Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

Profiling

In[∘]:=
```
Once[
  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Invariant"];
  << KnotTheory`;
  << "../Profile/Profile.m";
 ];
BeginProfile[];
Once@PopupWindow[Button["Show Profile Monitor"],
  Dynamic[PrintProfile[], UpdateInterval → 3, TrackedSymbols → {}]]
```

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: June 2018. Original version: July 1994.
```

Out[∘]= Show Profile Monitor

External Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

Program

Canonical Form:

Program

In[∘]:=
```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
      Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
   ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

Program

The Kronecker δ:

Program

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_{$k_} := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

In[∘]:=
```
𝔼3@𝔼[ω_, L_, Q_, Ps_] := CF /@ 𝔼[L, ω^-1 Q, ω^-1 (ω^-4 ϵ)^(-1+Range@Length@Ps).Ps]_{$k};
𝔼4@𝔼[L_, Q_, P_] := Module[
   {ω = Normal[P]^-1 /. ϵ → 0, Ps = CoefficientList[P, ϵ]},
   CF /@ 𝔼[ω, L, ω Q, ω^(-3+4 Range@Length@Ps) Ps]];
𝔼3@𝔼_{sp___}[as___] := 𝔼3@𝔼[as] /. 𝔼 → 𝔼_{sp};
𝔼4@𝔼_{sp___}[as___] := 𝔼4@𝔼[as] /. 𝔼 → 𝔼_{sp};
```

Program

Zip and Bind

Program

Variables and their duals:

Program

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})* := (u*)_i;
```

Program

Finite Zips:

Program

In[∘]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := PP_{Collect}@Collect[ℰ, ζ];
Zip_{{}}[P_] := P;
Zip_{ζs_}[Ps_List] := Zip_{ζs} /@ Ps;
Zip_{{ζ_,ζs___}}[P_] := PP_{Zip}[
  (collect[P // Zip_{{ζs}}, ζ] /. f_. ζ^{d_.} ↦ ∂_{{ζ*,d}} f) /. ζ* → 0]
```

Program

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$ and/or on $\mathbb{E}(\omega, L, Q, P) = \omega^{-1} e^{L+\omega^{-1} Q} P(\omega^{-4} \epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde q|\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i}\Big|_{z_i\to\tilde q^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde q| e^{c+\eta^i\tilde q^k_i y_k}\left\langle P\left(\tilde q^k_i(z_k+y_k),\zeta^j+\eta^i\tilde q^j_i\right)\right\rangle.$$

Program

In[ ]:=
```
$QZipFail = False;
QZip_{ζs_List}@𝔼[L_, Q_, P_] := PP_{QZip}@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    out = CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]];
    If[¬ ($QZipFail ∨ TrueQ[out ≡ 𝔼3@QZip_{ζs}@𝔼4@𝔼[L, Q, P]]),
     $QZipFail = True; Print["QZip4 fail at {L,Q,P}=", {L, Q, P}]
    ];
    out
   ];
```

Program

In[ ]:=
```
$QZipFail = False;
QZip_{ζs_List}@𝔼[ω_, L_, Q_, Ps_] := PP_{QZip4}@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[ω Det[qt / ω], L, c + ηs.qt.ys, Zip_{ζs}[Ps /. (zrule ⋃ ζrule)]] ];
```

Program

Upper to lower and lower to Upper:

Program

In[ ]:=
```
U2l = {B_{i_}^{p_.} → e^{-p ħ γ b_i}, B^{p_.} → e^{-p ħ γ b}, T_{i_}^{p_.} → e^{p ħ t_i}, T^{p_.} → e^{p ħ t}, 𝒜_{i_}^{p_.} → e^{p γ α_i}, 𝒜^{p_.} → e^{p γ α}};
l2U = {e^{c_. b_{i_} + d_.} :→ B_i^{-c/(ħ γ)} e^d, e^{c_. b + d_.} :→ B^{-c/(ħ γ)} e^d,
    e^{c_. t_{i_} + d_.} :→ T_i^{c/ħ} e^d, e^{c_. t + d_.} :→ T^{c/ħ} e^d,
    e^{c_. α_{i_} + d_.} :→ 𝒜_i^{c/γ} e^d, e^{c_. α + d_.} :→ 𝒜^{c/γ} e^d,
    e^{ℰ_} :→ e^{Expand@ℰ}};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single"P". Here the z's are b and α and the ζ's are β and a.

Program

In[∘]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  PP_LZip@Module[{ζ, z, zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    Zrule = zrule /. r_Rule ⧴
       ((U = r⟦1⟧ /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U)); (* not used *)
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. U2l /. (zrule ⋃ ζrule);
    EEQ[ps___] := EEQ[ps] = PP_"EEQ"@(CF[e^-Q1 D[e^Q1, Sequence @@ Thread[{zs, {ps}}]] /.
          Alternatives @@ zs → 0 //. l2U);
    CF /@ ((*CF/@*)𝔼[
        c + ηs.lt.ys, Q1 /. Alternatives @@ zs → 0,
        Det[lt] (Zip_ζs[(EQ @@ zs) (P /. U2l /. (zrule ⋃ ζrule))] /.
            Derivative[ps___][EQ][___] ⧴ EEQ[ps] /. _EQ → 1)
       ] //. l2U)
   ];
```

Program

In[∘]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_B@Module[{n},
    Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
     ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

Program

𝔼 morphisms with domain and range.

Program

In[∘]:=
```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_{d_→r_}[L_, Q_, P_]_{$k_} := 𝔼_{d→r} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

Program

"Define" code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

Program

Booting Up

Program

In[∘]:=
```
$k = 2; (*ℏ=γ=1;*)
```

Program

In[∘]:=
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_$k,
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i}-1) η_j y_k}]_$k]
```

Program

In[∘]:=
```
Define[R_{i,j} = CF@𝔼_{{}→{i,j}}[ℏ a_j b_i, ℏ x_j y_i, e^(Sum_{k=2}^{$k+1} ((1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k)/(k (1 - e^{k γ ϵ ℏ})))]_$k,
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_$k[3] -
       (((R̄_{{i,j},0})_$k R_{1,2} (R̄_{{3,4},$k-1})_$k) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_$k[3] -
      (R_{1,2} // ((P_{{1,j},0})_$k (P_{{i,2},$k-1})_$k))[3]]]]
```

Program

In[∘]:=
```
Define[aS_j = R̄_{i,j} ~ B_i ~ P_{i,j},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_$k[3] -
       ((aS̄_{{i},0})_$k ~ B_i ~ aS_i ~ B_i ~ (aS̄_{{i},$k-1})_$k)[3]]]]
```

Program

In[∘]:=
```
Define[bS_i = R_{i,1} ~ B_1 ~ aS_1 ~ B_1 ~ P_{i,1},
  bS̄_i = R_{i,1} ~ B_1 ~ aS̄_1 ~ B_1 ~ P_{i,1},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

Program

In[∘]:=
```
Define[dm_{i,j→k} = (𝔼_{{i,j}→{i,j}}[β_i b_i + α_j a_j, η_i y_i + ξ_j x_j, 1]
     (aΔ_{i→1,2} // aΔ_{2→2,3} // aS̄_3) (bΔ_{j→-1,-2} // bΔ_{-2→-2,-3})) // (P_{-1,3} P_{-3,1} am_{2,j→k} bm_{i,-2→k}),
  dS_i = 𝔼_{{i}→{1,2}}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] // (bS̄_1 aS_2) // dm_{2,1→i},
  dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j})]
```

Program

In[∘]:=
```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} ℯ^{-ℏ ϵ a_i/2}]_{$k},
  C̄_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} ℯ^{ℏ ϵ a_i/2}]_{$k},
  Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
  Kink̄_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
```

Program

Note. $t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

Program

In[∘]:=
```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i - β_i t_i/γ, ξ_i x_i + η_i y_i, ℯ^{ϵ β_i a_i/γ}]_{$k},
  t2b_i = 𝔼_{{i}→{i}}[α_i a_i - τ_i γ b_i, ξ_i x_i + η_i y_i, ℯ^{ϵ τ_i a_i}]_{$k}]
```

Testing

In[∘]:=
```
Block[{$k = 1}, {
    am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
    aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
    dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

QZip4 fail at {L,Q,P} = $\Big\{\hbar a_3 b_1,$

$$\hbar x_3 y_{n\$12199[1]} + y_1 \eta_{n\$12199[1]} + y_1 \eta_{n\$12199[2]} + x_1 \xi_{n\$12199[1]} + \frac{(1-B_1)\,\eta_{n\$12199[2]}\,\xi_{n\$12199[1]}}{\hbar} + x_1 \xi_{n\$12199[2]},$$

$$\frac{1}{\sqrt{B_1}} + \Bigg(\frac{\hbar a_1}{2\sqrt{B_1}} - \frac{\gamma \hbar^3 x_3^2 y_{n\$12199[1]}^2}{4\sqrt{B_1}} - \frac{\hbar a_3 y_1 \eta_{n\$12199[2]}}{\sqrt{B_1}} - \frac{\gamma \hbar x_1 \xi_{n\$12199[1]}}{\sqrt{B_1}} + a_1\sqrt{B_1}\,\eta_{n\$12199[2]}\,\xi_{n\$12199[1]} + \frac{\gamma \hbar x_1 y_1 \eta_{n\$12199[2]} \xi_{n\$12199[1]}}{\sqrt{B_1}} + \frac{(\gamma - 3\gamma B_1)\, y_1 \eta_{n\$12199[2]}^2 \xi_{n\$12199[1]}}{2\sqrt{B_1}} + \frac{(\gamma - 3\gamma B_1)\, x_1 \eta_{n\$12199[2]} \xi_{n\$12199[1]}^2}{2\sqrt{B_1}} + \frac{(\gamma - 4\gamma B_1 + 3\gamma B_1^2)\, \eta_{n\$12199[2]}^2 \xi_{n\$12199[1]}^2}{4\hbar\sqrt{B_1}}\Bigg)\epsilon + O[\epsilon]^2\Big\}$$

Out[ ]=

$$\text{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j),\ x_k\left(e^{-\gamma\,\alpha_j}\,\xi_i+\xi_j\right),\ 1\right]$$

$$\text{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,(\beta_i+\beta_j),\ y_k\,(\eta_i+\eta_j),\ 1-y_k\,\beta_i\,\eta_j\,\epsilon+O[\epsilon]^2\right]$$

$$\text{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,\alpha_i+a_k\,\alpha_j+b_k\,\beta_i+b_k\,\beta_j,\ y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,$$
$$1+\left(-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\gamma\,\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(\gamma-3\,\gamma\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(\gamma-3\,\gamma\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{\left(\gamma-4\,\gamma\,B_k+3\,\gamma\,B_k^2\right)\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon+$$
$$O[\epsilon]^2\Big]$$

$$\text{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1-\frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{\text{R}} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1+\left(-\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\gamma\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right)\epsilon+O[\epsilon]^2\right]$$

$$\text{P} \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1+\frac{\gamma\,\eta_i^2\,\xi_j^2\,\epsilon}{4\,\hbar}+O[\epsilon]^2\right]$$

$$\text{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\left(-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{\text{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\left(\gamma\,\hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\text{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1+\left(-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{\text{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1+\left(\frac{\gamma\,\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon+O[\epsilon]^2\right]$$

$$\text{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i-b_i\,\beta_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},$$
$$1+\Big(\frac{\gamma\,\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\gamma\,\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-$$
$$\frac{\gamma\,\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}+\frac{(-\gamma\,\mathcal{A}_i+\gamma\,B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i}+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}+\frac{y_i\left(3\,\gamma\,\mathcal{A}_i^2-\gamma\,B_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2\,B_i^2}-$$
$$\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\left(3\,\gamma\,\mathcal{A}_i^2-\gamma\,B_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2\,B_i}+\frac{\left(-3\,\gamma\,\mathcal{A}_i^2+4\,\gamma\,B_i\,\mathcal{A}_i^2-\gamma\,B_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\Big)\epsilon+O[\epsilon]^2\Big]$$

$$\text{a}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i,\ x_j\,\xi_i+x_k\,\xi_i,\ 1+\left(-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\text{b}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[b_j\,\beta_i+b_k\,\beta_i,\ B_k\,y_j\,\eta_i+y_k\,\eta_i,\ 1+\frac{1}{2}\,\gamma\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\,\epsilon+O[\epsilon]^2\right]$$

$$\text{d}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i+a_k\,\alpha_i+b_j\,\beta_i+b_k\,\beta_i,\ y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,$$
$$1+\left(\frac{1}{2}\,\gamma\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon+O[\epsilon]^2\Big]$$

$$\text{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{B_i}-\frac{1}{2}\left(\hbar\,a_i\,\sqrt{B_i}\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{\text{C}} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}}+\frac{\hbar\,a_i\,\epsilon}{2\,\sqrt{B_i}}+O[\epsilon]^2\right]$$

$$\text{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,b_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{B_i}}+\left(\frac{\hbar\,a_i}{2\,\sqrt{B_i}}-\frac{\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,\sqrt{B_i}}\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{\text{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,b_i,\ -\frac{\hbar\,x_i\,y_i}{B_i},\ \sqrt{B_i}+\left(-\frac{1}{2}\,\hbar\,a_i\,\sqrt{B_i}-\frac{\hbar^2\,a_i\,x_i\,y_i}{\sqrt{B_i}}-\frac{3\,\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^{3/2}}\right)\epsilon+O[\epsilon]^2\right]$$

$$\text{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-\frac{t_i\,\beta_i}{\gamma},\ y_i\,\eta_i+x_i\,\xi_i,\ 1+\frac{a_i\,\beta_i\,\epsilon}{\gamma}+O[\epsilon]^2\right]$$

$$\text{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-\gamma\,b_i\,\tau_i,\ y_i\,\eta_i+x_i\,\xi_i,\ 1+a_i\,\tau_i\,\epsilon+O[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

In[∘]:=
```
Timing@
  {{"[a,x]" → ((𝔼_{}→{1,2}[0, 0, a2 x1] // am1,2→1)[3] - (𝔼_{}→{1,2}[0, 0, a1 x2] // am1,2→1)[3]),
     "[b,y]" → ((𝔼_{}→{1,2}[0, 0, y2 b1] // bm1,2→1)[3] - (𝔼_{}→{1,2}[0, 0, y1 b2] // bm1,2→1)[3])} /.
    z_1 → z,
   {"Δ[y]" → Last[𝔼_{}→{1}[0, 0, y1] ~B1~ bΔ1→1,2],
    "Δ[b]" → Last[𝔼_{}→{1}[0, 0, b1] ~B1~ bΔ1→1,2],
    "Δ[a]" → Last[𝔼_{}→{1}[0, 0, a1] ~B1~ aΔ1→1,2],
    "Δ[x]" → Last[𝔼_{}→{1}[0, 0, x1] ~B1~ aΔ1→1,2]},
   {
     "S(a)" → ((𝔼_{}→{1}[0, 0, a1] ~B1~ aS1)[3]),
     "S(x)" → ((𝔼_{}→{1}[0, 0, x1] ~B1~ aS1)[3]),
     "S(b)" → ((𝔼_{}→{1}[0, 0, b1] ~B1~ bS1)[3]),
     "S(y)" → ((𝔼_{}→{1}[0, 0, y1] ~B1~ bS1)[3])
   } /. z_1 → z}
```

Out[∘]= $\{0.96875, \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\,\epsilon + O[\epsilon]^3\},\ \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,$
$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar\, a_1 x_2\, \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2\, \epsilon^2 + O[\epsilon]^3\},\ \{S(a) \to -a + O[\epsilon]^3,$
$S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}(a^2 x \hbar^2)\,\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
 ]
```

Out[∘]= {0.15625, {True, True}}

R and *P* are inverses:

In[∘]:=
```
Timing@Block[{$k = 3}, {Ri,j, Pi,k, HL[(Ri,j // Pi,k) ≡ 𝔼_{k}→{j}[aj αk, xj ξk, 1]]}]
```

Out[∘]= $\{0.140625, \{\mathbb{E}_{\{\}\to\{i,j\}}[\hbar\, a_j b_i,\ \hbar\, x_j y_i,\ 1 - \frac{1}{4}(\gamma \hbar^3 x_j^2 y_i^2)\,\epsilon + \left(\frac{1}{9}\gamma^2\hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2\hbar^6 x_j^4 y_i^4\right)\epsilon^2 +$
$\left(\frac{1}{48}\gamma^3\hbar^5 x_j^2 y_i^2 - \frac{1}{16}\gamma^3\hbar^7 x_j^4 y_i^4 - \frac{1}{36}\gamma^3\hbar^8 x_j^5 y_i^5 - \frac{1}{384}\gamma^3\hbar^9 x_j^6 y_i^6\right)\epsilon^3 + O[\epsilon]^4],$
$\mathbb{E}_{\{i,k\}\to\{\}}[\frac{\alpha_k \beta_i}{\hbar},\ \frac{\eta_i \xi_k}{\hbar},\ 1 + \frac{\gamma\,\eta_i^2 \xi_k^2\,\epsilon}{4\hbar} + \frac{(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\,\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4)\,\epsilon^2}{288\hbar^2} +$
$\left(\frac{1}{24}\gamma^3\hbar\,\eta_i^2\xi_k^2 + \frac{1}{6}\gamma^3\eta_i^3\xi_k^3 + \frac{13\gamma^3\eta_i^4\xi_k^4}{96\hbar} + \frac{5\gamma^3\eta_i^5\xi_k^5}{144\hbar^2} + \frac{\gamma^3\eta_i^6\xi_k^6}{384\hbar^3}\right)\epsilon^3 + O[\epsilon]^4],\ \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

In[∘]:=
```
Timing[HL /@ {(aS̄1 // aS1) ≡ 𝔼_{1}→{1}[a1 α1, x1 ξ1, 1], (bS̄1 // bS1) ≡ 𝔼_{1}→{1}[b1 β1, y1 η1, 1]}]
```

Out[∘]= {0.359375, {True, True}}

(co)-associativity on both sides

In[∘]:= Timing[
 HL /@ {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
 (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]

Out[∘]= {0.4375, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:= Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
 (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]

Out[∘]= {0.546875, {True, True}}

An explicit formula for aS_i

In[∘]:= Timing@Block[{$k = 4}, HL[aS_i ≡ (𝔼_{{i}→{i,j}}[-α_i a_j, -ξ_i x_i,
 Sum[Expand[$\frac{e^{\xi_i x_i} (-\hbar \gamma \epsilon)^k}{2^k k!}$ Nest[Expand[$x_i^2 \partial_{\{x_i,2\}}$ #] &, $e^{-\xi_i e^{\hbar \epsilon a_i} x_i}$, k]], {k, 0, $k}]]_{$k} //
 am_{i,j→i})]]

Out[∘]= {2.5625, True}

S is convolution inverse of id

In[∘]:= Timing[HL[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@ {
 (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{1,2→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{1,2→1},
 (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{1,2→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{1,2→1}}]

Out[∘]= {0.4375, {True, True, True, True}}

But not with the opposite product:

In[∘]:= Timing[Short[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@ {
 (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{2,1→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{2,1→1},
 (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{2,1→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{2,1→1}}]

Out[∘]= $\{0.515625, \{\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2\epsilon^2\hbar^2 x_1 \mathcal{A}_1 \xi_1 - 2\gamma\epsilon^2\hbar^2 a_1 x_1 \mathcal{A}_1 \xi_1 + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2\right) == 0,$

$\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \xi_1 - \gamma^2\epsilon^2\hbar^2 x_1 \xi_1 + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \xi_1^2\right) == 0,$

$\frac{1}{2}\left(-2\gamma\epsilon\hbar y_1 \eta_1 - \gamma^2\epsilon^2\hbar^2 y_1 \eta_1 + 2\gamma^2\epsilon^2\hbar^2 y_1^2 \eta_1^2\right) == 0,$

$\frac{-2\gamma\epsilon\hbar B_1 y_1 \eta_1 + \ll 3 \gg + 2\gamma^2\epsilon^2\hbar^2 y_1^2 \eta_1^2}{2 B_1^2} == 0\}\}$

S is an algebra anti-(co)morphism

In[∘]:= Timing[HL /@ {am_{1,2→1} ~ B_1 ~ aS_1 ≡ (aS_1 aS_2) ~ B_{1,2} ~ am_{2,1→1}, bm_{1,2→1} ~ B_1 ~ bS_1 ≡ (bS_1 bS_2) ~ B_{1,2} ~ bm_{2,1→1},
 aS_1 ~ B_1 ~ aΔ_{1→1,2} ≡ aΔ_{1→2,1} ~ B_{1,2} ~ (aS_1 aS_2), bS_1 ~ B_1 ~ bΔ_{1→1,2} ≡ bΔ_{1→2,1} ~ B_{1,2} ~ (bS_1 bS_2)}]

Out[∘]= {0.65625, {True, True, True, True}}

Pairing axioms

```
In[•]:= Timing[HL /@ {(bm1,2→1 𝔼{3}→{3}[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
          (𝔼{1}→{1}[β1 b1, η1 y1, 1] 𝔼{2}→{2}[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
        (bΔ1→1,2 𝔼{3}→{3}[α3 a3, ξ3 x3, 1] 𝔼{4}→{4}[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
          (𝔼{1}→{1}[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
Out[•]= {0.328125, {True, True}}
```

```
In[•]:= Timing[HL /@ {((bS1 𝔼{2}→{2}[α2 a2, ξ2 x2, 1]) // P1,2) ≡ ((𝔼{1}→{1}[β1 b1, η1 y1, 1] aS2) // P1,2),
        (b̄S̄1 𝔼{2}→{2}[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼{1}→{1}[β1 b1, η1 y1, 1] āS̄2) ~ B1,2 ~ P1,2}]
Out[•]= {0.234375, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[•]:= Timing@{{
          "[a,y]" →
           ((𝔼{}→{1,2}[0, 0, y2 a1] ~ B1,2 ~ dm1,2→1) [3] - (𝔼{}→{1,2}[0, 0, y1 a2] ~ B1,2 ~ dm1,2→1) [3]),
          "[b,x]"  → ((𝔼{}→{1,2}[0, 0, x2 b1] ~ B1,2 ~ dm1,2→1) [3] -
             (𝔼{}→{1,2}[0, 0, x1 b2] ~ B1,2 ~ dm1,2→1) [3]),
          "xy-qyx" → ((𝔼{}→{1,2}[0, 0, x1 y2] ~ B1,2 ~ dm1,2→1) [3] -
             (1 + ϵ) (𝔼{}→{1,2}[0, 0, y1 x2] ~ B1,2 ~ dm1,2→1) [3])
        } /. {z_1 → z} // Expand // Factor,
      {
        "Δ(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ dΔ1→1,2) [3]),
        "Δ(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ dΔ1→1,2) [3]),
        "Δ(b)" → ((𝔼{}→{1}[0, 0, b1] ~ B1 ~ dΔ1→1,2) [3]),
        "Δ(y)" → ((𝔼{}→{1}[0, 0, y1] ~ B1 ~ dΔ1→1,2) [3])
       } // Simplify,
      {
         "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ dS1) [3]),
         "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ dS1) [3]),
         "S(b)" → ((𝔼{}→{1}[0, 0, b1] ~ B1 ~ dS1) [3]),
         "S(y)" → ((𝔼{}→{1}[0, 0, y1] ~ B1 ~ dS1) [3])
        } /. {z_1 → z} // Simplify
     }
```

Out[•]=

$$\Big\{3.29688, \Big\{\Big\{[a,y] \to -y\,\gamma + O[\epsilon]^3,\ [b,x] \to x\,\epsilon + O[\epsilon]^3,$$

$$xy{-}qyx \to \frac{1-B}{\hbar} + (a\,B - x\,y + x\,y\,\gamma\,\hbar)\,\epsilon + \left(-\frac{1}{2}a^2\,B\,\hbar + \frac{1}{2}x\,y\,\gamma^2\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3\Big\},$$

$$\Big\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\epsilon^2 + O[\epsilon]^3,$$

$$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1\,y_2) + O[\epsilon]^3\Big\},$$

$$\Big\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}(a^2\,x\,\hbar^2)\,\epsilon^2 + O[\epsilon]^3,$$

$$S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\,\gamma\,\hbar\,\epsilon}{B} - \frac{(y\,\gamma^2\,\hbar^2)\,\epsilon^2}{2\,B} + O[\epsilon]^3\Big\}\Big\}\Big\}$$

(co)-associativity

```
In[∘]:= Timing[
         HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2->1)}]
Out[∘]= {1.79688, {True, True}}
```

Δ is an algebra morphism

```
In[∘]:= Timing@HL[dm1,2→1 ~ B1 ~ dΔ1→1,2 ≡ (dΔ1→1,3 dΔ2→2,4) ~ B1,2,3,4 ~ (dm3,4→2 dm1,2→1)]
Out[∘]= {1.82813, True}
```

S_2 inverts R, but not S_1:

```
In[∘]:= Timing@{R1,2 ~ B1 ~ dS1 ≡ R̄1,2, HL[R1,2 ~ B2 ~ dS2 ≡ R̄1,2]}
```

Out[∘]= $\left\{0.34375, \left\{\frac{1}{4 B_1^3}\left(4 \gamma \epsilon \hbar^2 B_1^2 x_2 y_1 - 2 \gamma^2 \epsilon^2 \hbar^3 B_1^2 x_2 y_1 + 4 \gamma \epsilon^2 \hbar^3 a_2 B_1^2 x_2 y_1 + 8 \gamma^2 \epsilon^2 \hbar^4 B_1 x_2^2 y_1^2 - 4 \gamma \epsilon^2 \hbar^4 a_2 B_1 x_2^2 y_1^2 - 3 \gamma^2 \epsilon^2 \hbar^5 x_2^3 y_1^3\right) == 0, \text{True}\right\}\right\}$

S is convolution inverse of id

```
In[∘]:= Timing[HL[# ≡ 𝔼{1}→{1}[0, 0, 1]] & /@
          {(dΔ1→1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2→1, (dΔ1→1,2 ~ B2 ~ dS2) // dm1,2→1}]
Out[∘]= {3.54688, {True, True}}
```

S is a (co)-algebra anti-morphism

```
In[∘]:= Timing[HL /@
          Expand /@ {dm1,2→1 ~ B1 ~ dS1 ≡ (dS1 dS2) ~ B1,2 ~ dm2,1→1, dS1 ~ B1 ~ dΔ1→1,2 ≡ dΔ1→2,1 ~ B1,2 ~ (dS1 dS2)}]
Out[∘]= {8.51563, {True, True}}
```

Quasi-triangular axiom 1:

```
In[∘]:= Timing@HL[R1,2 ~ B1 ~ dΔ1->1,3 ≡ (R1,4 R3,2) ~ B2,4 ~ dm2,4→2]
Out[∘]= {0.203125, True}
```

Quasi-triangular axiom 2:

```
In[∘]:= Timing@HL[((dΔ1->1,2 R3,4) ~ B1,2,3,4 ~ (dm1,3→1 dm2,4→2)) ≡ ((dΔ1→2,1 R3,4) ~ B1,2,3,4 ~ (dm3,1→1 dm4,2→2))]
Out[∘]= {1.76563, True}
```

The Drinfel'd element inverse property, $(u_1\, \overline{u}_2) \sim B_{1,2} \sim \mathrm{dm}_{1,2\to 1} \equiv \mathbb{E}[0, 0, 1]$:

```
In[∘]:= Timing@HL[((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1→i) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1→j)) ~ Bi,j ~ dmi,j→i ≡
          𝔼{}→{i}[0, 0, 1]]
Out[∘]= {1.8125, True}
```

The ribbon element v satisfies $v^2 = S(u)\, u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\, u^{-1}$ which is something easy.

In[∘]:= **Timing@Block[{$k = 2},**
(((R₁,₂ ~ B₁ ~ dS₁ ~ B₁,₂ ~ dm₂,₁→ᵢ) ~ Bᵢ ~ dSᵢ) (R₁,₂ ~ B₂ ~ dS₂ ~ B₂ ~ dS₂ ~ B₁,₂ ~ dm₂,₁→ⱼ)) ~ Bᵢ,ⱼ ~ dmᵢ,ⱼ→ᵢ]

$$\text{Out}= \left\{2.46875,\ \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{B_i}+\frac{\hbar\, a_i\, \epsilon}{B_i}+\frac{\hbar^2 a_i^2 \epsilon^2}{2 B_i}+O[\epsilon]^3\right]\right\}$$

In[∘]:= **Timing@Block[{$k = 2}, HL /@ {(Cᵢ C̄ⱼ) ~ Bᵢ,ⱼ ~ dmᵢ,ⱼ→ᵢ ≡ 𝔼{}→{i}[0, 0, 1], (C̄ᵢ C̄ⱼ) ~ Bᵢ,ⱼ ~ dmᵢ,ⱼ→ᵢ ≡**
(((R₁,₂ ~ B₁ ~ dS₁ ~ B₁,₂ ~ dm₂,₁→ᵢ) ~ Bᵢ ~ dSᵢ) (R₁,₂ ~ B₂ ~ dS₂ ~ B₂ ~ dS₂ ~ B₁,₂ ~ dm₂,₁→ⱼ)) ~ Bᵢ,ⱼ ~ dmᵢ,ⱼ→ᵢ}]

Out[∘]= {2.54688, {True, True}}

Reidemeister 2:

In[∘]:= **Timing[HL[# ≡ 𝔼{}→{1,2}[0, 0, 1]] & /@**
{(R̄₁,₂ R₃,₄) ~ B₁,₂,₃,₄ ~ (dm₁,₃→₁ dm₂,₄→₂), (R₁,₂ R̄₃,₄) ~ B₁,₂,₃,₄ ~ (dm₁,₃→₁ dm₂,₄→₂)}]

Out[∘]= {1.4375, {True, True}}

Cyclic Reidemeister 2:

In[∘]:= **Timing@HL[(R₁,₄ R̄₅,₂ C̄₃) ~ B₂,₄ ~ dm₂,₄→₂ ~ B₁,₃ ~ dm₁,₃→₁ ~ B₁,₅ ~ dm₁,₅→₁ ≡ C̄₁ 𝔼{}→{2}[0, 0, 1]]**

Out[∘]= {1.10938, True}

Reidemeister 3:

In[∘]:= **Timing@HL[((R₁,₂ R₄,₃ R₅,₆) ~ B₁,₄ ~ dm₁,₄→₁ ~ B₂,₅ ~ dm₂,₅→₂ ~ B₃,₆ ~ dm₃,₆→₃) ≡**
((R₁,₆ R₂,₃ R₄,₅) ~ B₁,₄ ~ dm₁,₄→₁ ~ B₂,₅ ~ dm₂,₅→₂ ~ B₃,₆ ~ dm₃,₆→₃)]

Out[∘]= {1.34375, True}

Relations between the four kinks:

In[∘]:= **Timing[HL /@ {Kinkᵢ ≡ (R₃,₁ C₂) ~ B₁,₂ ~ dm₁,₂→₁ ~ B₁,₃ ~ dm₁,₃→ᵢ,**
K̄inkⱼ ≡ (R̄₃,₁ C̄₂) ~ B₁,₂ ~ dm₁,₂→₁ ~ B₁,₃ ~ dm₁,₃→ⱼ, (Kinkᵢ K̄inkⱼ) ~ Bᵢ,ⱼ ~ dmᵢ,ⱼ→₁ ≡ 𝔼{}→{1}[0, 0, 1]}]

Out[∘]= {2.89063, {True, True, True}}

The Trefoil

In[∘]:= **Timing@Block[{$k = 1},**
Z = R₁,₅ R₆,₂ R₃,₇ C̄₄ K̄ink₈ K̄ink₉ K̄ink₁₀;
Do[Z = Z ~ B₁,ᵣ ~ dm₁,ᵣ→₁, {r, 2, 10}];
{Simplify /@ Z, Simplify /@ (Z ~ B₁ ~ b2t₁ /. T₁ → T)}]

$$\text{Out}= \left\{2.21875,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{B_1}{1-B_1+B_1^2}-\left(\hbar B_1\left(-a_1\left(-1+B_1-B_1^3+B_1^4\right)+\gamma\left(B_1-2B_1^2-2B_1^4+2\hbar x_1 y_1+B_1^3\left(3+2\hbar x_1 y_1\right)\right)\right)\epsilon\right)\Big/\left(1-B_1+B_1^2\right)^3+O[\epsilon]^2\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T}{1-T+T^2}+\left(T\hbar\left(T\left(-1+2T-3T^2+2T^3\right)\gamma+2\left(-1+T-T^3+T^4\right)a_1-2\left(1+T^3\right)\gamma\hbar x_1 y_1\right)\epsilon\right)\Big/\left(1-T+T^2\right)^3+O[\epsilon]^2\right]\right\}\right\}$$

Program

In[∘]:=
```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
  kR̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
  km_{i,j→k} = (t2b_i t2b_j) // dm_{i,j→k} // b2t_k /. {t_k → t, T_k → T, τ_{i|j} → 0},
  kC_i = C_i // b2t_i /. T_i → T,
  kC̄_i = C̄_i // b2t_i /. T_i → T,
  kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
  kKink̄_i = Kink̄_i // b2t_i /. {t_i → t, T_i → T}]
```

In[∘]:=
```
Timing@Block[{$k = 1},
  Z = kR_{1,5} kR_{6,2} kR_{3,7} kC̄_4 kKink̄_8 kKink̄_9 kKink̄_10;
  Do[Z = Z~B_{1,r}~km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z]
```

Out[∘]= $\left\{1.32813, \mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{T}{1-T+T^2} + \left(T\hbar\left(T\left(-1+2T-3T^2+2T^3\right)\gamma + 2\left(-1+T-T^3+T^4\right)a_1 - 2\left(1+T^3\right)\gamma\hbar x_1 y_1\right)\epsilon\right) \Big/ \left(1-T+T^2\right)^3 + O[\epsilon]^2\right]\right\}$

RVK, rot, Z from 2016-09/OneSmidgen.nb. See also local version in this folder.

Program

Some details of the code below are at http://drorbn.net/bbs/show?shot=Dror-160920-151350.jpg.

Program

In[∘]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
    xs and a length 2n list of rotation numbers rots. Crossing
    sites are indexed 1 through 2n, and rots⟦k⟧ is the rotation
    between site k-1 and site k. RVK is also a casting operator
    converting to the RVK presentation from other knot presentations.";
```

Program

In[∘]:=
```
RVK[pd_PD] := PP_RVK@Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X ⧴ { Xp[x⟦4⟧, x⟦1⟧]   PositiveQ@x
                          Xm[x⟦2⟧, x⟦1⟧]   True       ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] ⧴ {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] ⧴ (++rots⟦l⟧; {1 - l, k + 1, l})
            })],
      Cases[front, k | -k] /. {k, -k} ⧴ --rots⟦k + 1⟧;
     ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
xs = Cases[pd, x_X ⧴ If[PositiveQ@x, Xp[x⟦4⟧, x⟦1⟧], Xm[x⟦2⟧, x⟦1⟧]]];
```

```
In[]:= RVK[Knot[10, 100]]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[]= RVK[{Xp[1, 6], Xp[5, 18], Xm[13, 20], Xm[7, 14], Xm[3, 10], Xm[9, 16], Xm[11, 4], Xm[15, 8],
  Xm[19, 12], Xp[17, 2]}, {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}]
```

Program

```
In[]:= rot[i_, 0] := 𝔼_{{}→{i}}[0, 0, 1];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC̄_j] // km_{i,j→i}];
```

Program

```
In[]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Monitor[ PP"Z"@Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
    {todo, rots} = List @@ rvk;
    AppendTo[rots, 0];
    n = Length[todo];
    ξ = 𝔼_{{}→{0}}[0, 0, 1];
    done = {0};
    st = Range[0, 2 n + 1];
    While[{} =!= ($M = todo),
     {cx} = MaximalBy[todo, Length[done ⋂ {#⟦1⟧, #⟦2⟧, #⟦1⟧ - 1, #⟦2⟧ - 1}] &, 1];
     {i, j} = List @@ cx;
     ξ1 = Switch[Head[cx],
       Xp, (kR_{i,j} kKink̄_k) // km_{j,k→j},
       Xm, (kR̄_{i,j} kKink_k) // km_{j,k→j}
      ];
     ξ1 = (rot[k, rots⟦i⟧] ξ1) // km_{k,i→i}; rots⟦i⟧ = 0;
     ξ1 = (ξ1 rot[k, rots⟦i + 1⟧]) // km_{i,k→i}; rots⟦i + 1⟧ = 0;
     ξ1 = (rot[k, rots⟦j⟧] ξ1) // km_{k,j→j}; rots⟦j⟧ = 0;
     ξ1 = (ξ1 rot[k, rots⟦j + 1⟧]) // km_{j,k→j}; rots⟦j + 1⟧ = 0;
     ξ *= ξ1;
     If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st⟦i + 2⟧ → st⟦i + 1⟧];
     If[MemberQ[done, i - 1], ξ = ξ // km_{st⟦i⟧,i→st⟦i⟧}; st = st /. st⟦i + 1⟧ → st⟦i⟧]];
     If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st⟦j + 2⟧ → st⟦j + 1⟧];
     If[MemberQ[done, j - 1], ξ = ξ // km_{st⟦j⟧,j→st⟦j⟧}; st = st /. st⟦j + 1⟧ → st⟦j⟧]];
     done = done ⋃ {i - 1, i, j - 1, j};
     todo = DeleteCases[todo, cx]
    ];
    Simplify /@ (ξ /. {x_0 → x, y_0 → y, a_0 → a})
   ], $M]
```

Knot

In[∘]:= **$k = 1; Timing@Z@Knot[10, 100]**

Knot

Out[∘]= $\{43.2656, \mathbb{E}_{\{\}\to\{0\}}[0, 0, T^4 / (1 - 4 T + 9 T^2 - 12 T^3 + 13 T^4 - 12 T^5 + 9 T^6 - 4 T^7 + T^8) + (T^4 \hbar (4 a (-2 + 14 T - 51 T^2 + 120 T^3 - 203 T^4 + 258 T^5 - 246 T^6 + 152 T^7 - 152 T^9 + 246 T^{10} - 258 T^{11} + 203 T^{12} - 120 T^{13} + 51 T^{14} - 14 T^{15} + 2 T^{16}) + \gamma (-6 + 2 T^{16} - 8 x y \hbar - 440 T^9 (-1 + x y \hbar) - 4 T^{15} (3 + 2 x y \hbar) + 8 T^8 (-97 + 21 x y \hbar) + 8 T^7 (131 + 21 x y \hbar) - 20 T^6 (57 + 22 x y \hbar) + T^{14} (37 + 48 x y \hbar) + T (44 + 48 x y \hbar) - 8 T^{11} (2 + 61 x y \hbar) + 8 T^5 (127 + 68 x y \hbar) - 2 T^{13} (35 + 78 x y \hbar) + 4 T^{10} (-39 + 136 x y \hbar) - T^2 (167 + 156 x y \hbar) + T^{12} (79 + 324 x y \hbar) + T^3 (410 + 324 x y \hbar) - T^4 (733 + 488 x y \hbar))) \epsilon) / (1 - 4 T + 9 T^2 - 12 T^3 + 13 T^4 - 12 T^5 + 9 T^6 - 4 T^7 + T^8)^3 + O[\epsilon]^2]\}$

In[∘]:= **EndProfile[];**

Profile

In[∘]:= **PrintProfile[]**

Profile

Out[∘]=
```
ProfileRoot is root. Profiled time: 91.231
  (     1)   0.186/ 43.265 above Z
  (   157)   0.482/ 38.527 above B
  (    37)   0.174/  9.315 above Boot
  (   147)   0.046/  0.124 above CF
  (     2)       0/      0 above RVK
CF: called 13451 times, time in 29.327/70.775
  (  1047)   1.184/  4.148 under EEQ
  (    47)   0.015/  0.092 under Boot
  (  1347)   7.321/ 21.491 under LZip
  (   147)   0.046/  0.124 under ProfileRoot
  (  9875)  20.401/ 44.107 under QZip
  (   988)   0.360/  0.813 under QZip4
  ( 36730)  13.710/ 41.448 above CCF
Together: called 37864 times, time in 20.974/28.018
  ( 37864)  20.974/ 28.018 under CCF
  ( 37864)   6.267/  7.044 above Exp
CCF: called 37864 times, time in 14.207/42.225
  ( 36730)  13.710/ 41.448 under CF
  (  1134)   0.497/  0.777 under Exp
  ( 37864)  20.974/ 28.018 above Together
Zip: called 2851 times, time in 8.918/41.124
  (   294)   1.013/  6.783 under LZip
  (   294)   0.988/  4.665 under QZip
  (    44)   0.015/  0.031 under QZip4
  (  2219)   6.902/ 29.645 under Zip
  (  2851)   2.561/  2.561 above Collect
  (  2219)   6.902/ 29.645 above Zip
Exp: called 37864 times, time in 6.267/7.044
  ( 37864)   6.267/  7.044 under Together
  (  1134)   0.497/  0.777 above CCF
LZip: called 294 times, time in 5.529/38.359
  (   294)   5.529/ 38.359 under B
  (  1047)   0.408/  4.556 above EEQ
```

```
  (  1347)   7.321/ 21.491 above CF
  (   294)   1.013/  6.783 above Zip
Collect: called 2851 times, time in 2.561/2.561
  (  2851)   2.561/  2.561 under Zip
QZip: called 294 times, time in 1.73/51.439
  (   294)   1.730/ 51.439 under B
  (  9875)  20.401/ 44.107 above CF
  (    22)   0.093/  0.937 above QZip4
  (   294)   0.988/  4.665 above Zip
B: called 294 times, time in 0.67/90.468
  (    72)   0.109/ 42.986 under Z
  (    65)   0.079/  8.955 under Boot
  (   157)   0.482/ 38.527 under ProfileRoot
  (   294)   5.529/ 38.359 above LZip
  (   294)   1.730/ 51.439 above QZip
EEQ: called 1047 times, time in 0.408/4.556
  (  1047)   0.408/  4.556 under LZip
  (  1047)   1.184/  4.148 above CF
Boot: called 59 times, time in 0.361/14.218
  (     3)       0/  0.093 under Z
  (    19)   0.187/  4.810 under Boot
  (    37)   0.174/  9.315 under ProfileRoot
  (    65)   0.079/  8.955 above B
  (    19)   0.187/  4.810 above Boot
  (    47)   0.015/  0.092 above CF
Z: called 1 times, time in 0.186/43.265
  (     1)   0.186/ 43.265 under ProfileRoot
  (    72)   0.109/ 42.986 above B
  (     3)       0/  0.093 above Boot
QZip4: called 22 times, time in 0.093/0.937
  (    22)   0.093/  0.937 under QZip
  (   988)   0.360/  0.813 above CF
  (    44)   0.015/  0.031 above Zip
RVK: called 2 times, time in 0./0.
  (     2)       0/      0 under ProfileRoot
```