\$63941]} \}$
- ” $\{ B_{n\$64760[i]} \rightarrow B_{n\$64760[i]}, T_{n\$64760[i]} \rightarrow T_{n\$64760[i]}, \mathcal{A}_{n\$64760[i]} \rightarrow \mathcal{A}_{n\$64760[i]},$
 $B_{n\$64760[j\$63941]} \rightarrow B_{n\$64760[j\$63941]}, T_{n\$64760[j\$63941]} \rightarrow T_{n\$64760[j\$63941]}, \mathcal{A}_{n\$64760[j\$63941]} \rightarrow \mathcal{A}_{n\$64760[j\$63941]} \}$
- » $\left\{ b_{n\$64970[i]} \rightarrow b_{n\$64970[i]} - \frac{t_i}{\gamma}, t_{n\$64970[i]} \rightarrow t_{n\$64970[i]}, \alpha_{n\$64970[i]} \rightarrow \alpha_{n\$64970[i]} \right\}$
- “ $\{ B_{n\$64970[i]} \rightarrow B_{n\$64970[i]} T_i, T_{n\$64970[i]} \rightarrow T_{n\$64970[i]}, \mathcal{A}_{n\$64970[i]} \rightarrow \mathcal{A}_{n\$64970[i]} \}$
- ” $\{ b_{n\$65002[i]} \rightarrow b_{n\$65002[i]}, t_{n\$65002[i]} \rightarrow t_{n\$65002[i]}, \alpha_{n\$65002[i]} \rightarrow \alpha_{n\$65002[i]},$
 $b_{n\$65002[j\$64969]} \rightarrow b_{n\$65002[j\$64969]}, t_{n\$65002[j\$64969]} \rightarrow t_{n\$65002[j\$64969]}, \alpha_{n\$65002[j\$64969]} \rightarrow \alpha_{n\$65002[j\$64969]} \}$
- ” $\{ B_{n\$65002[i]} \rightarrow B_{n\$65002[i]}, T_{n\$65002[i]} \rightarrow T_{n\$65002[i]}, \mathcal{A}_{n\$65002[i]} \rightarrow \mathcal{A}_{n\$65002[i]},$
 $B_{n\$65002[j\$64969]} \rightarrow B_{n\$65002[j\$64969]}, T_{n\$65002[j\$64969]} \rightarrow T_{n\$65002[j\$64969]}, \mathcal{A}_{n\$65002[j\$64969]} \rightarrow \mathcal{A}_{n\$65002[j\$64969]} \}$
- ” $\{ b_{n\$65199[i]} \rightarrow b_{n\$65199[i]}, t_{n\$65199[i]} \rightarrow t_{n\$65199[i]}, \alpha_{n\$65199[i]} \rightarrow \alpha_{n\$65199[i]},$
 $b_{n\$65199[j\$64968]} \rightarrow b_{n\$65199[j\$64968]}, t_{n\$65199[j\$64968]} \rightarrow t_{n\$65199[j\$64968]}, \alpha_{n\$65199[j\$64968]} \rightarrow \alpha_{n\$65199[j\$64968]} \}$
- ” $\{ B_{n\$65199[i]} \rightarrow B_{n\$65199[i]}, T_{n\$65199[i]} \rightarrow T_{n\$65199[i]}, \mathcal{A}_{n\$65199[i]} \rightarrow \mathcal{A}_{n\$65199[i]},$
 $B_{n\$65199[j\$64968]} \rightarrow B_{n\$65199[j\$64968]}, T_{n\$65199[j\$64968]} \rightarrow T_{n\$65199[j\$64968]}, \mathcal{A}_{n\$65199[j\$64968]} \rightarrow \mathcal{A}_{n\$65199[j\$64968]} \}$
- ” $\{ b_{n\$65348[i]} \rightarrow b_{n\$65348[i]}, t_{n\$65348[i]} \rightarrow t_{n\$65348[i]}, \alpha_{n\$65348[i]} \rightarrow \alpha_{n\$65348[i]},$
 $b_{n\$65348[j\$64967]} \rightarrow b_{n\$65348[j\$64967]}, t_{n\$65348[j\$64967]} \rightarrow t_{n\$65348[j\$64967]}, \alpha_{n\$65348[j\$64967]} \rightarrow \alpha_{n\$65348[j\$64967]} \}$
- ” $\{ B_{n\$65348[i]} \rightarrow B_{n\$65348[i]}, T_{n\$65348[i]} \rightarrow T_{n\$65348[i]}, \mathcal{A}_{n\$65348[i]} \rightarrow \mathcal{A}_{n\$65348[i]},$
 $B_{n\$65348[j\$64967]} \rightarrow B_{n\$65348[j\$64967]}, T_{n\$65348[j\$64967]} \rightarrow T_{n\$65348[j\$64967]}, \mathcal{A}_{n\$65348[j\$64967]} \rightarrow \mathcal{A}_{n\$65348[j\$64967]} \}$

Out[] = $\left\{ \mathbb{E} \left[\mathbf{0}, \mathbf{0}, T^{3/2} - 3 \left(T^{3/2} \hbar a_i \right) \epsilon + \frac{9}{2} T^{3/2} \hbar^2 a_i^2 \epsilon^2 + O[\epsilon]^3 \right], \mathbb{E} \left[\mathbf{0}, \mathbf{0}, \frac{1}{T^{3/2}} + \frac{3 \hbar a_i \epsilon}{T^{3/2}} + \frac{9 \hbar^2 a_i^2 \epsilon^2}{2 T^{3/2}} + O[\epsilon]^3 \right] \right\}$

Program

```

In[ ]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] ==)
PP`z`@Module[{todo, n, rots, z, done, st, cx, z1, i, j, k, k1, k2, k3},
  {todo, rots} = List@@rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  z = E[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[todo != {},
    {cx} = MaximalBy[todo, Length[done ∩ {#[1], #[2], #[1] - 1, #[2] - 1}] &, 1];
    {i, j} = List@@cx;
    z1 = Switch[Head[cx],
      Xp, (kRi,j kKinkk) ~ Bj,k ~ kmj,k→j,
      Xm, (kRi,j kKinkk) ~ Bj,k ~ kmj,k→j
    ];
    z1 = (rot[k, rots[[i]] z1) ~ Bk,i ~ kmk,i→i; rots[[i]] = 0;
    z1 = (z1 rot[k, rots[[i + 1]]) ~ Bi,k ~ kmi,k→i; rots[[i + 1]] = 0;
    z1 = (rot[k, rots[[j]] z1) ~ Bk,j ~ kmk,j→j; rots[[j]] = 0;
    z1 = (z1 rot[k, rots[[j + 1]]) ~ Bj,k ~ kmj,k→j; rots[[j + 1]] = 0;
    z *= z1;
    If[MemberQ[done, i], z = z ~ Bi,i+1 ~ kmi,i+1→i; st = st /. st[[i + 2]] → st[[i + 1]];
    If[MemberQ[done, i - 1], z = z ~ Bst[[i],i] ~ kmst[[i],i→st[[i]]; st = st /. st[[i + 1]] → st[[i]];
    If[MemberQ[done, j], z = z ~ Bj,j+1 ~ kmj,j+1→j; st = st /. st[[j + 2]] → st[[j + 1]];
    If[MemberQ[done, j - 1], z = z ~ Bst[[j],j] ~ kmst[[j],j→st[[j]]; st = st /. st[[j + 1]] → st[[j]];
    done = done ∪ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, cx];
  ];
  Simplify /@ (z /. {x0 → x, y0 → y, a0 → a})
]

```

Knot

```

In[ ]:= $k = 1; Timing@Z@Knot[10, 100]

```

Knot

$$\gg \left\{ b_{n\$65565[6]} \rightarrow b_{n\$65565[6]}, t_{n\$65565[6]} \rightarrow t_{n\$65565[6]}, \alpha_{n\$65565[6]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$65565[6]}, \right. \\
 \left. b_{n\$65565[k\$65562]} \rightarrow b_{n\$65565[k\$65562]}, t_{n\$65565[k\$65562]} \rightarrow t_{n\$65565[k\$65562]}, \alpha_{n\$65565[k\$65562]} \rightarrow \frac{t \hbar}{\gamma} + \alpha_{n\$65565[k\$65562]} \right\}$$

Knot

$$\gg \left\{ B_{n\$65565[6]} \rightarrow B_{n\$65565[6]}, T_{n\$65565[6]} \rightarrow T_{n\$65565[6]}, \mathcal{A}_{n\$65565[6]} \rightarrow \frac{\mathcal{A}_{n\$65565[6]}}{T}, \right. \\
 \left. B_{n\$65565[k\$65562]} \rightarrow B_{n\$65565[k\$65562]}, T_{n\$65565[k\$65562]} \rightarrow T_{n\$65565[k\$65562]}, \mathcal{A}_{n\$65565[k\$65562]} \rightarrow T \mathcal{A}_{n\$65565[k\$65562]} \right\}$$

Knot

$$\gg \left\{ b_{n\$65632[k\$65562]} \rightarrow b_{n\$65632[k\$65562]}, t_{n\$65632[k\$65562]} \rightarrow t_{n\$65632[k\$65562]}, \right. \\
 \left. \alpha_{n\$65632[k\$65562]} \rightarrow \alpha_{n\$65632[k\$65562]}, b_{n\$65632[1]} \rightarrow b_{n\$65632[1]}, t_{n\$65632[1]} \rightarrow t_{n\$65632[1]}, \alpha_{n\$65632[1]} \rightarrow \alpha_{n\$65632[1]} \right\}$$

Knot

$$\gg \left\{ B_{n\$65632[k\$65562]} \rightarrow B_{n\$65632[k\$65562]}, T_{n\$65632[k\$65562]} \rightarrow T_{n\$65632[k\$65562]}, \right. \\
 \left. \mathcal{A}_{n\$65632[k\$65562]} \rightarrow \mathcal{A}_{n\$65632[k\$65562]}, B_{n\$65632[1]} \rightarrow B_{n\$65632[1]}, T_{n\$65632[1]} \rightarrow T_{n\$65632[1]}, \mathcal{A}_{n\$65632[1]} \rightarrow \mathcal{A}_{n\$65632[1]} \right\}$$

Knot " { $b_n\$65691[1] \rightarrow b_n\$65691[1]$, $t_n\$65691[1] \rightarrow t_n\$65691[1]$, $\alpha_n\$65691[1] \rightarrow \alpha_n\$65691[1]$,
 $b_n\$65691[k\$65562] \rightarrow b_n\$65691[k\$65562]$, $t_n\$65691[k\$65562] \rightarrow t_n\$65691[k\$65562]$, $\alpha_n\$65691[k\$65562] \rightarrow \alpha_n\$65691[k\$65562]$ }

Knot " { $B_n\$65691[1] \rightarrow B_n\$65691[1]$, $T_n\$65691[1] \rightarrow T_n\$65691[1]$, $\mathcal{A}_n\$65691[1] \rightarrow \mathcal{A}_n\$65691[1]$,
 $B_n\$65691[k\$65562] \rightarrow B_n\$65691[k\$65562]$, $T_n\$65691[k\$65562] \rightarrow T_n\$65691[k\$65562]$, $\mathcal{A}_n\$65691[k\$65562] \rightarrow \mathcal{A}_n\$65691[k\$65562]$ }

Knot " { $b_n\$65751[k\$65562] \rightarrow b_n\$65751[k\$65562]$, $t_n\$65751[k\$65562] \rightarrow t_n\$65751[k\$65562]$, $\alpha_n\$65751[k\$65562] \rightarrow \alpha_n\$65751[k\$65562]$,
 $b_n\$65751[j\$65750] \rightarrow b_n\$65751[j\$65750]$, $t_n\$65751[j\$65750] \rightarrow t_n\$65751[j\$65750]$, $\alpha_n\$65751[j\$65750] \rightarrow \alpha_n\$65751[j\$65750]$ }

Knot " { $B_n\$65751[k\$65562] \rightarrow B_n\$65751[k\$65562]$, $T_n\$65751[k\$65562] \rightarrow T_n\$65751[k\$65562]$, $\mathcal{A}_n\$65751[k\$65562] \rightarrow \mathcal{A}_n\$65751[k\$65562]$,
 $B_n\$65751[j\$65750] \rightarrow B_n\$65751[j\$65750]$, $T_n\$65751[j\$65750] \rightarrow T_n\$65751[j\$65750]$, $\mathcal{A}_n\$65751[j\$65750] \rightarrow \mathcal{A}_n\$65751[j\$65750]$ }

Knot " { $b_n\$65796[k\$65562] \rightarrow b_n\$65796[k\$65562]$, $t_n\$65796[k\$65562] \rightarrow t_n\$65796[k\$65562]$,
 $\alpha_n\$65796[k\$65562] \rightarrow \alpha_n\$65796[k\$65562]$, $b_n\$65796[6] \rightarrow b_n\$65796[6]$, $t_n\$65796[6] \rightarrow t_n\$65796[6]$, $\alpha_n\$65796[6] \rightarrow \alpha_n\$65796[6]$ }

Knot " { $B_n\$65796[k\$65562] \rightarrow B_n\$65796[k\$65562]$, $T_n\$65796[k\$65562] \rightarrow T_n\$65796[k\$65562]$,
 $\mathcal{A}_n\$65796[k\$65562] \rightarrow \mathcal{A}_n\$65796[k\$65562]$, $B_n\$65796[6] \rightarrow B_n\$65796[6]$, $T_n\$65796[6] \rightarrow T_n\$65796[6]$, $\mathcal{A}_n\$65796[6] \rightarrow \mathcal{A}_n\$65796[6]$ }

Knot " { $b_n\$65827[6] \rightarrow b_n\$65827[6]$, $t_n\$65827[6] \rightarrow t_n\$65827[6]$, $\alpha_n\$65827[6] \rightarrow \alpha_n\$65827[6]$,
 $b_n\$65827[k\$65562] \rightarrow b_n\$65827[k\$65562]$, $t_n\$65827[k\$65562] \rightarrow t_n\$65827[k\$65562]$, $\alpha_n\$65827[k\$65562] \rightarrow \alpha_n\$65827[k\$65562]$ }

Knot " { $B_n\$65827[6] \rightarrow B_n\$65827[6]$, $T_n\$65827[6] \rightarrow T_n\$65827[6]$, $\mathcal{A}_n\$65827[6] \rightarrow \mathcal{A}_n\$65827[6]$,
 $B_n\$65827[k\$65562] \rightarrow B_n\$65827[k\$65562]$, $T_n\$65827[k\$65562] \rightarrow T_n\$65827[k\$65562]$, $\mathcal{A}_n\$65827[k\$65562] \rightarrow \mathcal{A}_n\$65827[k\$65562]$ }

Knot " { $b_n\$65858[0] \rightarrow b_n\$65858[0]$, $t_n\$65858[0] \rightarrow t_n\$65858[0]$, $\alpha_n\$65858[0] \rightarrow \alpha_n\$65858[0]$,
 $b_n\$65858[1] \rightarrow b_n\$65858[1]$, $t_n\$65858[1] \rightarrow t_n\$65858[1]$, $\alpha_n\$65858[1] \rightarrow \alpha_n\$65858[1]$ }

Knot " { $B_n\$65858[0] \rightarrow B_n\$65858[0]$, $T_n\$65858[0] \rightarrow T_n\$65858[0]$, $\mathcal{A}_n\$65858[0] \rightarrow \mathcal{A}_n\$65858[0]$,
 $B_n\$65858[1] \rightarrow B_n\$65858[1]$, $T_n\$65858[1] \rightarrow T_n\$65858[1]$, $\mathcal{A}_n\$65858[1] \rightarrow \mathcal{A}_n\$65858[1]$ }

Knot " { $b_n\$65890[18] \rightarrow b_n\$65890[18]$, $t_n\$65890[18] \rightarrow t_n\$65890[18]$, $\alpha_n\$65890[18] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$65890[18]$,
 $b_n\$65890[k\$65562] \rightarrow b_n\$65890[k\$65562]$, $t_n\$65890[k\$65562] \rightarrow t_n\$65890[k\$65562]$, $\alpha_n\$65890[k\$65562] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$65890[k\$65562]$ }

Knot " { $B_n\$65890[18] \rightarrow B_n\$65890[18]$, $T_n\$65890[18] \rightarrow T_n\$65890[18]$, $\mathcal{A}_n\$65890[18] \rightarrow \frac{\mathcal{A}_n\$65890[18]}{T}$,
 $B_n\$65890[k\$65562] \rightarrow B_n\$65890[k\$65562]$, $T_n\$65890[k\$65562] \rightarrow T_n\$65890[k\$65562]$, $\mathcal{A}_n\$65890[k\$65562] \rightarrow T \mathcal{A}_n\$65890[k\$65562]$ }

Knot " { $b_n\$65957[k\$65562] \rightarrow b_n\$65957[k\$65562]$, $t_n\$65957[k\$65562] \rightarrow t_n\$65957[k\$65562]$,
 $\alpha_n\$65957[k\$65562] \rightarrow \alpha_n\$65957[k\$65562]$, $b_n\$65957[5] \rightarrow b_n\$65957[5]$, $t_n\$65957[5] \rightarrow t_n\$65957[5]$, $\alpha_n\$65957[5] \rightarrow \alpha_n\$65957[5]$ }

Knot " { $B_n\$65957[k\$65562] \rightarrow B_n\$65957[k\$65562]$, $T_n\$65957[k\$65562] \rightarrow T_n\$65957[k\$65562]$,
 $\mathcal{A}_n\$65957[k\$65562] \rightarrow \mathcal{A}_n\$65957[k\$65562]$, $B_n\$65957[5] \rightarrow B_n\$65957[5]$, $T_n\$65957[5] \rightarrow T_n\$65957[5]$, $\mathcal{A}_n\$65957[5] \rightarrow \mathcal{A}_n\$65957[5]$ }

Knot " { $b_n\$66016[5] \rightarrow b_n\$66016[5]$, $t_n\$66016[5] \rightarrow t_n\$66016[5]$, $\alpha_n\$66016[5] \rightarrow \alpha_n\$66016[5]$,
 $b_n\$66016[k\$65562] \rightarrow b_n\$66016[k\$65562]$, $t_n\$66016[k\$65562] \rightarrow t_n\$66016[k\$65562]$, $\alpha_n\$66016[k\$65562] \rightarrow \alpha_n\$66016[k\$65562]$ }

Knot " { $B_n\$66016[5] \rightarrow B_n\$66016[5]$, $T_n\$66016[5] \rightarrow T_n\$66016[5]$, $\mathcal{A}_n\$66016[5] \rightarrow \mathcal{A}_n\$66016[5]$,
 $B_n\$66016[k\$65562] \rightarrow B_n\$66016[k\$65562]$, $T_n\$66016[k\$65562] \rightarrow T_n\$66016[k\$65562]$, $\mathcal{A}_n\$66016[k\$65562] \rightarrow \mathcal{A}_n\$66016[k\$65562]$ }

Knot " { $b_n\$66075[k\$65562] \rightarrow b_n\$66075[k\$65562]$, $t_n\$66075[k\$65562] \rightarrow t_n\$66075[k\$65562]$, $\alpha_n\$66075[k\$65562] \rightarrow \alpha_n\$66075[k\$65562]$, $b_n\$66075[18] \rightarrow b_n\$66075[18]$, $t_n\$66075[18] \rightarrow t_n\$66075[18]$, $\alpha_n\$66075[18] \rightarrow \alpha_n\$66075[18]$ }

Knot " { $B_n\$66075[k\$65562] \rightarrow B_n\$66075[k\$65562]$, $T_n\$66075[k\$65562] \rightarrow T_n\$66075[k\$65562]$, $\mathcal{A}_n\$66075[k\$65562] \rightarrow \mathcal{A}_n\$66075[k\$65562]$, $B_n\$66075[18] \rightarrow B_n\$66075[18]$, $T_n\$66075[18] \rightarrow T_n\$66075[18]$, $\mathcal{A}_n\$66075[18] \rightarrow \mathcal{A}_n\$66075[18]$ }

Knot " { $b_n\$66138[18] \rightarrow b_n\$66138[18]$, $t_n\$66138[18] \rightarrow t_n\$66138[18]$, $\alpha_n\$66138[18] \rightarrow \alpha_n\$66138[18]$, $b_n\$66138[k\$65562] \rightarrow b_n\$66138[k\$65562]$, $t_n\$66138[k\$65562] \rightarrow t_n\$66138[k\$65562]$, $\alpha_n\$66138[k\$65562] \rightarrow \alpha_n\$66138[k\$65562]$ }

Knot " { $B_n\$66138[18] \rightarrow B_n\$66138[18]$, $T_n\$66138[18] \rightarrow T_n\$66138[18]$, $\mathcal{A}_n\$66138[18] \rightarrow \mathcal{A}_n\$66138[18]$, $B_n\$66138[k\$65562] \rightarrow B_n\$66138[k\$65562]$, $T_n\$66138[k\$65562] \rightarrow T_n\$66138[k\$65562]$, $\mathcal{A}_n\$66138[k\$65562] \rightarrow \mathcal{A}_n\$66138[k\$65562]$ }

Knot " { $b_n\$66201[5] \rightarrow b_n\$66201[5]$, $t_n\$66201[5] \rightarrow t_n\$66201[5]$, $\alpha_n\$66201[5] \rightarrow \alpha_n\$66201[5]$, $b_n\$66201[6] \rightarrow b_n\$66201[6]$, $t_n\$66201[6] \rightarrow t_n\$66201[6]$, $\alpha_n\$66201[6] \rightarrow \alpha_n\$66201[6]$ }

Knot " { $B_n\$66201[5] \rightarrow B_n\$66201[5]$, $T_n\$66201[5] \rightarrow T_n\$66201[5]$, $\mathcal{A}_n\$66201[5] \rightarrow \mathcal{A}_n\$66201[5]$, $B_n\$66201[6] \rightarrow B_n\$66201[6]$, $T_n\$66201[6] \rightarrow T_n\$66201[6]$, $\mathcal{A}_n\$66201[6] \rightarrow \mathcal{A}_n\$66201[6]$ }

Knot " { $b_n\$66285[2] \rightarrow b_n\$66285[2]$, $t_n\$66285[2] \rightarrow t_n\$66285[2]$, $\alpha_n\$66285[2] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$66285[2]$, $b_n\$66285[k\$65562] \rightarrow b_n\$66285[k\$65562]$, $t_n\$66285[k\$65562] \rightarrow t_n\$66285[k\$65562]$, $\alpha_n\$66285[k\$65562] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$66285[k\$65562]$ }

Knot " { $B_n\$66285[2] \rightarrow B_n\$66285[2]$, $T_n\$66285[2] \rightarrow T_n\$66285[2]$, $\mathcal{A}_n\$66285[2] \rightarrow \frac{\mathcal{A}_n\$66285[2]}{T}$, $B_n\$66285[k\$65562] \rightarrow B_n\$66285[k\$65562]$, $T_n\$66285[k\$65562] \rightarrow T_n\$66285[k\$65562]$, $\mathcal{A}_n\$66285[k\$65562] \rightarrow T \mathcal{A}_n\$66285[k\$65562]$ }

Knot " { $b_n\$66352[k\$65562] \rightarrow b_n\$66352[k\$65562]$, $t_n\$66352[k\$65562] \rightarrow t_n\$66352[k\$65562]$, $\alpha_n\$66352[k\$65562] \rightarrow \alpha_n\$66352[k\$65562]$, $b_n\$66352[17] \rightarrow b_n\$66352[17]$, $t_n\$66352[17] \rightarrow t_n\$66352[17]$, $\alpha_n\$66352[17] \rightarrow \alpha_n\$66352[17]$ }

Knot " { $B_n\$66352[k\$65562] \rightarrow B_n\$66352[k\$65562]$, $T_n\$66352[k\$65562] \rightarrow T_n\$66352[k\$65562]$, $\mathcal{A}_n\$66352[k\$65562] \rightarrow \mathcal{A}_n\$66352[k\$65562]$, $B_n\$66352[17] \rightarrow B_n\$66352[17]$, $T_n\$66352[17] \rightarrow T_n\$66352[17]$, $\mathcal{A}_n\$66352[17] \rightarrow \mathcal{A}_n\$66352[17]$ }

Knot " { $b_n\$66411[17] \rightarrow b_n\$66411[17]$, $t_n\$66411[17] \rightarrow t_n\$66411[17]$, $\alpha_n\$66411[17] \rightarrow \alpha_n\$66411[17]$, $b_n\$66411[k\$65562] \rightarrow b_n\$66411[k\$65562]$, $t_n\$66411[k\$65562] \rightarrow t_n\$66411[k\$65562]$, $\alpha_n\$66411[k\$65562] \rightarrow \alpha_n\$66411[k\$65562]$ }

Knot " { $B_n\$66411[17] \rightarrow B_n\$66411[17]$, $T_n\$66411[17] \rightarrow T_n\$66411[17]$, $\mathcal{A}_n\$66411[17] \rightarrow \mathcal{A}_n\$66411[17]$, $B_n\$66411[k\$65562] \rightarrow B_n\$66411[k\$65562]$, $T_n\$66411[k\$65562] \rightarrow T_n\$66411[k\$65562]$, $\mathcal{A}_n\$66411[k\$65562] \rightarrow \mathcal{A}_n\$66411[k\$65562]$ }

Knot " { $b_n\$66470[k\$65562] \rightarrow b_n\$66470[k\$65562]$, $t_n\$66470[k\$65562] \rightarrow t_n\$66470[k\$65562]$, $\alpha_n\$66470[k\$65562] \rightarrow \alpha_n\$66470[k\$65562]$, $b_n\$66470[2] \rightarrow b_n\$66470[2]$, $t_n\$66470[2] \rightarrow t_n\$66470[2]$, $\alpha_n\$66470[2] \rightarrow \alpha_n\$66470[2]$ }

Knot " { $B_n\$66470[k\$65562] \rightarrow B_n\$66470[k\$65562]$, $T_n\$66470[k\$65562] \rightarrow T_n\$66470[k\$65562]$, $\mathcal{A}_n\$66470[k\$65562] \rightarrow \mathcal{A}_n\$66470[k\$65562]$, $B_n\$66470[2] \rightarrow B_n\$66470[2]$, $T_n\$66470[2] \rightarrow T_n\$66470[2]$, $\mathcal{A}_n\$66470[2] \rightarrow \mathcal{A}_n\$66470[2]$ }

Knot " { $b_n\$66533[2] \rightarrow b_n\$66533[2]$, $t_n\$66533[2] \rightarrow t_n\$66533[2]$, $\alpha_n\$66533[2] \rightarrow \alpha_n\$66533[2]$, $b_n\$66533[k\$65562] \rightarrow b_n\$66533[k\$65562]$, $t_n\$66533[k\$65562] \rightarrow t_n\$66533[k\$65562]$, $\alpha_n\$66533[k\$65562] \rightarrow \alpha_n\$66533[k\$65562]$ }

Knot " { $B_n\$66533[2] \rightarrow B_n\$66533[2]$, $T_n\$66533[2] \rightarrow T_n\$66533[2]$, $\mathcal{A}_n\$66533[2] \rightarrow \mathcal{A}_n\$66533[2]$, $B_n\$66533[k\$65562] \rightarrow B_n\$66533[k\$65562]$, $T_n\$66533[k\$65562] \rightarrow T_n\$66533[k\$65562]$, $\mathcal{A}_n\$66533[k\$65562] \rightarrow \mathcal{A}_n\$66533[k\$65562]$ }

Knot
 " $\{ b_{n\$66596[17]} \rightarrow b_{n\$66596[17]}, t_{n\$66596[17]} \rightarrow t_{n\$66596[17]}, \alpha_{n\$66596[17]} \rightarrow \alpha_{n\$66596[17]},$
 $b_{n\$66596[18]} \rightarrow b_{n\$66596[18]}, t_{n\$66596[18]} \rightarrow t_{n\$66596[18]}, \alpha_{n\$66596[18]} \rightarrow \alpha_{n\$66596[18]} \}$

Knot
 " $\{ B_{n\$66596[17]} \rightarrow B_{n\$66596[17]}, T_{n\$66596[17]} \rightarrow T_{n\$66596[17]}, \mathcal{A}_{n\$66596[17]} \rightarrow \mathcal{A}_{n\$66596[17]},$
 $B_{n\$66596[18]} \rightarrow B_{n\$66596[18]}, T_{n\$66596[18]} \rightarrow T_{n\$66596[18]}, \mathcal{A}_{n\$66596[18]} \rightarrow \mathcal{A}_{n\$66596[18]} \}$

Knot
 " $\{ b_{n\$66697[0]} \rightarrow b_{n\$66697[0]}, t_{n\$66697[0]} \rightarrow t_{n\$66697[0]}, \alpha_{n\$66697[0]} \rightarrow \alpha_{n\$66697[0]},$
 $b_{n\$66697[2]} \rightarrow b_{n\$66697[2]}, t_{n\$66697[2]} \rightarrow t_{n\$66697[2]}, \alpha_{n\$66697[2]} \rightarrow \alpha_{n\$66697[2]} \}$

Knot
 " $\{ B_{n\$66697[0]} \rightarrow B_{n\$66697[0]}, T_{n\$66697[0]} \rightarrow T_{n\$66697[0]}, \mathcal{A}_{n\$66697[0]} \rightarrow \mathcal{A}_{n\$66697[0]},$
 $B_{n\$66697[2]} \rightarrow B_{n\$66697[2]}, T_{n\$66697[2]} \rightarrow T_{n\$66697[2]}, \mathcal{A}_{n\$66697[2]} \rightarrow \mathcal{A}_{n\$66697[2]} \}$

Knot
 " $\left\{ b_{n\$66803[i]} \rightarrow b_{n\$66803[i]} - \frac{t_i}{\gamma}, t_{n\$66803[i]} \rightarrow t_{n\$66803[i]}, \alpha_{n\$66803[i]} \rightarrow \alpha_{n\$66803[i]}, \right.$
 $b_{n\$66803[j]} \rightarrow b_{n\$66803[j]} - \frac{t_j}{\gamma}, t_{n\$66803[j]} \rightarrow t_{n\$66803[j]}, \alpha_{n\$66803[j]} \rightarrow -\hbar \left(b_{n\$66803[i]} - \frac{t_i}{\gamma} \right) + \alpha_{n\$66803[j]} \left. \right\}$

Knot
 " $\{ B_{n\$66803[i]} \rightarrow B_{n\$66803[i]} T_i, T_{n\$66803[i]} \rightarrow T_{n\$66803[i]}, \mathcal{A}_{n\$66803[i]} \rightarrow \mathcal{A}_{n\$66803[i]},$
 $B_{n\$66803[j]} \rightarrow B_{n\$66803[j]} T_j, T_{n\$66803[j]} \rightarrow T_{n\$66803[j]}, \mathcal{A}_{n\$66803[j]} \rightarrow B_{n\$66803[i]} T_i \mathcal{A}_{n\$66803[j]} \}$

Knot
 " $\left\{ b_{n\$66830[i]} \rightarrow b_{n\$66830[i]} - \frac{t_i}{\gamma}, t_{n\$66830[i]} \rightarrow t_{n\$66830[i]}, \alpha_{n\$66830[i]} \rightarrow \hbar \left(b_{n\$66830[i]} - \frac{t_i}{\gamma} \right) + \alpha_{n\$66830[i]} \right\}$

Knot
 " $\left\{ B_{n\$66830[i]} \rightarrow B_{n\$66830[i]} T_i, T_{n\$66830[i]} \rightarrow T_{n\$66830[i]}, \mathcal{A}_{n\$66830[i]} \rightarrow \frac{\mathcal{A}_{n\$66830[i]}}{B_{n\$66830[i]} T_i} \right\}$

Knot
 " $\left\{ b_{n\$66857[14]} \rightarrow b_{n\$66857[14]}, t_{n\$66857[14]} \rightarrow t_{n\$66857[14]}, \alpha_{n\$66857[14]} \rightarrow \frac{t \hbar}{\gamma} + \alpha_{n\$66857[14]}, \right.$
 $b_{n\$66857[k\$65562]} \rightarrow b_{n\$66857[k\$65562]}, t_{n\$66857[k\$65562]} \rightarrow t_{n\$66857[k\$65562]}, \alpha_{n\$66857[k\$65562]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$66857[k\$65562]} \left. \right\}$

Knot
 " $\left\{ B_{n\$66857[14]} \rightarrow B_{n\$66857[14]}, T_{n\$66857[14]} \rightarrow T_{n\$66857[14]}, \mathcal{A}_{n\$66857[14]} \rightarrow T \mathcal{A}_{n\$66857[14]}, \right.$
 $B_{n\$66857[k\$65562]} \rightarrow B_{n\$66857[k\$65562]}, T_{n\$66857[k\$65562]} \rightarrow T_{n\$66857[k\$65562]}, \mathcal{A}_{n\$66857[k\$65562]} \rightarrow \frac{\mathcal{A}_{n\$66857[k\$65562]}}{T} \left. \right\}$

Knot
 " $\{ b_{n\$66926[k\$65562]} \rightarrow b_{n\$66926[k\$65562]}, t_{n\$66926[k\$65562]} \rightarrow t_{n\$66926[k\$65562]},$
 $\alpha_{n\$66926[k\$65562]} \rightarrow \alpha_{n\$66926[k\$65562]}, b_{n\$66926[7]} \rightarrow b_{n\$66926[7]}, t_{n\$66926[7]} \rightarrow t_{n\$66926[7]}, \alpha_{n\$66926[7]} \rightarrow \alpha_{n\$66926[7]} \}$

Knot
 " $\{ B_{n\$66926[k\$65562]} \rightarrow B_{n\$66926[k\$65562]}, T_{n\$66926[k\$65562]} \rightarrow T_{n\$66926[k\$65562]},$
 $\mathcal{A}_{n\$66926[k\$65562]} \rightarrow \mathcal{A}_{n\$66926[k\$65562]}, B_{n\$66926[7]} \rightarrow B_{n\$66926[7]}, T_{n\$66926[7]} \rightarrow T_{n\$66926[7]}, \mathcal{A}_{n\$66926[7]} \rightarrow \mathcal{A}_{n\$66926[7]} \}$

Knot
 " $\{ b_{n\$66995[7]} \rightarrow b_{n\$66995[7]}, t_{n\$66995[7]} \rightarrow t_{n\$66995[7]}, \alpha_{n\$66995[7]} \rightarrow \alpha_{n\$66995[7]},$
 $b_{n\$66995[k\$65562]} \rightarrow b_{n\$66995[k\$65562]}, t_{n\$66995[k\$65562]} \rightarrow t_{n\$66995[k\$65562]}, \alpha_{n\$66995[k\$65562]} \rightarrow \alpha_{n\$66995[k\$65562]} \}$

Knot
 " $\{ B_{n\$66995[7]} \rightarrow B_{n\$66995[7]}, T_{n\$66995[7]} \rightarrow T_{n\$66995[7]}, \mathcal{A}_{n\$66995[7]} \rightarrow \mathcal{A}_{n\$66995[7]},$
 $B_{n\$66995[k\$65562]} \rightarrow B_{n\$66995[k\$65562]}, T_{n\$66995[k\$65562]} \rightarrow T_{n\$66995[k\$65562]}, \mathcal{A}_{n\$66995[k\$65562]} \rightarrow \mathcal{A}_{n\$66995[k\$65562]} \}$

Knot
 " $\left\{ b_{n\$67065[i]} \rightarrow b_{n\$67065[i]} - \frac{t_i}{\gamma}, t_{n\$67065[i]} \rightarrow t_{n\$67065[i]}, \alpha_{n\$67065[i]} \rightarrow \alpha_{n\$67065[i]} \right\}$

Knot " { $B_n\$67065[i] \rightarrow B_n\$67065[i] T_i, T_n\$67065[i] \rightarrow T_n\$67065[i], \mathcal{A}_n\$67065[i] \rightarrow \mathcal{A}_n\$67065[i] \}$

Knot " { $b_n\$67088[k\$65562] \rightarrow b_n\$67088[k\$65562], t_n\$67088[k\$65562] \rightarrow t_n\$67088[k\$65562], \alpha_n\$67088[k\$65562] \rightarrow \alpha_n\$67088[k\$65562],$
 $b_n\$67088[j\$67064] \rightarrow b_n\$67088[j\$67064], t_n\$67088[j\$67064] \rightarrow t_n\$67088[j\$67064], \alpha_n\$67088[j\$67064] \rightarrow \alpha_n\$67088[j\$67064] \}$

Knot " { $B_n\$67088[k\$65562] \rightarrow B_n\$67088[k\$65562], T_n\$67088[k\$65562] \rightarrow T_n\$67088[k\$65562], \mathcal{A}_n\$67088[k\$65562] \rightarrow \mathcal{A}_n\$67088[k\$65562],$
 $B_n\$67088[j\$67064] \rightarrow B_n\$67088[j\$67064], T_n\$67088[j\$67064] \rightarrow T_n\$67088[j\$67064], \mathcal{A}_n\$67088[j\$67064] \rightarrow \mathcal{A}_n\$67088[j\$67064] \}$

Knot " { $b_n\$67133[k\$65562] \rightarrow b_n\$67133[k\$65562], t_n\$67133[k\$65562] \rightarrow t_n\$67133[k\$65562], \alpha_n\$67133[k\$65562] \rightarrow \alpha_n\$67133[k\$65562],$
 $b_n\$67133[14] \rightarrow b_n\$67133[14], t_n\$67133[14] \rightarrow t_n\$67133[14], \alpha_n\$67133[14] \rightarrow \alpha_n\$67133[14] \}$

Knot " { $B_n\$67133[k\$65562] \rightarrow B_n\$67133[k\$65562], T_n\$67133[k\$65562] \rightarrow T_n\$67133[k\$65562], \mathcal{A}_n\$67133[k\$65562] \rightarrow \mathcal{A}_n\$67133[k\$65562],$
 $B_n\$67133[14] \rightarrow B_n\$67133[14], T_n\$67133[14] \rightarrow T_n\$67133[14], \mathcal{A}_n\$67133[14] \rightarrow \mathcal{A}_n\$67133[14] \}$

Knot " { $b_n\$67182[14] \rightarrow b_n\$67182[14], t_n\$67182[14] \rightarrow t_n\$67182[14], \alpha_n\$67182[14] \rightarrow \alpha_n\$67182[14],$
 $b_n\$67182[k\$65562] \rightarrow b_n\$67182[k\$65562], t_n\$67182[k\$65562] \rightarrow t_n\$67182[k\$65562], \alpha_n\$67182[k\$65562] \rightarrow \alpha_n\$67182[k\$65562] \}$

Knot " { $B_n\$67182[14] \rightarrow B_n\$67182[14], T_n\$67182[14] \rightarrow T_n\$67182[14], \mathcal{A}_n\$67182[14] \rightarrow \mathcal{A}_n\$67182[14],$
 $B_n\$67182[k\$65562] \rightarrow B_n\$67182[k\$65562], T_n\$67182[k\$65562] \rightarrow T_n\$67182[k\$65562], \mathcal{A}_n\$67182[k\$65562] \rightarrow \mathcal{A}_n\$67182[k\$65562] \}$

Knot " { $b_n\$67233[5] \rightarrow b_n\$67233[5], t_n\$67233[5] \rightarrow t_n\$67233[5], \alpha_n\$67233[5] \rightarrow \alpha_n\$67233[5],$
 $b_n\$67233[7] \rightarrow b_n\$67233[7], t_n\$67233[7] \rightarrow t_n\$67233[7], \alpha_n\$67233[7] \rightarrow \alpha_n\$67233[7] \}$

Knot " { $B_n\$67233[5] \rightarrow B_n\$67233[5], T_n\$67233[5] \rightarrow T_n\$67233[5], \mathcal{A}_n\$67233[5] \rightarrow \mathcal{A}_n\$67233[5],$
 $B_n\$67233[7] \rightarrow B_n\$67233[7], T_n\$67233[7] \rightarrow T_n\$67233[7], \mathcal{A}_n\$67233[7] \rightarrow \mathcal{A}_n\$67233[7] \}$

Knot " { $b_n\$67361[8] \rightarrow b_n\$67361[8], t_n\$67361[8] \rightarrow t_n\$67361[8], \alpha_n\$67361[8] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$67361[8],$
 $b_n\$67361[k\$65562] \rightarrow b_n\$67361[k\$65562], t_n\$67361[k\$65562] \rightarrow t_n\$67361[k\$65562], \alpha_n\$67361[k\$65562] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$67361[k\$65562] \}$

Knot " { $B_n\$67361[8] \rightarrow B_n\$67361[8], T_n\$67361[8] \rightarrow T_n\$67361[8], \mathcal{A}_n\$67361[8] \rightarrow T \mathcal{A}_n\$67361[8],$
 $B_n\$67361[k\$65562] \rightarrow B_n\$67361[k\$65562], T_n\$67361[k\$65562] \rightarrow T_n\$67361[k\$65562], \mathcal{A}_n\$67361[k\$65562] \rightarrow \frac{\mathcal{A}_n\$67361[k\$65562]}{T} \}$

Knot " { $b_n\$67430[k\$65562] \rightarrow b_n\$67430[k\$65562], t_n\$67430[k\$65562] \rightarrow t_n\$67430[k\$65562], \alpha_n\$67430[k\$65562] \rightarrow \alpha_n\$67430[k\$65562],$
 $b_n\$67430[15] \rightarrow b_n\$67430[15], t_n\$67430[15] \rightarrow t_n\$67430[15], \alpha_n\$67430[15] \rightarrow \alpha_n\$67430[15] \}$

Knot " { $B_n\$67430[k\$65562] \rightarrow B_n\$67430[k\$65562], T_n\$67430[k\$65562] \rightarrow T_n\$67430[k\$65562], \mathcal{A}_n\$67430[k\$65562] \rightarrow \mathcal{A}_n\$67430[k\$65562],$
 $B_n\$67430[15] \rightarrow B_n\$67430[15], T_n\$67430[15] \rightarrow T_n\$67430[15], \mathcal{A}_n\$67430[15] \rightarrow \mathcal{A}_n\$67430[15] \}$

Knot " { $b_n\$67499[15] \rightarrow b_n\$67499[15], t_n\$67499[15] \rightarrow t_n\$67499[15], \alpha_n\$67499[15] \rightarrow \alpha_n\$67499[15],$
 $b_n\$67499[k\$65562] \rightarrow b_n\$67499[k\$65562], t_n\$67499[k\$65562] \rightarrow t_n\$67499[k\$65562], \alpha_n\$67499[k\$65562] \rightarrow \alpha_n\$67499[k\$65562] \}$

Knot " { $B_n\$67499[15] \rightarrow B_n\$67499[15], T_n\$67499[15] \rightarrow T_n\$67499[15], \mathcal{A}_n\$67499[15] \rightarrow \mathcal{A}_n\$67499[15],$
 $B_n\$67499[k\$65562] \rightarrow B_n\$67499[k\$65562], T_n\$67499[k\$65562] \rightarrow T_n\$67499[k\$65562], \mathcal{A}_n\$67499[k\$65562] \rightarrow \mathcal{A}_n\$67499[k\$65562] \}$

Knot " { $b_n\$67568[k\$65562] \rightarrow b_n\$67568[k\$65562], t_n\$67568[k\$65562] \rightarrow t_n\$67568[k\$65562],$
 $\alpha_n\$67568[k\$65562] \rightarrow \alpha_n\$67568[k\$65562], b_n\$67568[8] \rightarrow b_n\$67568[8], t_n\$67568[8] \rightarrow t_n\$67568[8], \alpha_n\$67568[8] \rightarrow \alpha_n\$67568[8] \}$

Knot " { $B_n\$67568[k\$65562] \rightarrow B_n\$67568[k\$65562]$, $T_n\$67568[k\$65562] \rightarrow T_n\$67568[k\$65562]$,
 $\mathcal{A}_n\$67568[k\$65562] \rightarrow \mathcal{A}_n\$67568[k\$65562]$, $B_n\$67568[8] \rightarrow B_n\$67568[8]$, $T_n\$67568[8] \rightarrow T_n\$67568[8]$, $\mathcal{A}_n\$67568[8] \rightarrow \mathcal{A}_n\$67568[8]$ }

Knot " { $b_n\$67639[8] \rightarrow b_n\$67639[8]$, $t_n\$67639[8] \rightarrow t_n\$67639[8]$, $\alpha_n\$67639[8] \rightarrow \alpha_n\$67639[8]$,
 $b_n\$67639[k\$65562] \rightarrow b_n\$67639[k\$65562]$, $t_n\$67639[k\$65562] \rightarrow t_n\$67639[k\$65562]$, $\alpha_n\$67639[k\$65562] \rightarrow \alpha_n\$67639[k\$65562]$ }

Knot " { $B_n\$67639[8] \rightarrow B_n\$67639[8]$, $T_n\$67639[8] \rightarrow T_n\$67639[8]$, $\mathcal{A}_n\$67639[8] \rightarrow \mathcal{A}_n\$67639[8]$,
 $B_n\$67639[k\$65562] \rightarrow B_n\$67639[k\$65562]$, $T_n\$67639[k\$65562] \rightarrow T_n\$67639[k\$65562]$, $\mathcal{A}_n\$67639[k\$65562] \rightarrow \mathcal{A}_n\$67639[k\$65562]$ }

Knot " { $b_n\$67710[14] \rightarrow b_n\$67710[14]$, $t_n\$67710[14] \rightarrow t_n\$67710[14]$, $\alpha_n\$67710[14] \rightarrow \alpha_n\$67710[14]$,
 $b_n\$67710[15] \rightarrow b_n\$67710[15]$, $t_n\$67710[15] \rightarrow t_n\$67710[15]$, $\alpha_n\$67710[15] \rightarrow \alpha_n\$67710[15]$ }

Knot " { $B_n\$67710[14] \rightarrow B_n\$67710[14]$, $T_n\$67710[14] \rightarrow T_n\$67710[14]$, $\mathcal{A}_n\$67710[14] \rightarrow \mathcal{A}_n\$67710[14]$,
 $B_n\$67710[15] \rightarrow B_n\$67710[15]$, $T_n\$67710[15] \rightarrow T_n\$67710[15]$, $\mathcal{A}_n\$67710[15] \rightarrow \mathcal{A}_n\$67710[15]$ }

Knot " { $b_n\$67919[5] \rightarrow b_n\$67919[5]$, $t_n\$67919[5] \rightarrow t_n\$67919[5]$, $\alpha_n\$67919[5] \rightarrow \alpha_n\$67919[5]$,
 $b_n\$67919[8] \rightarrow b_n\$67919[8]$, $t_n\$67919[8] \rightarrow t_n\$67919[8]$, $\alpha_n\$67919[8] \rightarrow \alpha_n\$67919[8]$ }

Knot " { $B_n\$67919[5] \rightarrow B_n\$67919[5]$, $T_n\$67919[5] \rightarrow T_n\$67919[5]$, $\mathcal{A}_n\$67919[5] \rightarrow \mathcal{A}_n\$67919[5]$,
 $B_n\$67919[8] \rightarrow B_n\$67919[8]$, $T_n\$67919[8] \rightarrow T_n\$67919[8]$, $\mathcal{A}_n\$67919[8] \rightarrow \mathcal{A}_n\$67919[8]$ }

Knot " { $b_n\$68291[16] \rightarrow b_n\$68291[16]$, $t_n\$68291[16] \rightarrow t_n\$68291[16]$, $\alpha_n\$68291[16] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$68291[16]$,
 $b_n\$68291[k\$65562] \rightarrow b_n\$68291[k\$65562]$, $t_n\$68291[k\$65562] \rightarrow t_n\$68291[k\$65562]$, $\alpha_n\$68291[k\$65562] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$68291[k\$65562]$ }

Knot " { $B_n\$68291[16] \rightarrow B_n\$68291[16]$, $T_n\$68291[16] \rightarrow T_n\$68291[16]$, $\mathcal{A}_n\$68291[16] \rightarrow T \mathcal{A}_n\$68291[16]$,
 $B_n\$68291[k\$65562] \rightarrow B_n\$68291[k\$65562]$, $T_n\$68291[k\$65562] \rightarrow T_n\$68291[k\$65562]$, $\mathcal{A}_n\$68291[k\$65562] \rightarrow \frac{\mathcal{A}_n\$68291[k\$65562]}{T}$ }

Knot " { $b_n\$68360[k\$65562] \rightarrow b_n\$68360[k\$65562]$, $t_n\$68360[k\$65562] \rightarrow t_n\$68360[k\$65562]$,
 $\alpha_n\$68360[k\$65562] \rightarrow \alpha_n\$68360[k\$65562]$, $b_n\$68360[9] \rightarrow b_n\$68360[9]$, $t_n\$68360[9] \rightarrow t_n\$68360[9]$, $\alpha_n\$68360[9] \rightarrow \alpha_n\$68360[9]$ }

Knot " { $B_n\$68360[k\$65562] \rightarrow B_n\$68360[k\$65562]$, $T_n\$68360[k\$65562] \rightarrow T_n\$68360[k\$65562]$,
 $\mathcal{A}_n\$68360[k\$65562] \rightarrow \mathcal{A}_n\$68360[k\$65562]$, $B_n\$68360[9] \rightarrow B_n\$68360[9]$, $T_n\$68360[9] \rightarrow T_n\$68360[9]$, $\mathcal{A}_n\$68360[9] \rightarrow \mathcal{A}_n\$68360[9]$ }

Knot " { $b_n\$68429[9] \rightarrow b_n\$68429[9]$, $t_n\$68429[9] \rightarrow t_n\$68429[9]$, $\alpha_n\$68429[9] \rightarrow \alpha_n\$68429[9]$,
 $b_n\$68429[k\$65562] \rightarrow b_n\$68429[k\$65562]$, $t_n\$68429[k\$65562] \rightarrow t_n\$68429[k\$65562]$, $\alpha_n\$68429[k\$65562] \rightarrow \alpha_n\$68429[k\$65562]$ }

Knot " { $B_n\$68429[9] \rightarrow B_n\$68429[9]$, $T_n\$68429[9] \rightarrow T_n\$68429[9]$, $\mathcal{A}_n\$68429[9] \rightarrow \mathcal{A}_n\$68429[9]$,
 $B_n\$68429[k\$65562] \rightarrow B_n\$68429[k\$65562]$, $T_n\$68429[k\$65562] \rightarrow T_n\$68429[k\$65562]$, $\mathcal{A}_n\$68429[k\$65562] \rightarrow \mathcal{A}_n\$68429[k\$65562]$ }

Knot " { $b_n\$68498[k\$65562] \rightarrow b_n\$68498[k\$65562]$, $t_n\$68498[k\$65562] \rightarrow t_n\$68498[k\$65562]$, $\alpha_n\$68498[k\$65562] \rightarrow \alpha_n\$68498[k\$65562]$,
 $b_n\$68498[16] \rightarrow b_n\$68498[16]$, $t_n\$68498[16] \rightarrow t_n\$68498[16]$, $\alpha_n\$68498[16] \rightarrow \alpha_n\$68498[16]$ }

Knot " { $B_n\$68498[k\$65562] \rightarrow B_n\$68498[k\$65562]$, $T_n\$68498[k\$65562] \rightarrow T_n\$68498[k\$65562]$, $\mathcal{A}_n\$68498[k\$65562] \rightarrow \mathcal{A}_n\$68498[k\$65562]$,
 $B_n\$68498[16] \rightarrow B_n\$68498[16]$, $T_n\$68498[16] \rightarrow T_n\$68498[16]$, $\mathcal{A}_n\$68498[16] \rightarrow \mathcal{A}_n\$68498[16]$ }

Knot " { $b_n\$68569[16] \rightarrow b_n\$68569[16]$, $t_n\$68569[16] \rightarrow t_n\$68569[16]$, $\alpha_n\$68569[16] \rightarrow \alpha_n\$68569[16]$,
 $b_n\$68569[k\$65562] \rightarrow b_n\$68569[k\$65562]$, $t_n\$68569[k\$65562] \rightarrow t_n\$68569[k\$65562]$, $\alpha_n\$68569[k\$65562] \rightarrow \alpha_n\$68569[k\$65562]$ }

Knot " { $B_n\$68569[16] \rightarrow B_n\$68569[16]$, $T_n\$68569[16] \rightarrow T_n\$68569[16]$, $\mathcal{A}_n\$68569[16] \rightarrow \mathcal{A}_n\$68569[16]$,
 $B_n\$68569[k\$65562] \rightarrow B_n\$68569[k\$65562]$, $T_n\$68569[k\$65562] \rightarrow T_n\$68569[k\$65562]$, $\mathcal{A}_n\$68569[k\$65562] \rightarrow \mathcal{A}_n\$68569[k\$65562]$ }

Knot " { $b_n\$68640[5] \rightarrow b_n\$68640[5]$, $t_n\$68640[5] \rightarrow t_n\$68640[5]$, $\alpha_n\$68640[5] \rightarrow \alpha_n\$68640[5]$,
 $b_n\$68640[9] \rightarrow b_n\$68640[9]$, $t_n\$68640[9] \rightarrow t_n\$68640[9]$, $\alpha_n\$68640[9] \rightarrow \alpha_n\$68640[9]$ }

Knot " { $B_n\$68640[5] \rightarrow B_n\$68640[5]$, $T_n\$68640[5] \rightarrow T_n\$68640[5]$, $\mathcal{A}_n\$68640[5] \rightarrow \mathcal{A}_n\$68640[5]$,
 $B_n\$68640[9] \rightarrow B_n\$68640[9]$, $T_n\$68640[9] \rightarrow T_n\$68640[9]$, $\mathcal{A}_n\$68640[9] \rightarrow \mathcal{A}_n\$68640[9]$ }

Knot " { $b_n\$68813[16] \rightarrow b_n\$68813[16]$, $t_n\$68813[16] \rightarrow t_n\$68813[16]$, $\alpha_n\$68813[16] \rightarrow \alpha_n\$68813[16]$,
 $b_n\$68813[17] \rightarrow b_n\$68813[17]$, $t_n\$68813[17] \rightarrow t_n\$68813[17]$, $\alpha_n\$68813[17] \rightarrow \alpha_n\$68813[17]$ }

Knot " { $B_n\$68813[16] \rightarrow B_n\$68813[16]$, $T_n\$68813[16] \rightarrow T_n\$68813[16]$, $\mathcal{A}_n\$68813[16] \rightarrow \mathcal{A}_n\$68813[16]$,
 $B_n\$68813[17] \rightarrow B_n\$68813[17]$, $T_n\$68813[17] \rightarrow T_n\$68813[17]$, $\mathcal{A}_n\$68813[17] \rightarrow \mathcal{A}_n\$68813[17]$ }

Knot " { $b_n\$69216[14] \rightarrow b_n\$69216[14]$, $t_n\$69216[14] \rightarrow t_n\$69216[14]$, $\alpha_n\$69216[14] \rightarrow \alpha_n\$69216[14]$,
 $b_n\$69216[16] \rightarrow b_n\$69216[16]$, $t_n\$69216[16] \rightarrow t_n\$69216[16]$, $\alpha_n\$69216[16] \rightarrow \alpha_n\$69216[16]$ }

Knot " { $B_n\$69216[14] \rightarrow B_n\$69216[14]$, $T_n\$69216[14] \rightarrow T_n\$69216[14]$, $\mathcal{A}_n\$69216[14] \rightarrow \mathcal{A}_n\$69216[14]$,
 $B_n\$69216[16] \rightarrow B_n\$69216[16]$, $T_n\$69216[16] \rightarrow T_n\$69216[16]$, $\mathcal{A}_n\$69216[16] \rightarrow \mathcal{A}_n\$69216[16]$ }

Knot " { $b_n\$69916[10] \rightarrow b_n\$69916[10]$, $t_n\$69916[10] \rightarrow t_n\$69916[10]$, $\alpha_n\$69916[10] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$69916[10]$,
 $b_n\$69916[k\$65562] \rightarrow b_n\$69916[k\$65562]$, $t_n\$69916[k\$65562] \rightarrow t_n\$69916[k\$65562]$, $\alpha_n\$69916[k\$65562] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$69916[k\$65562]$ }

Knot " { $B_n\$69916[10] \rightarrow B_n\$69916[10]$, $T_n\$69916[10] \rightarrow T_n\$69916[10]$, $\mathcal{A}_n\$69916[10] \rightarrow T \mathcal{A}_n\$69916[10]$,
 $B_n\$69916[k\$65562] \rightarrow B_n\$69916[k\$65562]$, $T_n\$69916[k\$65562] \rightarrow T_n\$69916[k\$65562]$, $\mathcal{A}_n\$69916[k\$65562] \rightarrow \frac{\mathcal{A}_n\$69916[k\$65562]}{T}$ }

Knot " { $b_n\$69985[k\$65562] \rightarrow b_n\$69985[k\$65562]$, $t_n\$69985[k\$65562] \rightarrow t_n\$69985[k\$65562]$,
 $\alpha_n\$69985[k\$65562] \rightarrow \alpha_n\$69985[k\$65562]$, $b_n\$69985[3] \rightarrow b_n\$69985[3]$, $t_n\$69985[3] \rightarrow t_n\$69985[3]$, $\alpha_n\$69985[3] \rightarrow \alpha_n\$69985[3]$ }

Knot " { $B_n\$69985[k\$65562] \rightarrow B_n\$69985[k\$65562]$, $T_n\$69985[k\$65562] \rightarrow T_n\$69985[k\$65562]$,
 $\mathcal{A}_n\$69985[k\$65562] \rightarrow \mathcal{A}_n\$69985[k\$65562]$, $B_n\$69985[3] \rightarrow B_n\$69985[3]$, $T_n\$69985[3] \rightarrow T_n\$69985[3]$, $\mathcal{A}_n\$69985[3] \rightarrow \mathcal{A}_n\$69985[3]$ }

Knot " { $b_n\$70054[3] \rightarrow b_n\$70054[3]$, $t_n\$70054[3] \rightarrow t_n\$70054[3]$, $\alpha_n\$70054[3] \rightarrow \alpha_n\$70054[3]$,
 $b_n\$70054[k\$65562] \rightarrow b_n\$70054[k\$65562]$, $t_n\$70054[k\$65562] \rightarrow t_n\$70054[k\$65562]$, $\alpha_n\$70054[k\$65562] \rightarrow \alpha_n\$70054[k\$65562]$ }

Knot " { $B_n\$70054[3] \rightarrow B_n\$70054[3]$, $T_n\$70054[3] \rightarrow T_n\$70054[3]$, $\mathcal{A}_n\$70054[3] \rightarrow \mathcal{A}_n\$70054[3]$,
 $B_n\$70054[k\$65562] \rightarrow B_n\$70054[k\$65562]$, $T_n\$70054[k\$65562] \rightarrow T_n\$70054[k\$65562]$, $\mathcal{A}_n\$70054[k\$65562] \rightarrow \mathcal{A}_n\$70054[k\$65562]$ }

Knot " { $b_n\$70123[k\$65562] \rightarrow b_n\$70123[k\$65562]$, $t_n\$70123[k\$65562] \rightarrow t_n\$70123[k\$65562]$, $\alpha_n\$70123[k\$65562] \rightarrow \alpha_n\$70123[k\$65562]$,
 $b_n\$70123[10] \rightarrow b_n\$70123[10]$, $t_n\$70123[10] \rightarrow t_n\$70123[10]$, $\alpha_n\$70123[10] \rightarrow \alpha_n\$70123[10]$ }

Knot " { $B_n\$70123[k\$65562] \rightarrow B_n\$70123[k\$65562]$, $T_n\$70123[k\$65562] \rightarrow T_n\$70123[k\$65562]$, $\mathcal{A}_n\$70123[k\$65562] \rightarrow \mathcal{A}_n\$70123[k\$65562]$,
 $B_n\$70123[10] \rightarrow B_n\$70123[10]$, $T_n\$70123[10] \rightarrow T_n\$70123[10]$, $\mathcal{A}_n\$70123[10] \rightarrow \mathcal{A}_n\$70123[10]$ }

Knot " { $b_n\$70194[10] \rightarrow b_n\$70194[10]$, $t_n\$70194[10] \rightarrow t_n\$70194[10]$, $\alpha_n\$70194[10] \rightarrow \alpha_n\$70194[10]$,
 $b_n\$70194[k\$65562] \rightarrow b_n\$70194[k\$65562]$, $t_n\$70194[k\$65562] \rightarrow t_n\$70194[k\$65562]$, $\alpha_n\$70194[k\$65562] \rightarrow \alpha_n\$70194[k\$65562]$ }

Knot " { $B_{n\$70194[10]} \rightarrow B_{n\$70194[10]}$, $T_{n\$70194[10]} \rightarrow T_{n\$70194[10]}$, $\mathcal{A}_{n\$70194[10]} \rightarrow \mathcal{A}_{n\$70194[10]}$,
 $B_{n\$70194[k\$65562]} \rightarrow B_{n\$70194[k\$65562]}$, $T_{n\$70194[k\$65562]} \rightarrow T_{n\$70194[k\$65562]}$, $\mathcal{A}_{n\$70194[k\$65562]} \rightarrow \mathcal{A}_{n\$70194[k\$65562]}$ }

Knot " { $b_{n\$70265[0]} \rightarrow b_{n\$70265[0]}$, $t_{n\$70265[0]} \rightarrow t_{n\$70265[0]}$, $\alpha_{n\$70265[0]} \rightarrow \alpha_{n\$70265[0]}$,
 $b_{n\$70265[3]} \rightarrow b_{n\$70265[3]}$, $t_{n\$70265[3]} \rightarrow t_{n\$70265[3]}$, $\alpha_{n\$70265[3]} \rightarrow \alpha_{n\$70265[3]}$ }

Knot " { $B_{n\$70265[0]} \rightarrow B_{n\$70265[0]}$, $T_{n\$70265[0]} \rightarrow T_{n\$70265[0]}$, $\mathcal{A}_{n\$70265[0]} \rightarrow \mathcal{A}_{n\$70265[0]}$,
 $B_{n\$70265[3]} \rightarrow B_{n\$70265[3]}$, $T_{n\$70265[3]} \rightarrow T_{n\$70265[3]}$, $\mathcal{A}_{n\$70265[3]} \rightarrow \mathcal{A}_{n\$70265[3]}$ }

Knot " { $b_{n\$70864[5]} \rightarrow b_{n\$70864[5]}$, $t_{n\$70864[5]} \rightarrow t_{n\$70864[5]}$, $\alpha_{n\$70864[5]} \rightarrow \alpha_{n\$70864[5]}$,
 $b_{n\$70864[10]} \rightarrow b_{n\$70864[10]}$, $t_{n\$70864[10]} \rightarrow t_{n\$70864[10]}$, $\alpha_{n\$70864[10]} \rightarrow \alpha_{n\$70864[10]}$ }

Knot " { $B_{n\$70864[5]} \rightarrow B_{n\$70864[5]}$, $T_{n\$70864[5]} \rightarrow T_{n\$70864[5]}$, $\mathcal{A}_{n\$70864[5]} \rightarrow \mathcal{A}_{n\$70864[5]}$,
 $B_{n\$70864[10]} \rightarrow B_{n\$70864[10]}$, $T_{n\$70864[10]} \rightarrow T_{n\$70864[10]}$, $\mathcal{A}_{n\$70864[10]} \rightarrow \mathcal{A}_{n\$70864[10]}$ }

Knot " { $b_{n\$72044[4]} \rightarrow b_{n\$72044[4]}$, $t_{n\$72044[4]} \rightarrow t_{n\$72044[4]}$, $\alpha_{n\$72044[4]} \rightarrow \frac{t \hbar}{\gamma} + \alpha_{n\$72044[4]}$,
 $b_{n\$72044[k\$65562]} \rightarrow b_{n\$72044[k\$65562]}$, $t_{n\$72044[k\$65562]} \rightarrow t_{n\$72044[k\$65562]}$, $\alpha_{n\$72044[k\$65562]} \rightarrow -\frac{t \hbar}{\gamma} + \alpha_{n\$72044[k\$65562]}$ }

Knot " { $B_{n\$72044[4]} \rightarrow B_{n\$72044[4]}$, $T_{n\$72044[4]} \rightarrow T_{n\$72044[4]}$, $\mathcal{A}_{n\$72044[4]} \rightarrow T \mathcal{A}_{n\$72044[4]}$,
 $B_{n\$72044[k\$65562]} \rightarrow B_{n\$72044[k\$65562]}$, $T_{n\$72044[k\$65562]} \rightarrow T_{n\$72044[k\$65562]}$, $\mathcal{A}_{n\$72044[k\$65562]} \rightarrow \frac{\mathcal{A}_{n\$72044[k\$65562]}}{T}$ }

Knot " { $b_{n\$72113[k\$65562]} \rightarrow b_{n\$72113[k\$65562]}$, $t_{n\$72113[k\$65562]} \rightarrow t_{n\$72113[k\$65562]}$, $\alpha_{n\$72113[k\$65562]} \rightarrow \alpha_{n\$72113[k\$65562]}$,
 $b_{n\$72113[11]} \rightarrow b_{n\$72113[11]}$, $t_{n\$72113[11]} \rightarrow t_{n\$72113[11]}$, $\alpha_{n\$72113[11]} \rightarrow \alpha_{n\$72113[11]}$ }

Knot " { $B_{n\$72113[k\$65562]} \rightarrow B_{n\$72113[k\$65562]}$, $T_{n\$72113[k\$65562]} \rightarrow T_{n\$72113[k\$65562]}$, $\mathcal{A}_{n\$72113[k\$65562]} \rightarrow \mathcal{A}_{n\$72113[k\$65562]}$,
 $B_{n\$72113[11]} \rightarrow B_{n\$72113[11]}$, $T_{n\$72113[11]} \rightarrow T_{n\$72113[11]}$, $\mathcal{A}_{n\$72113[11]} \rightarrow \mathcal{A}_{n\$72113[11]}$ }

Knot " { $b_{n\$72182[11]} \rightarrow b_{n\$72182[11]}$, $t_{n\$72182[11]} \rightarrow t_{n\$72182[11]}$, $\alpha_{n\$72182[11]} \rightarrow \alpha_{n\$72182[11]}$,
 $b_{n\$72182[k\$65562]} \rightarrow b_{n\$72182[k\$65562]}$, $t_{n\$72182[k\$65562]} \rightarrow t_{n\$72182[k\$65562]}$, $\alpha_{n\$72182[k\$65562]} \rightarrow \alpha_{n\$72182[k\$65562]}$ }

Knot " { $B_{n\$72182[11]} \rightarrow B_{n\$72182[11]}$, $T_{n\$72182[11]} \rightarrow T_{n\$72182[11]}$, $\mathcal{A}_{n\$72182[11]} \rightarrow \mathcal{A}_{n\$72182[11]}$,
 $B_{n\$72182[k\$65562]} \rightarrow B_{n\$72182[k\$65562]}$, $T_{n\$72182[k\$65562]} \rightarrow T_{n\$72182[k\$65562]}$, $\mathcal{A}_{n\$72182[k\$65562]} \rightarrow \mathcal{A}_{n\$72182[k\$65562]}$ }

Knot " { $b_{n\$72251[k\$65562]} \rightarrow b_{n\$72251[k\$65562]}$, $t_{n\$72251[k\$65562]} \rightarrow t_{n\$72251[k\$65562]}$,
 $\alpha_{n\$72251[k\$65562]} \rightarrow \alpha_{n\$72251[k\$65562]}$, $b_{n\$72251[4]} \rightarrow b_{n\$72251[4]}$, $t_{n\$72251[4]} \rightarrow t_{n\$72251[4]}$, $\alpha_{n\$72251[4]} \rightarrow \alpha_{n\$72251[4]}$ }

Knot " { $B_{n\$72251[k\$65562]} \rightarrow B_{n\$72251[k\$65562]}$, $T_{n\$72251[k\$65562]} \rightarrow T_{n\$72251[k\$65562]}$,
 $\mathcal{A}_{n\$72251[k\$65562]} \rightarrow \mathcal{A}_{n\$72251[k\$65562]}$, $B_{n\$72251[4]} \rightarrow B_{n\$72251[4]}$, $T_{n\$72251[4]} \rightarrow T_{n\$72251[4]}$, $\mathcal{A}_{n\$72251[4]} \rightarrow \mathcal{A}_{n\$72251[4]}$ }

Knot " { $b_{n\$72322[4]} \rightarrow b_{n\$72322[4]}$, $t_{n\$72322[4]} \rightarrow t_{n\$72322[4]}$, $\alpha_{n\$72322[4]} \rightarrow \alpha_{n\$72322[4]}$,
 $b_{n\$72322[k\$65562]} \rightarrow b_{n\$72322[k\$65562]}$, $t_{n\$72322[k\$65562]} \rightarrow t_{n\$72322[k\$65562]}$, $\alpha_{n\$72322[k\$65562]} \rightarrow \alpha_{n\$72322[k\$65562]}$ }

Knot " { $B_{n\$72322[4]} \rightarrow B_{n\$72322[4]}$, $T_{n\$72322[4]} \rightarrow T_{n\$72322[4]}$, $\mathcal{A}_{n\$72322[4]} \rightarrow \mathcal{A}_{n\$72322[4]}$,
 $B_{n\$72322[k\$65562]} \rightarrow B_{n\$72322[k\$65562]}$, $T_{n\$72322[k\$65562]} \rightarrow T_{n\$72322[k\$65562]}$, $\mathcal{A}_{n\$72322[k\$65562]} \rightarrow \mathcal{A}_{n\$72322[k\$65562]}$ }

Knot " { $b_{n\$72393[5]} \rightarrow b_{n\$72393[5]}$, $t_{n\$72393[5]} \rightarrow t_{n\$72393[5]}$, $\alpha_{n\$72393[5]} \rightarrow \alpha_{n\$72393[5]}$,
 $b_{n\$72393[11]} \rightarrow b_{n\$72393[11]}$, $t_{n\$72393[11]} \rightarrow t_{n\$72393[11]}$, $\alpha_{n\$72393[11]} \rightarrow \alpha_{n\$72393[11]}$ }

Knot " { $B_n\$72393[5] \rightarrow B_n\$72393[5]$, $T_n\$72393[5] \rightarrow T_n\$72393[5]$, $\mathcal{A}_n\$72393[5] \rightarrow \mathcal{A}_n\$72393[5]$,
 $B_n\$72393[11] \rightarrow B_n\$72393[11]$, $T_n\$72393[11] \rightarrow T_n\$72393[11]$, $\mathcal{A}_n\$72393[11] \rightarrow \mathcal{A}_n\$72393[11]$ }

Knot " { $b_n\$73176[4] \rightarrow b_n\$73176[4]$, $t_n\$73176[4] \rightarrow t_n\$73176[4]$, $\alpha_n\$73176[4] \rightarrow \alpha_n\$73176[4]$,
 $b_n\$73176[5] \rightarrow b_n\$73176[5]$, $t_n\$73176[5] \rightarrow t_n\$73176[5]$, $\alpha_n\$73176[5] \rightarrow \alpha_n\$73176[5]$ }

Knot " { $B_n\$73176[4] \rightarrow B_n\$73176[4]$, $T_n\$73176[4] \rightarrow T_n\$73176[4]$, $\mathcal{A}_n\$73176[4] \rightarrow \mathcal{A}_n\$73176[4]$,
 $B_n\$73176[5] \rightarrow B_n\$73176[5]$, $T_n\$73176[5] \rightarrow T_n\$73176[5]$, $\mathcal{A}_n\$73176[5] \rightarrow \mathcal{A}_n\$73176[5]$ }

Knot " { $b_n\$74688[0] \rightarrow b_n\$74688[0]$, $t_n\$74688[0] \rightarrow t_n\$74688[0]$, $\alpha_n\$74688[0] \rightarrow \alpha_n\$74688[0]$,
 $b_n\$74688[4] \rightarrow b_n\$74688[4]$, $t_n\$74688[4] \rightarrow t_n\$74688[4]$, $\alpha_n\$74688[4] \rightarrow \alpha_n\$74688[4]$ }

Knot " { $B_n\$74688[0] \rightarrow B_n\$74688[0]$, $T_n\$74688[0] \rightarrow T_n\$74688[0]$, $\mathcal{A}_n\$74688[0] \rightarrow \mathcal{A}_n\$74688[0]$,
 $B_n\$74688[4] \rightarrow B_n\$74688[4]$, $T_n\$74688[4] \rightarrow T_n\$74688[4]$, $\mathcal{A}_n\$74688[4] \rightarrow \mathcal{A}_n\$74688[4]$ }

Knot " { $b_n\$203598[12] \rightarrow b_n\$203598[12]$, $t_n\$203598[12] \rightarrow t_n\$203598[12]$,
 $\alpha_n\$203598[12] \rightarrow \frac{t \hbar}{\gamma} + \alpha_n\$203598[12]$, $b_n\$203598[k\$65562] \rightarrow b_n\$203598[k\$65562]$,
 $t_n\$203598[k\$65562] \rightarrow t_n\$203598[k\$65562]$, $\alpha_n\$203598[k\$65562] \rightarrow -\frac{t \hbar}{\gamma} + \alpha_n\$203598[k\$65562]$ }

Knot " { $B_n\$203598[12] \rightarrow B_n\$203598[12]$, $T_n\$203598[12] \rightarrow T_n\$203598[12]$, $\mathcal{A}_n\$203598[12] \rightarrow \mathcal{A}_n\$203598[12]$,
 $B_n\$203598[k\$65562] \rightarrow B_n\$203598[k\$65562]$, $T_n\$203598[k\$65562] \rightarrow T_n\$203598[k\$65562]$, $\mathcal{A}_n\$203598[k\$65562] \rightarrow \frac{\mathcal{A}_n\$203598[k\$65562]}{T}$ }

Knot " { $b_n\$203667[k\$65562] \rightarrow b_n\$203667[k\$65562]$, $t_n\$203667[k\$65562] \rightarrow t_n\$203667[k\$65562]$, $\alpha_n\$203667[k\$65562] \rightarrow \alpha_n\$203667[k\$65562]$,
 $b_n\$203667[19] \rightarrow b_n\$203667[19]$, $t_n\$203667[19] \rightarrow t_n\$203667[19]$, $\alpha_n\$203667[19] \rightarrow \alpha_n\$203667[19]$ }

Knot " { $B_n\$203667[k\$65562] \rightarrow B_n\$203667[k\$65562]$, $T_n\$203667[k\$65562] \rightarrow T_n\$203667[k\$65562]$, $\mathcal{A}_n\$203667[k\$65562] \rightarrow \mathcal{A}_n\$203667[k\$65562]$,
 $B_n\$203667[19] \rightarrow B_n\$203667[19]$, $T_n\$203667[19] \rightarrow T_n\$203667[19]$, $\mathcal{A}_n\$203667[19] \rightarrow \mathcal{A}_n\$203667[19]$ }

Knot " { $b_n\$203736[19] \rightarrow b_n\$203736[19]$, $t_n\$203736[19] \rightarrow t_n\$203736[19]$, $\alpha_n\$203736[19] \rightarrow \alpha_n\$203736[19]$,
 $b_n\$203736[k\$65562] \rightarrow b_n\$203736[k\$65562]$, $t_n\$203736[k\$65562] \rightarrow t_n\$203736[k\$65562]$, $\alpha_n\$203736[k\$65562] \rightarrow \alpha_n\$203736[k\$65562]$ }

Knot " { $B_n\$203736[19] \rightarrow B_n\$203736[19]$, $T_n\$203736[19] \rightarrow T_n\$203736[19]$, $\mathcal{A}_n\$203736[19] \rightarrow \mathcal{A}_n\$203736[19]$,
 $B_n\$203736[k\$65562] \rightarrow B_n\$203736[k\$65562]$, $T_n\$203736[k\$65562] \rightarrow T_n\$203736[k\$65562]$, $\mathcal{A}_n\$203736[k\$65562] \rightarrow \mathcal{A}_n\$203736[k\$65562]$ }

Knot " { $b_n\$203805[k\$65562] \rightarrow b_n\$203805[k\$65562]$, $t_n\$203805[k\$65562] \rightarrow t_n\$203805[k\$65562]$, $\alpha_n\$203805[k\$65562] \rightarrow \alpha_n\$203805[k\$65562]$,
 $b_n\$203805[12] \rightarrow b_n\$203805[12]$, $t_n\$203805[12] \rightarrow t_n\$203805[12]$, $\alpha_n\$203805[12] \rightarrow \alpha_n\$203805[12]$ }

Knot " { $B_n\$203805[k\$65562] \rightarrow B_n\$203805[k\$65562]$, $T_n\$203805[k\$65562] \rightarrow T_n\$203805[k\$65562]$, $\mathcal{A}_n\$203805[k\$65562] \rightarrow \mathcal{A}_n\$203805[k\$65562]$,
 $B_n\$203805[12] \rightarrow B_n\$203805[12]$, $T_n\$203805[12] \rightarrow T_n\$203805[12]$, $\mathcal{A}_n\$203805[12] \rightarrow \mathcal{A}_n\$203805[12]$ }

Knot " { $b_n\$203876[12] \rightarrow b_n\$203876[12]$, $t_n\$203876[12] \rightarrow t_n\$203876[12]$, $\alpha_n\$203876[12] \rightarrow \alpha_n\$203876[12]$,
 $b_n\$203876[k\$65562] \rightarrow b_n\$203876[k\$65562]$, $t_n\$203876[k\$65562] \rightarrow t_n\$203876[k\$65562]$, $\alpha_n\$203876[k\$65562] \rightarrow \alpha_n\$203876[k\$65562]$ }

Knot " { $B_n\$203876[12] \rightarrow B_n\$203876[12]$, $T_n\$203876[12] \rightarrow T_n\$203876[12]$, $\mathcal{A}_n\$203876[12] \rightarrow \mathcal{A}_n\$203876[12]$,
 $B_n\$203876[k\$65562] \rightarrow B_n\$203876[k\$65562]$, $T_n\$203876[k\$65562] \rightarrow T_n\$203876[k\$65562]$, $\mathcal{A}_n\$203876[k\$65562] \rightarrow \mathcal{A}_n\$203876[k\$65562]$ }

Knot " { $b_n\$203947[14] \rightarrow b_n\$203947[14]$, $t_n\$203947[14] \rightarrow t_n\$203947[14]$, $\alpha_n\$203947[14] \rightarrow \alpha_n\$203947[14]$,
 $b_n\$203947[19] \rightarrow b_n\$203947[19]$, $t_n\$203947[19] \rightarrow t_n\$203947[19]$, $\alpha_n\$203947[19] \rightarrow \alpha_n\$203947[19]$ }

Knot " { $B_n\$203947[14] \rightarrow B_n\$203947[14]$, $T_n\$203947[14] \rightarrow T_n\$203947[14]$, $\mathcal{A}_n\$203947[14] \rightarrow \mathcal{A}_n\$203947[14]$,
 $B_n\$203947[19] \rightarrow B_n\$203947[19]$, $T_n\$203947[19] \rightarrow T_n\$203947[19]$, $\mathcal{A}_n\$203947[19] \rightarrow \mathcal{A}_n\$203947[19]$ }

Knot " { $b_n\$252522[0] \rightarrow b_n\$252522[0]$, $t_n\$252522[0] \rightarrow t_n\$252522[0]$, $\alpha_n\$252522[0] \rightarrow \alpha_n\$252522[0]$,
 $b_n\$252522[12] \rightarrow b_n\$252522[12]$, $t_n\$252522[12] \rightarrow t_n\$252522[12]$, $\alpha_n\$252522[12] \rightarrow \alpha_n\$252522[12]$ }

Knot " { $B_n\$252522[0] \rightarrow B_n\$252522[0]$, $T_n\$252522[0] \rightarrow T_n\$252522[0]$, $\mathcal{A}_n\$252522[0] \rightarrow \mathcal{A}_n\$252522[0]$,
 $B_n\$252522[12] \rightarrow B_n\$252522[12]$, $T_n\$252522[12] \rightarrow T_n\$252522[12]$, $\mathcal{A}_n\$252522[12] \rightarrow \mathcal{A}_n\$252522[12]$ }

Knot " { $b_n\$349289[20] \rightarrow b_n\$349289[20]$, $t_n\$349289[20] \rightarrow t_n\$349289[20]$,
 $\alpha_n\$349289[20] \rightarrow \frac{t_n \hbar}{\gamma} + \alpha_n\$349289[20]$, $b_n\$349289[k\$65562] \rightarrow b_n\$349289[k\$65562]$,
 $t_n\$349289[k\$65562] \rightarrow t_n\$349289[k\$65562]$, $\alpha_n\$349289[k\$65562] \rightarrow -\frac{t_n \hbar}{\gamma} + \alpha_n\$349289[k\$65562]$ }

Knot " { $B_n\$349289[20] \rightarrow B_n\$349289[20]$, $T_n\$349289[20] \rightarrow T_n\$349289[20]$, $\mathcal{A}_n\$349289[20] \rightarrow T_n \mathcal{A}_n\$349289[20]$,
 $B_n\$349289[k\$65562] \rightarrow B_n\$349289[k\$65562]$, $T_n\$349289[k\$65562] \rightarrow T_n\$349289[k\$65562]$, $\mathcal{A}_n\$349289[k\$65562] \rightarrow \frac{\mathcal{A}_n\$349289[k\$65562]}{T}$ }

Knot " { $b_n\$349358[k\$65562] \rightarrow b_n\$349358[k\$65562]$, $t_n\$349358[k\$65562] \rightarrow t_n\$349358[k\$65562]$, $\alpha_n\$349358[k\$65562] \rightarrow \alpha_n\$349358[k\$65562]$,
 $b_n\$349358[13] \rightarrow b_n\$349358[13]$, $t_n\$349358[13] \rightarrow t_n\$349358[13]$, $\alpha_n\$349358[13] \rightarrow \alpha_n\$349358[13]$ }

Knot " { $B_n\$349358[k\$65562] \rightarrow B_n\$349358[k\$65562]$, $T_n\$349358[k\$65562] \rightarrow T_n\$349358[k\$65562]$, $\mathcal{A}_n\$349358[k\$65562] \rightarrow \mathcal{A}_n\$349358[k\$65562]$,
 $B_n\$349358[13] \rightarrow B_n\$349358[13]$, $T_n\$349358[13] \rightarrow T_n\$349358[13]$, $\mathcal{A}_n\$349358[13] \rightarrow \mathcal{A}_n\$349358[13]$ }

Knot " { $b_n\$349427[13] \rightarrow b_n\$349427[13]$, $t_n\$349427[13] \rightarrow t_n\$349427[13]$, $\alpha_n\$349427[13] \rightarrow \alpha_n\$349427[13]$,
 $b_n\$349427[k\$65562] \rightarrow b_n\$349427[k\$65562]$, $t_n\$349427[k\$65562] \rightarrow t_n\$349427[k\$65562]$, $\alpha_n\$349427[k\$65562] \rightarrow \alpha_n\$349427[k\$65562]$ }

Knot " { $B_n\$349427[13] \rightarrow B_n\$349427[13]$, $T_n\$349427[13] \rightarrow T_n\$349427[13]$, $\mathcal{A}_n\$349427[13] \rightarrow \mathcal{A}_n\$349427[13]$,
 $B_n\$349427[k\$65562] \rightarrow B_n\$349427[k\$65562]$, $T_n\$349427[k\$65562] \rightarrow T_n\$349427[k\$65562]$, $\mathcal{A}_n\$349427[k\$65562] \rightarrow \mathcal{A}_n\$349427[k\$65562]$ }

Knot " { $b_n\$349496[k\$65562] \rightarrow b_n\$349496[k\$65562]$, $t_n\$349496[k\$65562] \rightarrow t_n\$349496[k\$65562]$, $\alpha_n\$349496[k\$65562] \rightarrow \alpha_n\$349496[k\$65562]$,
 $b_n\$349496[20] \rightarrow b_n\$349496[20]$, $t_n\$349496[20] \rightarrow t_n\$349496[20]$, $\alpha_n\$349496[20] \rightarrow \alpha_n\$349496[20]$ }

Knot " { $B_n\$349496[k\$65562] \rightarrow B_n\$349496[k\$65562]$, $T_n\$349496[k\$65562] \rightarrow T_n\$349496[k\$65562]$, $\mathcal{A}_n\$349496[k\$65562] \rightarrow \mathcal{A}_n\$349496[k\$65562]$,
 $B_n\$349496[20] \rightarrow B_n\$349496[20]$, $T_n\$349496[20] \rightarrow T_n\$349496[20]$, $\mathcal{A}_n\$349496[20] \rightarrow \mathcal{A}_n\$349496[20]$ }

Knot " { $b_n\$349567[20] \rightarrow b_n\$349567[20]$, $t_n\$349567[20] \rightarrow t_n\$349567[20]$, $\alpha_n\$349567[20] \rightarrow \alpha_n\$349567[20]$,
 $b_n\$349567[k\$65562] \rightarrow b_n\$349567[k\$65562]$, $t_n\$349567[k\$65562] \rightarrow t_n\$349567[k\$65562]$, $\alpha_n\$349567[k\$65562] \rightarrow \alpha_n\$349567[k\$65562]$ }

Knot " { $B_n\$349567[20] \rightarrow B_n\$349567[20]$, $T_n\$349567[20] \rightarrow T_n\$349567[20]$, $\mathcal{A}_n\$349567[20] \rightarrow \mathcal{A}_n\$349567[20]$,
 $B_n\$349567[k\$65562] \rightarrow B_n\$349567[k\$65562]$, $T_n\$349567[k\$65562] \rightarrow T_n\$349567[k\$65562]$, $\mathcal{A}_n\$349567[k\$65562] \rightarrow \mathcal{A}_n\$349567[k\$65562]$ }

Knot " { $b_n\$349638[13] \rightarrow b_n\$349638[13]$, $t_n\$349638[13] \rightarrow t_n\$349638[13]$, $\alpha_n\$349638[13] \rightarrow \alpha_n\$349638[13]$,
 $b_n\$349638[14] \rightarrow b_n\$349638[14]$, $t_n\$349638[14] \rightarrow t_n\$349638[14]$, $\alpha_n\$349638[14] \rightarrow \alpha_n\$349638[14]$ }

Knot " { Bn\$349638[13] → Bn\$349638[13], Tn\$349638[13] → Tn\$349638[13], An\$349638[13] → An\$349638[13],
 Bn\$349638[14] → Bn\$349638[14], Tn\$349638[14] → Tn\$349638[14], An\$349638[14] → An\$349638[14] }

Knot " { bn\$406025[0] → bn\$406025[0], tn\$406025[0] → tn\$406025[0], an\$406025[0] → an\$406025[0],
 bn\$406025[13] → bn\$406025[13], tn\$406025[13] → tn\$406025[13], an\$406025[13] → an\$406025[13] }

Knot " { Bn\$406025[0] → Bn\$406025[0], Tn\$406025[0] → Tn\$406025[0], An\$406025[0] → An\$406025[0],
 Bn\$406025[13] → Bn\$406025[13], Tn\$406025[13] → Tn\$406025[13], An\$406025[13] → An\$406025[13] }

Knot " { bn\$486721[0] → bn\$486721[0], tn\$486721[0] → tn\$486721[0], an\$486721[0] → an\$486721[0],
 bn\$486721[20] → bn\$486721[20], tn\$486721[20] → tn\$486721[20], an\$486721[20] → an\$486721[20] }

Knot " { Bn\$486721[0] → Bn\$486721[0], Tn\$486721[0] → Tn\$486721[0], An\$486721[0] → An\$486721[0],
 Bn\$486721[20] → Bn\$486721[20], Tn\$486721[20] → Tn\$486721[20], An\$486721[20] → An\$486721[20] }

Knot " Out[*]= { 1189.17, E [0, 0, T^4 / (1 - 4 T + 9 T^2 - 12 T^3 + 13 T^4 - 12 T^5 + 9 T^6 - 4 T^7 + T^8) +
 (T^4 ħ (4 a (-2 + 14 T - 51 T^2 + 120 T^3 - 203 T^4 + 258 T^5 - 246 T^6 + 152 T^7 -
 152 T^9 + 246 T^10 - 258 T^11 + 203 T^12 - 120 T^13 + 51 T^14 - 14 T^15 + 2 T^16) +
 γ (-6 + 2 T^16 - 8 x y ħ - 440 T^9 (-1 + x y ħ) - 4 T^15 (3 + 2 x y ħ) + 8 T^8 (-97 + 21 x y ħ) +
 8 T^7 (131 + 21 x y ħ) - 20 T^6 (57 + 22 x y ħ) + T^14 (37 + 48 x y ħ) + T (44 + 48 x y ħ) -
 8 T^11 (2 + 61 x y ħ) + 8 T^5 (127 + 68 x y ħ) - 2 T^13 (35 + 78 x y ħ) + 4 T^10 (-39 + 136 x y ħ) -
 T^2 (167 + 156 x y ħ) + T^12 (79 + 324 x y ħ) + T^3 (410 + 324 x y ħ) - T^4 (733 + 488 x y ħ)))
 ∈) / (1 - 4 T + 9 T^2 - 12 T^3 + 13 T^4 - 12 T^5 + 9 T^6 - 4 T^7 + T^8)^3 + O[ε]^2 }

In[*]:= EndProfile[];

Profile In[*]:= PrintProfile[]

Profile Out[*]= ProfileRoot is root. Profiled time: 1301.06

- (1) 0.155/ 1189.170 above Z
- (161) 0.777/ 96.375 above Bind
- (141) 0.015/ 0.030 above CF
- (2) 0/ 0 above RVK
- (17) 0.032/ 3.578 above Boot[1]
- (18) 0.091/ 8.876 above Boot[2]
- (4) 0.032/ 0.079 above Boot[3]
- (1) 0.016/ 2.953 above Boot[4]

Exp: called 247574 times, time in 773.698/843.477

- (247574) 773.698/ 843.477 under Together
- (245621) 35.895/ 69.779 above CF

QZip: called 302 times, time in 240.979/1069.37

- (302) 240.979/ 1069.370 under Bind
- (906) 0.873/ 791.484 above CF
- (302) 12.346/ 36.908 above Zip

Together: called 247574 times, time in 100.172/943.649

- (247574) 100.172/ 943.649 under CF
- (247574) 773.698/ 843.477 above Exp

CF: called 247574 times, time in 58.368/1002.02

- (245621) 35.895/ 69.779 under Exp

```

( 906) 21.585/ 140.724 under LZip
( 141) 0.015/ 0.030 under ProfileRoot
( 906) 0.873/ 791.484 under QZip
( 247574) 100.172/ 943.649 above Together
LZip: called 302 times, time in 56.957/229.995
( 302) 56.957/ 229.995 under Bind
( 906) 21.585/ 140.724 above CF
( 302) 6.715/ 32.314 above Zip
Zip: called 2750 times, time in 56.865/172.748
( 302) 6.715/ 32.314 under LZip
( 302) 12.346/ 36.908 under QZip
( 2146) 37.804/ 103.526 under Zip
( 2750) 12.357/ 12.357 above Collect
( 2146) 37.804/ 103.526 above Zip
Collect: called 2750 times, time in 12.357/12.357
( 2750) 12.357/ 12.357 under Zip
Bind: called 302 times, time in 1.217/1300.58
( 72) 0.284/ 1188.890 under Z
( 161) 0.777/ 96.375 under ProfileRoot
( 33) 0.047/ 3.672 under Boot[1]
( 29) 0.078/ 8.754 under Boot[2]
( 3) 0/ 0.781 under Boot[3]
( 4) 0.031/ 2.110 under Boot[4]
( 302) 56.957/ 229.995 above LZip
( 302) 240.979/ 1069.370 above QZip
Z: called 1 times, time in 0.155/1189.17
( 1) 0.155/ 1189.170 under ProfileRoot
( 72) 0.284/ 1188.890 above Bind
( 3) 0/ 0.126 above Boot[1]
Boot[2]: called 23 times, time in 0.122/9.282
( 18) 0.091/ 8.876 under ProfileRoot
( 5) 0.031/ 0.406 under Boot[2]
( 29) 0.078/ 8.754 above Bind
( 5) 0.031/ 0.406 above Boot[2]
Boot[4]: called 6 times, time in 0.093/5.687
( 1) 0.016/ 2.953 under ProfileRoot
( 3) 0.031/ 0.031 under Boot[3]
( 2) 0.046/ 2.703 under Boot[4]
( 4) 0.031/ 2.110 above Bind
( 1) 0.016/ 0.781 above Boot[3]
( 2) 0.046/ 2.703 above Boot[4]
Boot[3]: called 5 times, time in 0.048/0.86
( 4) 0.032/ 0.079 under ProfileRoot
( 1) 0.016/ 0.781 under Boot[4]
( 3) 0/ 0.781 above Bind
( 3) 0.031/ 0.031 above Boot[4]
Boot[1]: called 27 times, time in 0.032/5.235
( 3) 0/ 0.126 under Z
( 17) 0.032/ 3.578 under ProfileRoot
( 7) 0/ 1.531 under Boot[1]
( 33) 0.047/ 3.672 above Bind

```

```
( 3) 0/ 0 above Boot[0]
( 7) 0/ 1.531 above Boot[1]
Boot[0]: called 3 times, time in 0./0.
( 3) 0/ 0 under Boot[1]
RVK: called 2 times, time in 0./0.
( 2) 0/ 0 under ProfileRoot
```