Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb. Automated "Define" branch - not yet working.

The issue seems to be that $k gets converted to k below, for unknown reasons.

External Utilities

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Internal Utilities

Canonical Form:

```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. ⅇ^x_ ⅇ^y_ ⧴ ⅇ^(x+y) /. ⅇ^x_ ⧴ ⅇ^CF[x]];
```

The Kronecker δ:

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Finite Zips: (* Perhaps switch Expand to Collect[__, ζ]? *)

Program

In[∘]:=
```
expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ℰ_] := Expand[ℰ];
Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] := (expand[P // Zip{ζs}] /. f_. ζ^d_. :→ ∂{ζ*,d} f) /. ζ* → 0
```

Program

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

Program

In[∘]:=
```
QZip_{ζs_List,simp_} @𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → qt.(zs + ys)];
    Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L, Q2, Det[qt] e^{-Q2} Zip_ζs[e^{Q1} (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs,CF};
```

Program

Upper to lower and lower to Upper:

Program

In[∘]:=
```
U2l = {B_i_^p_. → e^{-p ħ γ b_i}, B^p_. → e^{-p ħ γ b}, T_i_^p_. → e^{p ħ t_i}, T^p_. → e^{p ħ t}, 𝒜_i_^p_. → e^{p γ α_i}, 𝒜^p_. → e^{p γ α}};
l2U = {e^{c_. b_i_+d_.} :→ B_i^{-c/(ħ γ)} e^d, e^{c_. b+d_.} :→ B^{-c/(ħ γ)} e^d,
    e^{c_. t_i_+d_.} :→ T_i^{c/ħ} e^d, e^{c_. t+d_.} :→ T^{c/ħ} e^d,
    e^{c_. α_i_+d_.} :→ 𝒜_i^{c/γ} e^d, e^{c_. α+d_.} :→ 𝒜^{c/γ} e^d,
    e^ℰ_ :→ e^{Expand@ℰ}};
```

Program

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

Program

In[∘]:=
```
LZip_{ζs_List,simp_} @𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L2, Q2, Det[lt] e^{-L2-Q2} Zip_ζs[e^{L1+Q1} (P /. U2l /. zrule)]] //. l2U ];
LZip_{ζs_List} := LZip_{ζs,CF};
```

Program

In[ ]:=
```
Bind{}[L_, R_] := L R;
Bind{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
      L /. Table[(v : b | B | t | T | a | x | y)i → vn@i, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)i → vn@i, {i, {is}}]
     ] // LZipFlatten@Table[{βn@i,τn@i,an@i},{i,{is}}] // QZipFlatten@Table[{ξn@i,yn@i},{i,{is}}] ];
BL_List[L_, R_] := BindL[L, R]; Bis___[L_, R_] := Bind{is}[L, R];
```

Program

The Fundamental Logoi

Program

In[ ]:=
```
$k = 2;
```

Program

Define[lhs = rhs] defines the lhs to be rhs, except that rhs is computed once and forever yet gets recomputed whenever $k changes. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

In[ ]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis},
  Block[{i, j, k, l, m, n, t1, t2, t3, h1, h2, h3},
   ReleaseHold@Echo[Hold[
       SD[opnisp,$k_Integer, Block[{i, j, k, l, m, n, t1, t2, t3, h1, h2, h3}, opisp,$k = ℰ;
         opnis,$k]];
       opisp := op{is},$k;
       opsis__ := op{sis};
      ] /. {
       isp → {is} /. {i → i_, j → j_, k → k_},
       nis → {is} /. {i → ii, j → jj, k → kk},
       nisp → {is} /. {i → ii_, j → jj_, k → kk_},
       SD → SetDelayed
      }]
  ]]
```

Program

In[ ]:=

```
Define[
  am_{i,j→k} = 𝔼[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k},
  bm_{i,j→k} = 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i} - 1) η_j y_k}]_{$k},
  R_{i,j} = 𝔼[ℏ a_j b_i, ℏ x_j y_i, e^(Sum_{k=2}^{$k+1} ((1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1 - e^{k γ ϵ ℏ})))]_{$k},
  P_{i,j} = If[$k == 0, 𝔼[β_i α_j / ℏ, η_i ξ_j / ℏ, 1]_0,
    MapAt[(# - ϵ^{$k} Coefficient[(R_{n,m} ~ B_{n,m} ~ ((P_{{n,j},0})_{$k} (P_{{i,m},$k-1})_{$k}))[[3]], ϵ, $k]) &,
      (P_{{i,j},$k-1})_{$k}, 3]],
]
```

Program

```
Hold[am_{ii$2830_,jj$2830_→kk$2830_},$k_Integer :=
  Block[{i, j, k, l, m, n, t1, t2, t3, h1, h2, h3}, am_{i_,j_→k_},$k = 𝔼[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k};
   am_{ii$2830,jj$2830→kk$2830},$k];
 am_{i_,j_→k_} := am_{i,j→k},$k;
 am_sis$__ := am_{sis$};]
```

Program

```
Hold[bm_{ii$2833_,jj$2833_→kk$2833_},$k_Integer := Block[
   {i, j, k, l, m, n, t1, t2, t3, h1, h2, h3}, bm_{i_,j_→k_},$k = 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i} - 1) η_j y_k}]_{$k};
   bm_{ii$2833,jj$2833→kk$2833},$k];
 bm_{i_,j_→k_} := bm_{i,j→k},$k;
 bm_sis$__ := bm_{sis$};]
```

Program

```
Hold[R_{ii$2834_,jj$2834_},$k_Integer :=
  Block[{i, j, k, l, m, n, t1, t2, t3, h1, h2, h3}, R_{i_,j_},$k = 𝔼[ℏ a_j b_i, ℏ x_j y_i, e^{Sum_{k=2}^{$k+1} ((1-e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1-e^{k γ ϵ ℏ}))}]_{$k};
   R_{ii$2834,jj$2834},$k];
 R_{i_,j_} := R_{i,j},$k;
 R_sis$__ := R_{sis$};]
```

Program

```
Hold[P_{ii$2836_,jj$2836_},$k_Integer :=
  Block[{i, j, k, l, m, n, t1, t2, t3, h1, h2, h3}, P_{i_,j_},$k = If[$k == 0, 𝔼[β_i α_j / ℏ, η_i ξ_j / ℏ, 1]_0,
     MapAt[#1 - ϵ^{$k} Coefficient[B_{n,m}[R_{n,m}, P_{{n,j},0$k} P_{{i,m},$k-1$k}][[3]], ϵ, $k] &, P_{{i,j},$k-1$k}, 3]];
   P_{ii$2836,jj$2836},$k];
 P_{i_,j_} := P_{i,j},$k;
 P_sis$__ := P_{sis$};]
```

The issue seems to be that \$k gets converted to \$k\$ above, for unknown reasons.

Testing

In[∘]:= $R_{1,2}$

Out[∘]= $\mathbb{E}\left[\hbar\, a_2 b_1,\ \hbar\, x_2 y_1,\ 1-\frac{1}{4}\left(\gamma \hbar^3 x_2^2 y_1^2\right)\epsilon+\left(\frac{1}{9}\gamma^2\hbar^5 x_2^3 y_1^3+\frac{1}{32}\gamma^2\hbar^6 x_2^4 y_1^4\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:= $\mathbb{E}\left[\hbar\, a_2 b_1,\ \hbar\, x_2 y_1,\ 1-\frac{1}{4}\left(\gamma \hbar^3 x_2^2 y_1^2\right)\epsilon+\left(\frac{1}{9}\gamma^2\hbar^5 x_2^3 y_1^3+\frac{1}{32}\gamma^2\hbar^6 x_2^4 y_1^4\right)\epsilon^2+O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}\left[\hbar\, a_2 b_1,\ \hbar\, x_2 y_1,\ 1-\frac{1}{4}\left(\gamma \hbar^3 x_2^2 y_1^2\right)\epsilon+\left(\frac{1}{9}\gamma^2\hbar^5 x_2^3 y_1^3+\frac{1}{32}\gamma^2\hbar^6 x_2^4 y_1^4\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:= $P_{1,2}$

Out[∘]= $\mathbb{E}\left[\frac{\alpha_2\beta_1}{\hbar},\ \frac{\eta_1\xi_2}{\hbar},\ 1+\frac{\gamma\eta_1^2\xi_2^2\epsilon}{4\hbar}+\frac{\left(36\gamma^2\hbar^2\eta_1^2\xi_2^2+40\gamma^2\hbar\,\eta_1^3\xi_2^3+9\gamma^2\eta_1^4\xi_2^4\right)\epsilon^2}{288\hbar^2}+O[\epsilon]^3\right]$

In[∘]:= `Block[{$k = 1}, ` $P_{1,2}$`]`

Out[∘]= $\mathbb{E}\left[\frac{\alpha_2\beta_1}{\hbar},\ \frac{\eta_1\xi_2}{\hbar},\ 1+\frac{\gamma\epsilon\eta_1^2\xi_2^2}{4\hbar}\right]$

In[∘]:= Block[{$k = 1}, {
am → $am_{i,j\to k}$, bm → $bm_{i,j\to k}$, dm → $dm_{i,j\to k}$, R → $R_{i,j}$, $\overline{R}$ → $\overline{R}_{i,j}$, P → $P_{i,j}$, aS → aS_i, $\overline{aS}$ → $\overline{aS}_i$, bS → bS_i,
$\overline{bS}$ → $\overline{bS}_i$, dS → dS_i, aΔ → $a\Delta_{i\to j,k}$, bΔ → $b\Delta_{i\to j,k}$, dΔ → $d\Delta_{i\to j,k}$, b2t → $b2t_i$, t2b → $t2b_i$
}] //
Column

... \$IterationLimit: Iteration limit of 4096 exceeded.

... \$IterationLimit: Iteration limit of 4096 exceeded.

... \$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
#1 − $\epsilon^{\$k}$ Coefficient[B_i[Subscript[≪2≫][≪2≫], Subscript[≪2≫]][[3]], ϵ, \$k] &.

... \$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
$\overline{aS}_{\{i\_\},0}$ = If[\$k == 0, 𝔼[−Subscript[≪2≫] α_i, −Subscript[≪2≫] $\mathcal{A}_i\ \xi_i$, 1]$_0$, MapAt[#1 − Times[≪2≫] &, $\overline{aS}_{\{i\},\$k-1\,\$k}$, 3]]; $\overline{aS}_{\{i\},0}$.

... \$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of Head[\$FrontEnd] === FrontEndObject.

... General: Further output of \$RecursionLimit::reclim2 will be suppressed during this calculation.

... \$IterationLimit: Iteration limit of 4096 exceeded.

... General: Further output of \$IterationLimit::itlim will be suppressed during this calculation.

MapAt::partw : Part {3} of
Hold[$\overline{aS}_{\{i\_\},0}$ = If[\$k == 0, 𝔼[Times[≪2≫], Times[≪3≫], 1]$_0$, MapAt[Plus[≪2≫] &, Subscript[≪3≫]$_{\$k}$, 3]]; $\overline{aS}_{\{i\},0}$]$_1$ does not exist.

Rule::rhs : Pattern i_ appears on the right-hand side of rule $\overline{aS}_{\{i\_\},0}$ → MapAt[#1 − $\epsilon^{\$k}$ Coefficient[Part[≪2≫], ϵ, \$k] &, Hold[$\overline{≪1≫}_{\{≪1≫\},0}$

MapAt::partw : Part {3} of
Hold[$\overline{aS}_{\{i\_\},0}$ = If[\$k == 0, 𝔼[Times[≪2≫], Times[≪3≫], 1]$_0$, MapAt[Plus[≪2≫] &, Subscript[≪3≫]$_{\$k}$, 3]]; $\overline{aS}_{\{i\},0}$]$_1$ does not exist.

MapAt::partw : Part {3} of
Hold[$\overline{aS}_{\{i\_\},0}$ = If[\$k == 0, 𝔼[Times[≪2≫], Times[≪3≫], 1]$_0$, MapAt[Plus[≪2≫] &, Subscript[≪3≫]$_{\$k}$, 3]]; $\overline{aS}_{\{i\},0}$]$_1$ does not exist.

Hold[General::stop : Further output of MapAt::partw will be suppressed during this calculation.]

Rule::rhs : Pattern i_ appears on the right−hand side of rule
$\overline{aS}_{\{i\_\},0}$ → MapAt[♯1 − $\epsilon^{\$k}$ Coefficient[Part[≪2≫], ϵ, $k] &, MapAt[♯1 − Times[≪2≫] &, Hold[Set[≪2≫]; Subscript[≪3≫]]$_1$, 3]$_1$, 3
].

... Rule: Pattern appears on the right−hand side of rule _

List[Pattern[i, Blank[]]],0

.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$IterationLimit::itlim, $IterationLimit::itlim, $RecursionLimit::reclim2, $RecursionLimit::reclim2, MapAt::partw, $RecursionLimit::reclim2, General::stop, $IterationLimit::itlim, General::stop, Rule::rhs, MapAt::partw, MapAt::partw, General::stop, Rule::rhs, Rule::rhs, General::stop}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of will be suppressed during this calculation.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, MapAt::partw}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

... MapAt: Part of _

List[Pattern[i,
Blank[]]],0

1
does not exist.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop,
MapAt::partw}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

... MapAt: Part of

1

does not exist.

... MapAt: Part of

1

does not exist.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop,
MapAt::partw, MapAt::partw}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop}]].

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

... $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

... General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

MapAt: Part of _

List[Pattern[i,
Blank[]]],0

1
does not exist.

$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop, MapAt::partw}]].

$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of
Message[Message::msgl, Hold[{$RecursionLimit::reclim2, $RecursionLimit::reclim2, $RecursionLimit::reclim2, General::stop}]].

$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {$RecursionLimit::reclim2}.

$RecursionLimit: Recursion depth of 1024 exceeded during evaluation of {OutputStream[Name: stdout Unique ID: 1]}.

General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

MapAt: Part of

1
does not exist.

MapAt: Part of

1
does not exist.

$IterationLimit: Iteration limit of 4096 exceeded.

$Aborted[]

$IterationLimit: Iteration limit of 4096 exceeded.

General: Further output of $IterationLimit::itlim will be suppressed during this calculation.

$Aborted[]

MapAt: Part of

1

does not exist.

General: Further output of will be suppressed during this calculation.

Rule: Pattern i_ appears on the right–hand side of rule

_

→

1

.

Rule: Pattern i_ appears on the right–hand side of
rule

$k

RowBox[{SubscriptBox[OverscriptBox[aS, _], RowBox[{RowBox[{{, i_, }}], ,, 0}]], →, RowBox[{MapAt,
[, RowBox[{RowBox[{RowBox[{Hold[RowBox[{Hold[MakeBoxes[Slot, StandardForm]],
[, Hold[MakeBoxes[1, StandardForm]],]}]], –, Hold[RowBox[{Hold[MakeBoxes[ϵ, StandardForm]],
, Hold[MakeBoxes[Coefficient[<< 2 >> [[]], ϵ, $k], StandardForm]]}]]}], &}],
,, SubscriptBox[RowBox[{MapAt,
[, RowBox[{Hold[RowBox[{Hold[MakeBoxes[#1 + Times[–1, Times[Skeleton[2]]], StandardForm]], &}]],
,, Hold[SubscriptBox[Hold[MakeBoxes[MapAt[+<< 2 >> &, Subscript[MapAt[Skeleton[3]], 1], 3],
StandardForm]], Hold[MakeBoxes[1, StandardForm]]]],
,, 3}],]}], 1], ,, 3}],]}]}]

.

Rule: Pattern i_ appears on the right–hand side of rule RowBox[{SubscriptBox[OverscriptBox[aS, _], RowBox[{RowBox[{{, i_, }}], ,,
0}]], →, RowBox[{MapAt, [, RowBox[{RowBox[{RowBox[{#1, –, RowBox[{SuperscriptBox[ϵ, $k],
, RowBox[{Coefficient, [, RowBox[{RowBox[{Part,
[, Hold[RowBox[{≪, Hold[MakeBoxes[2, StandardForm]], ≫}]],]}], ,, ϵ, ,, $k}],]}]}]}], &}],
,, SubscriptBox[RowBox[{MapAt, [, RowBox[{RowBox[{RowBox[{#1, –, RowBox[{Times,
[, Hold[RowBox[{≪, Hold[MakeBoxes[2, StandardForm]], ≫}]],]}]}]}], &}],
,, SubscriptBox[RowBox[{MapAt,
[, RowBox[{RowBox[{Hold[RowBox[{Hold[MakeBoxes[Plus, StandardForm]],
[, Hold[MakeBoxes[<< 2 >>, StandardForm]],]}]], &}],
,, SubscriptBox[Hold[RowBox[{Hold[MakeBoxes[MapAt, StandardForm]],
[, Hold[MakeBoxes[<< 3 >>, StandardForm]],]}]], 1], ,, 3}],]}], 1], ,, 3}],]}], 1], ,, 3}],
]}]}].

General: Further output of Rule::rhs will be suppressed during this calculation.

$RecursionLimit: Recursion depth of 4096 exceeded during evaluation of MakeBoxes[{{{{{{{≪1≫}}}}}}}, StandardForm].

$RecursionLimit: Recursion depth of 4096 exceeded during evaluation of {{, Hold[MakeBoxes[{{{{{≪1≫}}}}}, StandardForm]], }}.

$RecursionLimit: Recursion depth of 4096 exceeded during evaluation of MakeBoxes[{{{{{{{≪1≫}}}}}}}, StandardForm].

General: Further output of $RecursionLimit::reclim2 will be suppressed during this calculation.

Out[∘]=

```
⋯ 1 ⋯
large output   show less   show more   show all   set size limit...
```

Check that on the generators this agrees with our conventions in the handout:

In[∘]:=

```
Timing@{{"[a,x]" → ((𝔼[0, 0, a₂ x₁] ~B₁,₂~ am₁,₂→₁)⟦3⟧ - (𝔼[0, 0, a₁ x₂] ~B₁,₂~ am₁,₂→₁)⟦3⟧),
    "[b,y]" → ((𝔼[0, 0, y₂ b₁] ~B₁,₂~ bm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, y₁ b₂] ~B₁,₂~ bm₁,₂→₁)⟦3⟧)} /.
   z_1 → z,
  {"Δ[y]" → Last[𝔼[0, 0, y₁] ~B₁~ bΔ₁→₁,₂],
   "Δ[b]" → Last[𝔼[0, 0, b₁] ~B₁~ bΔ₁→₁,₂],
   "Δ[a]" → Last[𝔼[0, 0, a₁] ~B₁~ aΔ₁→₁,₂],
   "Δ[x]" → Last[𝔼[0, 0, x₁] ~B₁~ aΔ₁→₁,₂]},
  {
    "S(a)" → ((𝔼[0, 0, a₁] ~B₁~ aS₁)⟦3⟧),
    "S(x)" → ((𝔼[0, 0, x₁] ~B₁~ aS₁)⟦3⟧),
    "S(b)" → ((𝔼[0, 0, b₁] ~B₁~ bS₁)⟦3⟧),
    "S(y)" → ((𝔼[0, 0, y₁] ~B₁~ bS₁)⟦3⟧)
  } /. z_1 → z}
```

Booted @ $k=2 in 9. sec.

Out[∘]=

$$\{9.25,\ \{\{[a,x]\to -x\,\gamma,\ [b,y]\to -y\,\epsilon+O[\epsilon]^3\},\ \{\Delta[y]\to (B_2\,y_1+y_2)+O[\epsilon]^3,\ \Delta[b]\to (b_1+b_2)+O[\epsilon]^3,\ \Delta[a]\to (a_1+a_2)+O[\epsilon]^3,$$
$$\Delta[x]\to (x_1+x_2)-\hbar\,a_1\,x_2\,\epsilon+\frac{1}{2}\hbar^2\,a_1^2\,x_2\,\epsilon^2+O[\epsilon]^3\},\ \{S(a)\to -a+O[\epsilon]^3,$$
$$S(x)\to -x-a\,x\,\hbar\,\epsilon-\frac{1}{2}\left(a^2\,x\,\hbar^2\right)\epsilon^2+O[\epsilon]^3,\ S(b)\to -b+O[\epsilon]^3,\ S(y)\to -\frac{y}{B}+O[\epsilon]^3\}\}\}$$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[∘]:=

```
Timing@Block[{$k = 3},
  HL /@ {(am₁,₂→₁ ~B₁~ am₁,₃→₁) ≡ (am₂,₃→₂ ~B₂~ am₁,₂→₁), (bm₁,₂→₁ ~B₁~ bm₁,₃→₁) ≡ (bm₂,₃→₂ ~B₂~ bm₁,₂→₁)}
 ]
```

Booted @ $k=3 in 92.062 sec.

Out[∘]= {92.2188, {True, True}}

R and *P* are inverses:

In[∘]:=

```
Timing@Block[{$k = 3}, {Rᵢ,ⱼ, Pᵢ,ₖ, HL[Rᵢ,ⱼ ~Bᵢ~ Pᵢ,ₖ ≡ 𝔼[aⱼ αₖ, xⱼ ξₖ, 1]]}]
```

Out[∘]=

$$\{0.046875,\ \{\mathbb{E}\Big[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1-\frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon+\left(\frac{1}{9}\gamma^2\,\hbar^5\,x_j^3\,y_i^3+\frac{1}{32}\gamma^2\,\hbar^6\,x_j^4\,y_i^4\right)\epsilon^2+$$
$$\frac{1}{1152}\left(24\,\gamma^3\,\hbar^5\,x_j^2\,y_i^2-72\,\gamma^3\,\hbar^7\,x_j^4\,y_i^4-32\,\gamma^3\,\hbar^8\,x_j^5\,y_i^5-3\,\gamma^3\,\hbar^9\,x_j^6\,y_i^6\right)\epsilon^3+O[\epsilon]^4\Big],$$
$$\mathbb{E}\Big[\frac{\alpha_k\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_k}{\hbar},\ 1+\frac{\gamma\,\eta_i^2\,\xi_k^2\,\epsilon}{4\,\hbar}+\frac{\left(36\,\gamma^2\,\hbar^2\,\eta_i^2\,\xi_k^2+40\,\gamma^2\,\hbar\,\eta_i^3\,\xi_k^3+9\,\gamma^2\,\eta_i^4\,\xi_k^4\right)\epsilon^2}{288\,\hbar^2}+\frac{1}{1152\,\hbar^3}$$
$$\left(48\,\gamma^3\,\hbar^4\,\eta_i^2\,\xi_k^2+192\,\gamma^3\,\hbar^3\,\eta_i^3\,\xi_k^3+156\,\gamma^3\,\hbar^2\,\eta_i^4\,\xi_k^4+40\,\gamma^3\,\hbar\,\eta_i^5\,\xi_k^5+3\,\gamma^3\,\eta_i^6\,\xi_k^6\right)\epsilon^3+O[\epsilon]^4\Big],\ \text{True}\}\}$$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

In[∘]:= $$\text{Timing}\left[\text{HL} \mathbin{/@} \left\{\overline{\text{aS}}_1 \sim \text{B}_1 \sim \text{aS}_1 \equiv \mathbb{E}[\text{a}_1\,\alpha_1,\ \text{x}_1\,\xi_1,\ 1],\ \overline{\text{bS}}_1 \sim \text{B}_1 \sim \text{bS}_1 \equiv \mathbb{E}[\text{b}_1\,\beta_1,\ \text{y}_1\,\eta_1,\ 1]\right\}\right]$$

Out[∘]= {0.171875, {True, True}}

(co)-associativity on both sides

In[∘]:= $$\text{Timing}\Big[\text{HL} \mathbin{/@} \Big\{(\text{a}\Delta_{1\to1,2} \sim \text{B}_2 \sim \text{a}\Delta_{2\to2,3}) \equiv (\text{a}\Delta_{1\to1,3} \sim \text{B}_1 \sim \text{a}\Delta_{1\to1,2}),\ (\text{b}\Delta_{1\to1,2} \sim \text{B}_2 \sim \text{b}\Delta_{2\to2,3}) \equiv (\text{b}\Delta_{1\to1,3} \sim \text{B}_1 \sim \text{b}\Delta_{1\to1,2}),$$
$$(\text{am}_{1,2\to1} \sim \text{B}_1 \sim \text{am}_{1,3\to1}) \equiv (\text{am}_{2,3\to2} \sim \text{B}_2 \sim \text{am}_{1,2\to1}),\ (\text{bm}_{1,2\to1} \sim \text{B}_1 \sim \text{bm}_{1,3\to1}) \equiv (\text{bm}_{2,3\to2} \sim \text{B}_2 \sim \text{bm}_{1,2\to1})\Big\}\Big]$$

Out[∘]= {0.40625, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:= $$\text{Timing}\Big[\text{HL} \mathbin{/@} \Big\{\text{am}_{1,2\to1} \sim \text{B}_1 \sim \text{a}\Delta_{1\to1,2} \equiv (\text{a}\Delta_{1\to1,3}\ \text{a}\Delta_{2\to2,4}) \sim \text{B}_{1,2,3,4} \sim (\text{am}_{3,4\to2}\ \text{am}_{1,2\to1}),$$
$$\text{bm}_{1,2\to1} \sim \text{B}_1 \sim \text{b}\Delta_{1\to1,2} \equiv (\text{b}\Delta_{1\to1,3}\ \text{b}\Delta_{2\to2,4}) \sim \text{B}_{1,2,3,4} \sim (\text{bm}_{3,4\to2}\ \text{bm}_{1,2\to1})\Big\}\Big]$$

Out[∘]= {0.75, {True, True}}

S is convolution inverse of id

In[∘]:= $$\text{Timing}\Big[\text{HL}[\#\equiv \mathbb{E}[0,\ 0,\ 1]]\ \&\ \mathbin{/@} \Big\{$$
$$(\text{a}\Delta_{1\to1,2} \sim \text{B}_1 \sim \text{aS}_1) \sim \text{B}_{1,2} \sim \text{am}_{1,2\to1},\ (\text{a}\Delta_{1\to1,2} \sim \text{B}_2 \sim \text{aS}_2) \sim \text{B}_{1,2} \sim \text{am}_{1,2\to1},$$
$$(\text{b}\Delta_{1\to1,2} \sim \text{B}_1 \sim \text{bS}_1) \sim \text{B}_{1,2} \sim \text{bm}_{1,2\to1},\ (\text{b}\Delta_{1\to1,2} \sim \text{B}_2 \sim \text{bS}_2) \sim \text{B}_{1,2} \sim \text{bm}_{1,2\to1}\Big\}\Big]$$

Out[∘]= {0.671875, {True, True, True, True}}

S is an algebra anti-(co)morphism

In[∘]:= $$\text{Timing}\Big[\text{HL} \mathbin{/@} \Big\{\text{am}_{1,2\to1} \sim \text{B}_1 \sim \text{aS}_1 \equiv (\text{aS}_1\ \text{aS}_2) \sim \text{B}_{1,2} \sim \text{am}_{2,1\to1},\ \text{bm}_{1,2\to1} \sim \text{B}_1 \sim \text{bS}_1 \equiv (\text{bS}_1\ \text{bS}_2) \sim \text{B}_{1,2} \sim \text{bm}_{2,1\to1},$$
$$\text{aS}_1 \sim \text{B}_1 \sim \text{a}\Delta_{1\to1,2} \equiv \text{a}\Delta_{1\to2,1} \sim \text{B}_{1,2} \sim (\text{aS}_1\ \text{aS}_2),\ \text{bS}_1 \sim \text{B}_1 \sim \text{b}\Delta_{1\to1,2} \equiv \text{b}\Delta_{1\to2,1} \sim \text{B}_{1,2} \sim (\text{bS}_1\ \text{bS}_2)\Big\}\Big]$$

Out[∘]= {1.04688, {True, True, True, True}}

Pairing axioms

In[∘]:= $$\text{Timing}\Big[\text{HL} \mathbin{/@} \Big\{(\text{bm}_{1,2\to1}\ \mathbb{E}[\alpha_3\,\text{a}_3,\ \xi_3\,\text{x}_3,\ 1]) \sim \text{B}_{1,3} \sim \text{P}_{1,3} \equiv$$
$$(\mathbb{E}[\beta_1\,\text{b}_1,\ \eta_1\,\text{y}_1,\ 1]\ \mathbb{E}[\beta_2\,\text{b}_2,\ \eta_2\,\text{y}_2,\ 1]\ \text{a}\Delta_{3\to4,5}) \sim \text{B}_{1,4} \sim \text{P}_{1,4} \sim \text{B}_{2,5} \sim \text{P}_{2,5},$$
$$(\text{b}\Delta_{1\to1,2}\ \mathbb{E}[\alpha_3\,\text{a}_3,\ \xi_3\,\text{x}_3,\ 1]\ \mathbb{E}[\alpha_4\,\text{a}_4,\ \xi_4\,\text{x}_4,\ 1]) \sim \text{B}_{1,3} \sim \text{P}_{1,3} \sim \text{B}_{2,4} \sim \text{P}_{2,4} \equiv$$
$$(\mathbb{E}[\beta_1\,\text{b}_1,\ \eta_1\,\text{y}_1,\ 1]\ \text{am}_{3,4\to3}) \sim \text{B}_{1,3} \sim \text{P}_{1,3}\Big\}\Big]$$

Out[∘]= {0.484375, {True, True}}

In[∘]:= $$\text{Timing}\Big[\text{HL} \mathbin{/@} \Big\{(\text{bS}_1\ \mathbb{E}[\alpha_2\,\text{a}_2,\ \xi_2\,\text{x}_2,\ 1]) \sim \text{B}_{1,2} \sim \text{P}_{1,2} \equiv (\mathbb{E}[\beta_1\,\text{b}_1,\ \eta_1\,\text{y}_1,\ 1]\ \text{aS}_2) \sim \text{B}_{1,2} \sim \text{P}_{1,2},$$
$$(\overline{\text{bS}}_1\ \mathbb{E}[\alpha_2\,\text{a}_2,\ \xi_2\,\text{x}_2,\ 1]) \sim \text{B}_{1,2} \sim \text{P}_{1,2} \equiv (\mathbb{E}[\beta_1\,\text{b}_1,\ \eta_1\,\text{y}_1,\ 1]\ \overline{\text{aS}}_2) \sim \text{B}_{1,2} \sim \text{P}_{1,2}\Big\}\Big]$$

Out[∘]= {0.34375, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[∘]:= Timing@{{
        "[a,y]" → ((𝔼[0, 0, y₂ a₁] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, y₁ a₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧),
        "[b,x]"  → ((𝔼[0, 0, x₂ b₁] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, x₁ b₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧),
        "xy-qyx" → ((𝔼[0, 0, x₁ y₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ - (1 + ϵ) (𝔼[0, 0, y₁ x₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧)
      } /. {z_1 → z} // Expand // Factor,
    {
      "Δ(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
      "Δ(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
      "Δ(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
      "Δ(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧)
    } // Simplify,
    {
      "S(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ dS₁)⟦3⟧),
      "S(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ dS₁)⟦3⟧),
      "S(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ dS₁)⟦3⟧),
      "S(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ dS₁)⟦3⟧)
    } /. {z_1 → z} // Simplify
  }
```

Out[∘]= $\{3.90625, \{\{[a,y] \to -y\,\gamma + O[\epsilon]^3,\ [b,x] \to x\,\epsilon + O[\epsilon]^3,$

$\text{xy-qyx} \to \left(-x\,y + \frac{1 - B + x\,y\,\hbar}{\hbar}\right) + (a\,B - x\,y + x\,y\,\gamma\,\hbar)\,\epsilon + \frac{1}{2}\left(-a^2\,B\,\hbar + x\,y\,\gamma^2\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3\},$

$\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\epsilon^2 + O[\epsilon]^3,$

$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1\,y_2) + O[\epsilon]^3\},$

$\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}\left(a^2\,x\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3,$

$S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\,\gamma\,\hbar\,\epsilon}{B} - \frac{\left(y\,\gamma^2\,\hbar^2\right)\epsilon^2}{2\,B} + O[\epsilon]^3\}\}\}$

(co)-associativity

```
In[∘]:= Timing[HL /@
     {(dΔ₁→₁,₂ ~ B₂ ~ dΔ₂→₂,₃) ≡ (dΔ₁→₁,₃ ~ B₁ ~ dΔ₁→₁,₂), (dm₁,₂→₁ ~ B₁ ~ dm₁,₃→₁) ≡ (dm₂,₃→₂ ~ B₂ ~ dm₁,₂->₁)}]
```

Out[∘]= {8.15625, {True, True}}

Δ is an algebra morphism

```
In[∘]:= Timing@HL[dm₁,₂→₁ ~ B₁ ~ dΔ₁→₁,₂ ≡ (dΔ₁→₁,₃ dΔ₂→₂,₄) ~ B₁,₂,₃,₄ ~ (dm₃,₄→₂ dm₁,₂→₁)]
```

Out[∘]= {16.5, True}

S is convolution inverse of id

```
In[∘]:= Timing[
     HL[# ≡ 𝔼[0, 0, 1]] & /@ {(dΔ₁→₁,₂ ~ B₁ ~ dS₁) ~ B₁,₂ ~ dm₁,₂→₁, (dΔ₁→₁,₂ ~ B₂ ~ dS₂) ~ B₁,₂ ~ dm₁,₂→₁}]
```

Out[∘]= {12.5469, {True, True}}

S is a (co)-algebra anti-morphism

In[∘]:= $\text{Timing}[\text{HL} /@ \text{Expand} /@ \{dm_{1,2\to1} \sim B_1 \sim dS_1 \equiv (dS_1\, dS_2) \sim B_{1,2} \sim dm_{2,1\to1},\ dS_1 \sim B_1 \sim d\Delta_{1\to1,2} \equiv d\Delta_{1\to2,1} \sim B_{1,2} \sim (dS_1\, dS_2)\}]$

Out[∘]= {28.1563, {True, True}}

Quasi-triangular axiom 1:

In[∘]:= $\text{Timing@HL}[R_{1,2} \sim B_1 \sim d\Delta_{1\text{->}1,3} \equiv (R_{1,4}\, R_{3,2}) \sim B_{2,4} \sim dm_{2,4\to2}]$

Out[∘]= {0.765625, True}

Quasi-triangular axiom 2:

In[∘]:= $\text{Timing@HL}[((d\Delta_{1\text{->}1,2}\, R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\, dm_{2,4\to2})) \equiv ((d\Delta_{1\to2,1}\, R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1\to1}\, dm_{4,2\to2}))]$

Out[∘]= {12.4219, True}

Reidemeister 2:

In[∘]:= $\text{Timing}[\text{HL}[\# \equiv \mathbb{E}[0, 0, 1]] \,\&\, /@ \{(\overline{R}_{1,2}\, R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\, dm_{2,4\to2}),\ (R_{1,2}\, \overline{R}_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\, dm_{2,4\to2})\}]$

Out[∘]= {8.25, {True, True}}

Reidemeister 3:

In[∘]:= $\text{Timing@HL}[((R_{1,2}\, R_{4,3}\, R_{5,6}) \sim B_{1,4} \sim dm_{1,4\to1} \sim B_{2,5} \sim dm_{2,5\to2} \sim B_{3,6} \sim dm_{3,6\to3}) \equiv ((R_{1,6}\, R_{2,3}\, R_{4,5}) \sim B_{1,4} \sim dm_{1,4\to1} \sim B_{2,5} \sim dm_{2,5\to2} \sim B_{3,6} \sim dm_{3,6\to3})]$

Out[∘]= {5.90625, True}

Deriving the Drinfeld element *u* and its inverse $\overline{u}$

In[∘]:= $\text{Block}[\{i\}, \{$
$u_{i\_} = R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1\to i},$
$\overline{u}_{i\_} = R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1\to i}$
$\}]$

Out[∘]=
$$\Big\{\mathbb{E}\Big[-\hbar\, a_i\, b_i,\ -\frac{\hbar\, x_i\, y_i}{B_i},\ B_i + \frac{1}{4 B_i}\left(-4\hbar\, a_i\, B_i^2 - 4\gamma\hbar^2 B_i\, x_i\, y_i - 4\hbar^2 a_i\, B_i\, x_i\, y_i - 3\gamma\hbar^3 x_i^2\, y_i^2\right)\epsilon + \frac{1}{288 B_i^3}\left(144\hbar^2 a_i^2 B_i^4 - 144\gamma^2\hbar^3 B_i^3\, x_i\, y_i + 144\hbar^3 a_i^2 B_i^3\, x_i\, y_i - 144\gamma^2\hbar^4 B_i^2\, x_i^2\, y_i^2 + 72\gamma\hbar^4 a_i\, B_i^2\, x_i^2\, y_i^2 + 144\hbar^4 a_i^2 B_i^2\, x_i^2\, y_i^2 - 104\gamma^2\hbar^5 B_i\, x_i^3\, y_i^3 + 216\gamma\hbar^5 a_i\, B_i\, x_i^3\, y_i^3 + 81\gamma^2\hbar^6 x_i^4\, y_i^4\right)\epsilon^2 + O[\epsilon]^3\Big],$$
$$\mathbb{E}\Big[\hbar\, a_i\, b_i,\ \hbar\, x_i\, y_i,\ \frac{1}{B_i} + \frac{\left(4\hbar\, a_i - 4\gamma\hbar^2 x_i\, y_i - \gamma\hbar^3 x_i^2\, y_i^2\right)\epsilon}{4 B_i} + \frac{1}{288 B_i}\left(144\hbar^2 a_i^2 + 144\gamma^2\hbar^3 x_i\, y_i - 288\gamma\hbar^3 a_i\, x_i\, y_i + 288\gamma^2\hbar^4 x_i^2\, y_i^2 - 72\gamma\hbar^4 a_i\, x_i^2\, y_i^2 + 104\gamma^2\hbar^5 x_i^3\, y_i^3 + 9\gamma^2\hbar^6 x_i^4\, y_i^4\right)\epsilon^2 + O[\epsilon]^3\Big]\Big\}$$

u and $\overline{u}$ are inverses

In[∘]:= $\text{Timing@HL}[(u_1\, \overline{u}_2) \sim B_{1,2} \sim dm_{1,2\to1} \equiv \mathbb{E}[0, 0, 1]]$

Out[∘]= {1.48438, True}

The ribbon element v satisfies $v^2 = S(u)\, u$. The spinner $C = uv^{-1}$.

It is convenient to compute $z = S(u)\, u^{-1}$ which is something easy.

In[∘]:= $\text{Block}[\{\$k = 3\}, ((u_1 \sim B_1 \sim dS_1)\ \overline{u}_2) \sim B_{1,2} \sim dm_{1,2\to1}]$

Out[∘]= $\mathbb{E}\left[0, 0, \frac{1}{B_1} + \frac{\hbar a_1 \epsilon}{B_1} + \frac{\hbar^2 a_1^2 \epsilon^2}{2 B_1} + O[\epsilon]^3\right]$

(* Needs fixing! *) So in our case S(u) = u z so S(u)u = $u^2 z$ and v = $uz^{\frac{1}{2}}$ and finally C = uv^{-1} = $z^{-\frac{1}{2}} = e^{t_1/2}(1-\epsilon a_1)$.

In[∘]:= $\text{Block}[\{i\},$
$\{CC_{i\_} = \mathbb{E}[0, 0, B_i^{1/2} e^{-\epsilon a_i/2} + O[\epsilon]^2],$
$\overline{CC}_{i\_} = \mathbb{E}[0, 0, B_i^{-1/2} e^{\epsilon a_i/2} + O[\epsilon]^2]$
$\}]$

Out[∘]= $\left\{\mathbb{E}\left[0, 0, \sqrt{B_i} - \frac{1}{2}\left(a_i \sqrt{B_i}\right)\epsilon + O[\epsilon]^2\right], \mathbb{E}\left[0, 0, \frac{1}{\sqrt{B_i}} + \frac{a_i \epsilon}{2\sqrt{B_i}} + O[\epsilon]^2\right]\right\}$

In[∘]:= $\text{Block}[\{i, j\}, \{$
$Kink_{i\_} = (R_{1,3}\ \overline{CC}_2) \sim B_{1,2} \sim dm_{1,2\to1} \sim B_{1,3} \sim dm_{1,3\to i},$
$\overline{Kink}_{j\_} = (\overline{R}_{1,3}\ CC_2) \sim B_{1,2} \sim dm_{1,2\to1} \sim B_{1,3} \sim dm_{1,3\to j}$
$\}]$

Out[∘]= $\left\{\mathbb{E}\left[\hbar a_i b_i, \hbar x_i y_i, \frac{1}{\sqrt{B_i}} + \frac{(2 a_i - \gamma \hbar^3 x_i^2 y_i^2)\epsilon}{4\sqrt{B_i}} + O[\epsilon]^2\right],\right.$
$\left.\mathbb{E}\left[-\hbar a_j b_j, -\frac{\hbar x_j y_j}{B_j}, \sqrt{B_j} + \frac{(-2 a_j B_j^2 - 4\hbar^2 a_j B_j x_j y_j - 3\gamma\hbar^3 x_j^2 y_j^2)\epsilon}{4 B_j^{3/2}} + O[\epsilon]^2\right]\right\}$

In[∘]:= $k2 = (R_{3,1}\ CC_2) \sim B_{1,2} \sim dm_{1,2\to1} \sim B_{1,3} \sim dm_{1,3\to i}$ /. $e \to E$;
$k4 = (\overline{R}_{3,1}\ \overline{CC}_2) \sim B_{1,2} \sim dm_{1,2\to1} \sim B_{1,3} \sim dm_{1,3\to j}$ /. $e \to E$;
Simplify@$\{Kink_i \equiv k2, \overline{Kink}_j \equiv k4, (Kink_i\ \overline{Kink}_j) \sim B_{i,j} \sim dm_{i,j\to1}\}$

Out[∘]= $\left\{\frac{1}{\sqrt{B_i}}\epsilon\,(-1+\hbar)\,(\gamma + \gamma\hbar x_i y_i + B_i(-2 a_i + \gamma(-1 + \hbar x_i y_i))) == 0,\right.$
$\frac{1}{\sqrt{B_j}}\epsilon\,(-1+\hbar)\,(-\gamma B_j^2 + \gamma\hbar x_j y_j + B_j(\gamma + 2 a_j + \gamma\hbar x_j y_j)) == 0,$
$\left.\mathbb{E}\left[0, 0, 1 - \frac{(\gamma(-1+\hbar)\hbar(1+B_1) x_1 y_1)\epsilon}{2 B_1} + O[\epsilon]^2\right]\right\}$

Reidemeister 2:

In[∘]:= $(R_{1,2}\ \overline{R}_{3,4}) \sim B_{1,3} \sim dm_{1,3\to1} \sim B_{2,4} \sim dm_{2,4\to2}$

Out[∘]= $\mathbb{E}[0, 0, 1 + O[\epsilon]^3]$

Cyclic Reidemeister 2:

In[∘]:= $(R_{1,4}\ \overline{R}_{5,2}\ \overline{CC}_3) \sim B_{2,4} \sim dm_{2,4\to2} \sim B_{1,3} \sim dm_{1,3\to1} \sim B_{1,5} \sim dm_{1,5\to1} \equiv \overline{CC}_1$

Out[∘]= $\frac{\gamma\epsilon\hbar x_2 y_1 - \gamma\epsilon\hbar^2 x_2 y_1}{2\sqrt{B_1}} == 0$

Trefoil

In[∘]:=
$$Z = R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10};$$
$$\text{Do}[Z = Z \sim B_{1,r} \sim dm_{1,r\to 1},\ \{r, 2, 10\}];$$
$$\text{Simplify} /@ Z$$

Out[∘]= $Aborted

Out[∘]=
$$\mathbb{E}\Big[\hbar\,(a_7 b_1 + a_1 (b_1 + b_6) - a_8 b_8 - a_9 b_9 - a_{10} b_{10}),$$
$$\hbar\left(B_6 x_7 y_1 + x_7 y_6 - B_1 x_7 y_6 + x_1 (y_1 + B_1 y_6) - \frac{x_8 y_8}{B_8} - \frac{x_9 y_9}{B_9} - \frac{x_{10} y_{10}}{B_{10}}\right),$$
$$\frac{\sqrt{B_8}\sqrt{B_9}\sqrt{B_{10}}}{\sqrt{B_1}} + \frac{1}{4\sqrt{B_1}\, B_8^{3/2} B_9^{3/2} B_{10}^{3/2}}$$
$$\big(-2 a_9 B_8^2 B_9^2 B_{10}^2 - 2 a_{10} B_8^2 B_9^2 B_{10}^2 - \gamma\hbar^3 B_8^2 B_9^2 B_{10}^2 x_1^2 y_1^2 - \gamma\hbar^3 B_6^2 B_8^2 B_9^2 B_{10}^2 x_7^2 y_1^2 - 2\gamma\hbar B_1 B_8^2 B_9^2 B_{10}^2 x_1 y_6 -$$
$$2\gamma\hbar^2 B_1 B_8^2 B_9^2 B_{10}^2 x_1 y_6 - 4\hbar^2 a_7 B_1 B_8^2 B_9^2 B_{10}^2 x_1 y_6 + 8\gamma\hbar^3 B_1 B_6 B_8^2 B_9^2 B_{10}^2 x_1 x_7 y_1 y_6 -$$
$$4\gamma\hbar^3 B_1 B_6 B_8^2 B_9^2 B_{10}^2 x_7^2 y_1 y_6 - \gamma\hbar^3 B_1^2 B_8^2 B_9^2 B_{10}^2 x_1^2 y_6^2 - 4\gamma\hbar^3 B_1^2 B_8^2 B_9^2 B_{10}^2 x_1 x_7 y_6^2 -$$
$$\gamma\hbar^3 B_8^2 B_9^2 B_{10}^2 x_7^2 y_6^2 + \gamma\hbar^3 B_1^2 B_8^2 B_9^2 B_{10}^2 x_7^2 y_6^2 + 2 a_1 B_8^2 B_9^2 B_{10}^2 (1 - 2\hbar^2 B_6 x_7 y_1 + 2\hbar^2 B_1 x_7 y_6) -$$
$$3\gamma\hbar^3 B_9^2 B_{10}^2 x_8^2 y_8^2 - 2 a_8 B_8 B_9^2 B_{10}^2 (B_8 + 2\hbar^2 x_8 y_8) - 4\hbar^2 a_9 B_8^2 B_9 B_{10}^2 x_9 y_9 -$$
$$3\gamma\hbar^3 B_8^2 B_{10}^2 x_9^2 y_9^2 - 4\hbar^2 a_{10} B_8^2 B_9^2 B_{10} x_{10} y_{10} - 3\gamma\hbar^3 B_8^2 B_9^2 x_{10}^2 y_{10}^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

$$\text{Timing}\big[$$
$$Z = R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10};$$
$$\text{Do}[Z = Z \sim B_{1,r} \sim dm_{1,r\to 1},\ \{r, 2, 10\}];$$
$$\text{Simplify} /@ Z\ \big]$$

In[∘]:=
$$R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10}$$

Out[∘]=
$$\mathbb{E}\Big[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},$$
$$x_5 y_1 + x_7 y_3 + x_2 y_6 - \frac{x_8 y_8}{B_8} - \frac{x_9 y_9}{B_9} - \frac{x_{10} y_{10}}{B_{10}},\ \frac{\sqrt{B_8}\sqrt{B_9}\sqrt{B_{10}}}{\sqrt{B_4}} +$$
$$\left(\sqrt{B_{10}}\left(\sqrt{B_9}\left(\sqrt{B_8}\left(\frac{a_4}{2\sqrt{B_4}} - \frac{x_5^2 y_1^2}{4\sqrt{B_4}} - \frac{x_7^2 y_3^2}{4\sqrt{B_4}} - \frac{x_2^2 y_6^2}{4\sqrt{B_4}}\right) + \frac{-2 a_8 B_8^2 - 4 a_8 B_8 x_8 y_8 - 3 x_8^2 y_8^2}{4\sqrt{B_4}\, B_8^{3/2}}\right) +\right.\right.$$
$$\left.\frac{\sqrt{B_8}\,(-2 a_9 B_9^2 - 4 a_9 B_9 x_9 y_9 - 3 x_9^2 y_9^2)}{4\sqrt{B_4}\, B_9^{3/2}}\right) +$$
$$\left.\frac{1}{4\sqrt{B_4}\, B_{10}^{3/2}}\sqrt{B_8}\sqrt{B_9}\,(-2 a_{10} B_{10}^2 - 4 a_{10} B_{10} x_{10} y_{10} - 3 x_{10}^2 y_{10}^2)\right)\epsilon + O[\epsilon]^2\Big]$$

In[ ]:=
```
(R1,5 R6,2 R3,7 CC̄4 Kink̄8 Kink̄9 Kink̄10) ~B_Range[10] ~Product[b2t_i, {i, 10}]
```

Out[ ]=
$$\mathbb{E}\Big[-a_5 t_1 - a_7 t_3 - a_2 t_6 + a_8 t_8 + a_9 t_9 + a_{10} t_{10},\ \frac{1}{T_8 T_9 T_{10}}(T_8 T_9 T_{10} x_5 y_1 + T_8 T_9 T_{10} x_7 y_3 + T_8 T_9 T_{10} x_2 y_6 - T_9 T_{10} x_8 y_8 - T_8 T_{10} x_9 y_9 - T_8 T_9 x_{10} y_{10}),\ \frac{\sqrt{T_8}\sqrt{T_9}\sqrt{T_{10}}}{\sqrt{T_4}} + \frac{1}{4\sqrt{T_4}\, T_8^{3/2} T_9^{3/2} T_{10}^{3/2}}\big(4 a_4 T_8^2 T_9^2 T_{10}^2 + 4 a_1 a_5 T_8^2 T_9^2 T_{10}^2 + 4 a_2 a_6 T_8^2 T_9^2 T_{10}^2 + 4 a_3 a_7 T_8^2 T_9^2 T_{10}^2 - 4 a_8 T_8^2 T_9^2 T_{10}^2 - 4 a_8^2 T_8^2 T_9^2 T_{10}^2 - 4 a_9 T_8^2 T_9^2 T_{10}^2 - 4 a_9^2 T_8^2 T_9^2 T_{10}^2 - 4 a_{10} T_8^2 T_9^2 T_{10}^2 - 4 a_{10}^2 T_8^2 T_9^2 T_{10}^2 - T_8^2 T_9^2 T_{10}^2 x_5^2 y_1^2 - T_8^2 T_9^2 T_{10}^2 x_7^2 y_3^2 - T_8^2 T_9^2 T_{10}^2 x_2^2 y_6^2 - 8 a_8 T_8 T_9^2 T_{10}^2 x_8 y_8 - 3 T_9^2 T_{10}^2 x_8^2 y_8^2 - 8 a_9 T_8^2 T_9 T_{10}^2 x_9 y_9 - 3 T_8^2 T_{10}^2 x_9^2 y_9^2 - 8 a_{10} T_8^2 T_9^2 T_{10} x_{10} y_{10} - 3 T_8^2 T_9^2 x_{10}^2 y_{10}^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

In[ ]:=
```
Z = (((R1,5 R6,2 R3,7 CC̄4 Kink̄8 Kink̄9 Kink̄10) ~B_Range[10] ~Product[b2t_i, {i, 10}]) /. T_ → T1) ~
  B_Range[10] ~Product[t2b_i, {i, 10}]
```

Out[ ]=
$$\mathbb{E}\Big[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},\ \frac{1}{B_1}(B_1 x_5 y_1 + B_1 x_7 y_3 + B_1 x_2 y_6 - x_8 y_8 - x_9 y_9 - x_{10} y_{10}),\ B_1 + \frac{1}{4 B_1}\big(4 a_1 B_1^2 + 4 a_4 B_1^2 - 4 a_8 B_1^2 - 4 a_9 B_1^2 - 4 a_{10} B_1^2 - B_1^2 x_5^2 y_1^2 - B_1^2 x_7^2 y_3^2 - B_1^2 x_2^2 y_6^2 + 4 a_1 B_1 x_8 y_8 - 8 a_8 B_1 x_8 y_8 - 3 x_8^2 y_8^2 + 4 a_1 B_1 x_9 y_9 - 8 a_9 B_1 x_9 y_9 - 3 x_9^2 y_9^2 + 4 a_1 B_1 x_{10} y_{10} - 8 a_{10} B_1 x_{10} y_{10} - 3 x_{10}^2 y_{10}^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

```
Timing[
 Do[Z = Z~B1,r~dm1,r→1, {r, 2, 10}];
 Simplify@Z[[3]] ]
```

```
doing 2
doing 3
doing 4
doing 5
doing 6
doing 7
doing 8
doing 9
doing 10
```

Out[ ]=
$$\Big\{5.39063,\ \frac{B_1}{1 - B_1 + B_1^2} + \left(B_1\left(-B_1 + 2 B_1^2 + 2 B_1^4 + a_1\left(-1 + B_1 - B_1^3 + B_1^4\right) - 2 x_1 y_1 - B_1^3\left(3 + 2 x_1 y_1\right)\right)\epsilon\right) \Big/ \left(1 - B_1 + B_1^2\right)^3 + O[\epsilon]^2\Big\}$$

In[ ]:=
```
Timing[
 Z = R1,5 R6,2 R3,7 CC̄4 Kink̄8 Kink̄9 Kink̄10 /. B_ → B1;
 Do[Print["doing ", r]; Z = Z~B1,r~dm1,r→1 /. B_ → B1, {r, 2, 10}];
 Simplify@Z[[3]] ]
```

```
doing 2
doing 3
doing 4
doing 5
doing 6
doing 7
doing 8
doing 9
doing 10
```

Out[ ]=

$$\left\{5.3125, \frac{B_1}{1-B_1+B_1^2}+\frac{1}{2 B_1 \left(1-B_1+B_1^2\right)^3}\left(2 a_1 B_1^2 \left(-1+B_1-B_1^3+B_1^4\right)-6 x_1^2 y_1^2+4 B_1^7 x_1^2 y_1^2-2 B_1^8 x_1^2 y_1^2+B_1^2 x_1 y_1 (5-6 x_1 y_1)+3 B_1 x_1 y_1 \left(-1+2 x_1 y_1\right)+B_1^6 \left(3+3 x_1 y_1-6 x_1^2 y_1^2\right)-B_1^5 \left(4+13 x_1 y_1+2 x_1^2 y_1^2\right)+B_1^4 \left(2+15 x_1 y_1+4 x_1^2 y_1^2\right)-B_1^3 \left(1+15 x_1 y_1+6 x_1^2 y_1^2\right)\right) \epsilon+O[\epsilon]^2\right\}$$