

Pensieve header: The full Sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

Profiling

```
In[ ]:= Once[
  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Invariant"];
  << KnotTheory` ;
  << "../Profile/Profile.m";
];
BeginProfile[];
Once@PopupWindow[Button["Show Profile Monitor"],
  Dynamic[PrintProfile[], UpdateInterval -> 3, TrackedSymbols -> {}]]
```

Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.

Read more at <http://katlas.org/wiki/KnotTheory>.

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: June 2018. Original version: July 1994.

Out[]:=

External Utilities

```
In[ ]:= HL[ε_] := Style[ε, Background -> Yellow];
```

Program

Program

Internal Utilities

Program

Canonical Form:

Program

```
In[ ]:= CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := PPCF@ExpandDenominator@ExpandNumerator@PPTogether@Together[PPExp[
  Expand[ε] /. e^x_ e^y_ -> e^{x+y} /. e^x_ -> e^{CF[x]}]];
```

Program

The Kronecker δ :

Program

```
In[ ]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

```
In[*]:=
E /: E[L1_, Q1_, P1_] ≡ E[L2_, Q2_, P2_] :=
CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
E /: E[L1_, Q1_, P1_] E[L2_, Q2_, P2_] := E[L1 + L2, Q1 + Q2, P1 * P2];
E[L_, Q_, P_] $k_ := E[L, Q, Series[Normal@P, {ε, 0, $k}]];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

```
In[*]:=
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{-i})^* := (u^*)_i;
```

Program

Finite Zips:

Program

```
In[*]:=
collect[sd_SeriesData, ξ_] := MapAt[collect[#, ξ] &, sd, 3];
collect[ε_, ξ_] := PPCollect@Collect[ε, ξ];
Zip[_][P_] := P; Zip[ξ_, ξs___][P_] := PPZip[
(collect[P // Zip[ξs], ξ] /. f_. ξ^d_ -> ∂_{ξ*, d} f) /. ξ* -> 0]
```

Program

QZip implements the “Q-level zips” on $E(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Program

```
$T = 1;
QZip[ξs_List@E[L_, Q_, P_] :=
PPQZip@Module[{ξ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2, tin = TimeUsed[], out},
zs = Table[ξ*, {ξ, ξs}];
c = CF[Q /. Alternatives@@(ξs ∪ zs) -> 0];
ys = CF@Table[∂_ξ(Q /. Alternatives@@zs -> 0), {ξ, ξs}];
ηs = CF@Table[∂_z(Q /. Alternatives@@ξs -> 0), {z, zs}];
qt = CF@Inverse@Table[Kδ_{z, ξ*} - ∂_{z, ξ} Q, {ξ, ξs}, {z, zs}];
zrule = Thread[zs -> CF[qt.(zs + ys)]];
Q2 = CF[(Q1 = CF[c + ηs.zs /. zrule]) /. Alternatives@@zs -> 0];
out = CF /@ E[L, Q2, Det[qt] e^{-Q2} Zip[ξs][e^{Q1}(P /. zrule)]];
(*If[(tin=TimeUsed[]-tin)>$T, $T=2tin; Echo[{tin, ξs, L, Q, P}]];*) out];
```

Program

Upper to lower and lower to Upper:

Program

```
In[*]:=
U21 = {B_{i-}^{p-} -> e^{-p h γ b_i}, B_{-}^{p-} -> e^{-p h γ b}, T_{i-}^{p-} -> e^{p h t_i}, T_{-}^{p-} -> e^{p h t}, A_{i-}^{p-} -> e^{p γ α_i}, A_{-}^{p-} -> e^{p γ α}};
L2U = {e^{c_{-} b_i + d_{-}} -> B_i^{c/(h γ)} e^d, e^{c_{-} b + d_{-}} -> B^{-c/(h γ)} e^d,
e^{c_{-} t_i + d_{-}} -> T_i^{c/h} e^d, e^{c_{-} t + d_{-}} -> T^{c/h} e^d,
e^{c_{-} α_i + d_{-}} -> A_i^{c/γ} e^d, e^{c_{-} α + d_{-}} -> A^{c/γ} e^d,
e^{ε_{-}} -> e^{Expand@ε}};
```

Program

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = \mathbb{P}e^{L+Q}$. Such zips regard all of $\mathbb{P}e^Q$ as a single “P”. Here the z’s are b and α and the ζ ’s are β and a .

Program

```
In[*]:= LZip $\zeta$ s_List@E[L_, Q_, P_] := PP_LZip@Module[{ $\zeta$ , z, zs, c, ys,  $\eta$ s, lt, zrule, L1, L2, Q1, Q2},
  zs = Table[ $\zeta^*$ , { $\zeta$ ,  $\zeta$ s}];
  c = L /. Alternatives@@(z $\zeta$ s)  $\rightarrow$  0;
  ys = Table[ $\partial_{\zeta}$ (L /. Alternatives@@z $\zeta$ s  $\rightarrow$  0), { $\zeta$ ,  $\zeta$ s}];
   $\eta$ s = Table[ $\partial_z$ (L /. Alternatives@@z $\zeta$ s  $\rightarrow$  0), {z, zs}];
  lt = Inverse@Table[K $\delta_{z, \zeta^*} - \partial_{z, \zeta}L$ , { $\zeta$ ,  $\zeta$ s}, {z, zs}];
  zrule = Thread[z $\zeta$ s  $\rightarrow$  lt.(z $\zeta$ s + ys)];
  L2 = (L1 = c +  $\eta$ s.zs /. zrule) /. Alternatives@@z $\zeta$ s  $\rightarrow$  0;
  Q2 = (Q1 = Q /. U21 /. zrule) /. Alternatives@@z $\zeta$ s  $\rightarrow$  0;
  CF /@ (CF /@ E[L2, Q2, Det[lt] e $^{-L2-Q2}$  Zip $\zeta$ s[e $^{L1+Q1}$ (P /. U21 /. zrule)]] /. l2U)];
```

Program

```
In[*]:= B_{i} [L_, R_] := LR;
B_{is_} [L_E, R_E] := PP_B@Module[{n},
  Times[
    L /. Table[(v : b | B | t | T | a | x | y)_i  $\rightarrow$  v $_{nei}$ , {i, {is}}],
    R /. Table[(v :  $\beta$  |  $\tau$  |  $\alpha$  |  $\mathcal{A}$  |  $\xi$  |  $\eta$ )_i  $\rightarrow$  v $_{nei}$ , {i, {is}}]
  ] // LZJoin@Table[{ $\beta_{nei}$ ,  $\tau_{nei}$ ,  $\alpha_{nei}$ }, {i, {is}}] // QZJoin@Table[{ $\xi_{nei}$ ,  $\eta_{nei}$ }, {i, {is}}];
  B_{is_} [L_, R_] := B_{is} [L, R];
```

Program

E morphisms with domain and range.

Program

```
In[*]:= B_{is_List} [E $_{d1 \rightarrow r1}$  [L1_, Q1_, P1_], E $_{d2 \rightarrow r2}$  [L2_, Q2_, P2_]] :=
  E(d1UComplement[d2, is])  $\rightarrow$  (r2UComplement[r1, is]) @@ B_{is} [E [L1, Q1, P1], E [L2, Q2, P2]];
E $_{d1 \rightarrow r1}$  [L1_, Q1_, P1_] // E $_{d2 \rightarrow r2}$  [L2_, Q2_, P2_] :=
  B_{r1 \cap d2} [E $_{d1 \rightarrow r1}$  [L1, Q1, P1], E $_{d2 \rightarrow r2}$  [L2, Q2, P2]];
E $_{d1 \rightarrow r1}$  [L1_, Q1_, P1_]  $\equiv$  E $_{d2 \rightarrow r2}$  [L2_, Q2_, P2_]  $\wedge$  :=
  (d1 == d2)  $\wedge$  (r1 == r2)  $\wedge$  (E [L1, Q1, P1]  $\equiv$  E [L2, Q2, P2]);
E $_{d1 \rightarrow r1}$  [L1_, Q1_, P1_] E $_{d2 \rightarrow r2}$  [L2_, Q2_, P2_]  $\wedge$  :=
  E(d1Ud2)  $\rightarrow$  (r1Ur2) @@ (E [L1, Q1, P1] E [L2, Q2, P2]);
E $_{d \rightarrow r}$  [L_, Q_, P_] $k_ := E $_{d \rightarrow r}$  @@ E [L, Q, P] $k;
E_{S_} [i_] := {S} [[i]];
```

Program

“Define” code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of \$k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

```
In[ ]:=
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ = ε_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $k_Integer, PPBoot@$k@Block[{i, j, k}, op_isp, $k = ε; op_nis, $k]];
    SD[op_isp, op_{is}, $k]; SD[op_sis__, op_{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]
```

Program

Booting Up

Program

```
In[ ]:= $k = 2; (*ħ=γ=1;*)
```

Program

```
In[ ]:=
Define[am_{i,j} → k = E_{i,j} → {k} [(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1] $k,
  bm_{i,j} → k = E_{i,j} → {k} [(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ε β_i} - 1) η_j y_k}] $k]
```

Program

```
In[ ]:=
Define[R_{i,j} = E_{i,j} → {k} [ħ a_j b_i, ħ x_j y_i, e^{∑_{k=2}^{k+1} \frac{(1 - e^{γ ε ħ})^k (ħ y_i x_j)^k}{k (1 - e^{k γ ε ħ})}}] $k,
  R_{i,j} = E_{i,j} → {k} [-ħ a_j b_i, -ħ x_j y_i / B_i, 1 + If[$k == 0, 0, (R_{i,j}, $k-1) $k [3] -
    ((R_{i,j}, 0) $k R_{1,2} (R_{3,4}, $k-1) $k) // (bm_{i,1} → i am_{j,2} → j) // (bm_{i,3} → i am_{j,4} → j) [3] ]],
  P_{i,j} = E_{i,j} → {k} [β_i α_j / ħ, η_i ξ_j / ħ, 1 + If[$k == 0, 0, (P_{i,j}, $k-1) $k [3] -
    (R_{1,2} // ((P_{1,j}, 0) $k (P_{i,2}, $k-1) $k)) [3] ] ]]
```

Program

```
In[ ]:=
Define[aS_j = R_{i,j} ~ B_i ~ P_{i,j},
  aS_i = E_{i} → {i} [-a_i α_i, -x_i α_i ξ_i, 1 + If[$k == 0, 0, (aS_{i}, $k-1) $k [3] -
    ((aS_{i}, 0) $k ~ B_i ~ aS_i ~ B_i ~ (aS_{i}, $k-1) $k) [3] ] ]]
```

Program

```
In[ ]:=
Define[bS_i = R_{i,1} ~ B_1 ~ aS_1 ~ B_1 ~ P_{i,1},
  bS_i = R_{i,1} ~ B_1 ~ aS_1 ~ B_1 ~ P_{i,1},
  aΔ_{i,j,k} = (R_{1,j} R_{2,k}) // bm_{1,2} → 3 // P_{3,i},
  bΔ_{i,j,k} = (R_{j,1} R_{k,2}) // am_{1,2} → 3 // P_{i,3}]
```

Program

```
In[*]:= Define [dmi,j→k = (E{i,j}→{i,j} [βi bi + αj aj, ηi yi + ξj xj, 1]
    (aΔi→1,2 // aΔ2→2,3 // aS3) (bΔj→-1,-2 // bΔ-2→-2,-3) // (P-1,3 P-3,1 am2,j→k bmi,-2→k),
dSi = E{i}→{1,2} [βi b1 + αi a2, ηi y1 + ξi x2, 1] // (bS1 aS2) // dm2,1→i,
dΔi→j,k = (bΔi→3,1 aΔi→2,4) // (dm3,4→k dm1,2→j) ]
```

Program

```
In[*]:= Define [Ci = E{i}→{i} [0, 0, Bi1/2 e-ħ e ai/2] $k,
C̄i = E{i}→{i} [0, 0, Bi-1/2 eħ e ai/2] $k,
Kinki = (R1,3 C̄2) // dm1,2→1 // dm1,3→i,
K̄inki = (R̄1,3 C2) // dm1,2→1 // dm1,3→i ]
```

Program

Note. $t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

Program

```
In[*]:= Define [b2ti = E{i}→{i} [αi ai - βi ti/γ, ξi xi + ηi yi, eε βi ai/γ] $k,
t2bi = E{i}→{i} [αi ai - τi γ bi, ξi xi + ηi yi, eε τi ai] $k ]
```

Testing

```

In[ ]:= Block[{$k = 1}, {
  am → ami,j→k, bm → bmi,j→k, dm → dmi,j→k, R → Ri,j, R̄ → R̄i,j, P → Pi,j,
  aS → aSi, aS̄ → aS̄i, bS → bSi, bS̄ → bS̄i, dS → dSi, aΔ → aΔi→j,k, bΔ → bΔi→j,k,
  dΔ → dΔi→j,k, C → Ci, C̄ → C̄i, Kink → Kinki, K̄ink → K̄inki, b2t → b2ti, t2b → t2bi
}] //
Column

am → E{i,j}→{k} [ak (αi + αj), xk (e-γ αj ξi + ξj), 1]
bm → E{i,j}→{k} [bk (βi + βj), yk (ηi + ηj), 1 - yk βi ηj ∈ + O[ε]2]
dm → E{i,j}→{k} [ak αi + ak αj + bk βi + bk βj,  $\frac{1}{\hbar \mathcal{A}_i \mathcal{A}_j}$ 
  (ħ yk Ai Aj ηi + ħ yk Aj ηj + ħ xk Ai ξi + Ai Aj ηj ξi - Bk Ai Aj ηj ξi + ħ xk Ai Aj ξj),
  1 +  $\frac{1}{4 \hbar \mathcal{A}_i \mathcal{A}_j}$  (-4 ħ yk Aj βi ηj - 4 ħ xk Ai βj ξi + 4 γ ħ2 xk yk ηj ξi +
  4 ħ ak Bk Ai Aj ηj ξi + 2 γ ħ yk Aj ηj2 ξi - 6 γ ħ Bk yk Aj ηj2 ξi + 2 γ ħ xk Ai ηj ξi2 -
  6 γ ħ Bk xk Ai ηj ξi2 + γ Ai Aj ηj2 ξi2 - 4 γ Bk Ai Aj ηj2 ξi2 + 3 γ Bk2 Ai Aj ηj2 ξi2) ∈ + O[ε]2]
R → E{i}→{i,j} [ħ aj bi, ħ xj yi, 1 -  $\frac{1}{4}$  (γ ħ3 xj2 yi2) ∈ + O[ε]2]
R̄ → E{i}→{i,j} [-ħ aj bi, - $\frac{\hbar x_j y_i}{B_i}$ , 1 -  $\frac{(4 \hbar^2 a_j B_i x_j y_i + 3 \gamma \hbar^3 x_j^2 y_i^2) \epsilon}{4 B_i^2} \in + O[\epsilon]^2$ ]
P → E{i,j}→{i} [ $\frac{\alpha_j \beta_i}{\hbar}$ ,  $\frac{\eta_i \xi_j}{\hbar}$ , 1 +  $\frac{\gamma \eta_i^2 \xi_j^2 \epsilon}{4 \hbar} + O[\epsilon]^2$ ]
aS → E{i}→{i} [-ai αi, -xi Ai ξi, 1 +  $\frac{1}{2}$  (-2 ħ ai xi Ai ξi - γ ħ xi2 Ai2 ξi2) ∈ + O[ε]2]
aS̄ → E{i}→{i} [-ai αi, -xi Ai ξi, 1 +  $\frac{1}{2}$  (2 γ ħ xi Ai ξi - 2 ħ ai xi Ai ξi - γ ħ xi2 Ai2 ξi2) ∈ + O[ε]2]
bS → E{i}→{i} [-bi βi, - $\frac{y_i \eta_i}{B_i}$ , 1 +  $\frac{(-2 B_i y_i \beta_i \eta_i - \gamma \hbar y_i^2 \eta_i^2) \epsilon}{2 B_i^2} \in + O[\epsilon]^2$ ]
bS̄ → E{i}→{i} [-bi βi, - $\frac{y_i \eta_i}{B_i}$ , 1 +  $\frac{(2 \gamma \hbar B_i y_i \eta_i - 2 B_i y_i \beta_i \eta_i - \gamma \hbar y_i^2 \eta_i^2) \epsilon}{2 B_i^2} \in + O[\epsilon]^2$ ]
Out[ ]:= dS → E{i}→{i} [-ai αi - bi βi,  $\frac{-\hbar y_i \mathcal{A}_i \eta_i - \hbar B_i x_i \mathcal{A}_i \xi_i + \mathcal{A}_i \eta_i \xi_i - B_i \mathcal{A}_i \eta_i \xi_i}{\hbar B_i}$ ,
  1 +  $\frac{1}{4 \hbar B_i^2}$  (4 γ ħ2 Bi yi Ai ηi - 4 ħ Bi yi Ai βi ηi - 2 γ ħ2 yi2 Ai2 ηi2 - 4 ħ2 ai Bi2 xi Ai ξi - 4 ħ Bi2 xi Ai βi ξi -
  4 γ ħ Bi Ai ηi ξi + 4 ħ ai Bi Ai ηi ξi + 4 γ ħ Bi2 Ai ηi ξi - 4 γ ħ2 Bi xi yi Ai2 ηi ξi +
  4 Bi Ai βi ηi ξi - 4 Bi2 Ai βi ηi ξi + 6 γ ħ yi Ai2 ηi2 ξi - 2 γ ħ Bi yi Ai2 ηi2 ξi - 2 γ ħ2 Bi2 xi2 Ai2 ξi2 +
  6 γ ħ Bi xi Ai2 ηi ξi2 - 2 γ ħ Bi2 xi Ai2 ηi ξi2 - 3 γ Ai2 ηi2 ξi2 + 4 γ Bi Ai2 ηi2 ξi2 - γ Bi2 Ai2 ηi2 ξi2) ∈ + O[ε]2]
aΔ → E{i}→{j,k} [aj αi + ak αi, xj ξi + xk ξi, 1 +  $\frac{1}{2}$  (-2 ħ aj xk ξi + γ ħ xj xk ξi2) ∈ + O[ε]2]
bΔ → E{i}→{j,k} [bj βi + bk βi, Bk yj ηi + yk ηi, 1 +  $\frac{1}{2}$  γ ħ Bk yj yk ηi2 ∈ + O[ε]2]
dΔ → E{i}→{j,k} [aj αi + ak αi + bj βi + bk βi,
  yj ηi + Bj yk ηi + xj ξi + xk ξi, 1 +  $\frac{1}{2}$  (γ ħ Bj yj yk ηi2 - 2 ħ aj xk ξi + γ ħ xj xk ξi2) ∈ + O[ε]2]
C → E{i}→{i} [0, 0,  $\sqrt{B_i} - \frac{1}{2} (\hbar a_i \sqrt{B_i}) \in + O[\epsilon]^2$ ]
C̄ → E{i}→{i} [0, 0,  $\frac{1}{\sqrt{B_i}} + \frac{\hbar a_i \epsilon}{2 \sqrt{B_i}} + O[\epsilon]^2$ ]
Kink → E{i}→{i} [ħ ai bi, ħ xi yi,  $\frac{1}{\sqrt{B_i}} + \frac{(2 \hbar a_i - \gamma \hbar^3 x_i^2 y_i^2) \epsilon}{4 \sqrt{B_i}} + O[\epsilon]^2$ ]
K̄ink → E{i}→{i} [-ħ ai bi, - $\frac{\hbar x_i y_i}{B_i}$ ,  $\sqrt{B_i} + \frac{(-2 \hbar a_i B_i^2 - 4 \hbar^2 a_i B_i x_i y_i - 3 \gamma \hbar^3 x_i^2 y_i^2) \epsilon}{4 B_i^{3/2}} \in + O[\epsilon]^2$ ]
b2t → E{i}→{i} [ai αi -  $\frac{t_i \beta_i}{\gamma}$ , yi ηi + xi ξi, 1 +  $\frac{a_i \beta_i \epsilon}{\gamma} + O[\epsilon]^2$ ]
t2b → E{i}→{i} [ai αi - γ bi τi, yi ηi + xi ξi, 1 + ai τi ∈ + O[ε]2]

```

Check that on the generators this agrees with our conventions in the handout:

In[*]:= **Timing@**

```
{ {"[a,x]" -> ((E_{i->{1,2}}[0, 0, a_2 x_1] // am_{1,2->1})[3] - (E_{i->{1,2}}[0, 0, a_1 x_2] // am_{1,2->1})[3]),
  "[b,y]" -> ((E_{i->{1,2}}[0, 0, y_2 b_1] // bm_{1,2->1})[3] - (E_{i->{1,2}}[0, 0, y_1 b_2] // bm_{1,2->1})[3]) } /.
  z_{-1} -> z,
  {"Δ[y]" -> Last[E_{i->{1}}[0, 0, y_1] ~ B_1 ~ bΔ_{1->1,2}],
  "Δ[b]" -> Last[E_{i->{1}}[0, 0, b_1] ~ B_1 ~ bΔ_{1->1,2}],
  "Δ[a]" -> Last[E_{i->{1}}[0, 0, a_1] ~ B_1 ~ aΔ_{1->1,2}],
  "Δ[x]" -> Last[E_{i->{1}}[0, 0, x_1] ~ B_1 ~ aΔ_{1->1,2}]}
  {
  "S(a)" -> ((E_{i->{1}}[0, 0, a_1] ~ B_1 ~ aS_1)[3]),
  "S(x)" -> ((E_{i->{1}}[0, 0, x_1] ~ B_1 ~ aS_1)[3]),
  "S(b)" -> ((E_{i->{1}}[0, 0, b_1] ~ B_1 ~ bS_1)[3]),
  "S(y)" -> ((E_{i->{1}}[0, 0, y_1] ~ B_1 ~ bS_1)[3])
  } /. z_{-1} -> z}
```

Out[*]:= {1.51563,
 {{[a,x] -> -x γ, [b,y] -> -y ε + 0[ε]^3}, {Δ[y] -> (B_2 y_1 + y_2) + 0[ε]^3, Δ[b] -> (b_1 + b_2) + 0[ε]^3,
 Δ[a] -> (a_1 + a_2) + 0[ε]^3, Δ[x] -> (x_1 + x_2) - ħ a_1 x_2 ε + $\frac{1}{2} \hbar^2 a_1^2 x_2 \epsilon^2 + 0[\epsilon]^3$ }, {S(a) -> -a + 0[ε]^3,
 S(x) -> -x - a x ħ ε - $\frac{1}{2} (a^2 x \hbar^2) \epsilon^2 + 0[\epsilon]^3$, S(b) -> -b + 0[ε]^3, S(y) -> - $\frac{y}{B} + 0[\epsilon]^3$ }}}

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[*]:= **Timing@Block**[{**\$k = 3**},

```
HL /@ { (am_{1,2->1} // am_{1,3->1}) ≡ (am_{2,3->2} // am_{1,2->1}), (bm_{1,2->1} // bm_{1,3->1}) ≡ (bm_{2,3->2} // bm_{1,2->1})
}
```

Out[*]:= {0.15625, {True, True}}

R and P are inverses:

In[*]:= **Timing@Block**[{**\$k = 3**}, {R_{i,j}, P_{i,k}, HL[(R_{i,j} // P_{i,k}) ≡ E_{{k}→{j}}[a_j α_k, x_j ξ_k, 1]]}]

Out[*]:= {0.125, {E_{{i}→{i,j}}[ħ a_j b_i, ħ x_j y_i, 1 - $\frac{1}{4} (\gamma \hbar^3 x_j^2 y_i^2) \epsilon + \left(\frac{1}{9} \gamma^2 \hbar^5 x_j^3 y_i^3 + \frac{1}{32} \gamma^2 \hbar^6 x_j^4 y_i^4 \right) \epsilon^2 +$
 $\frac{1}{1152} (24 \gamma^3 \hbar^5 x_j^2 y_i^2 - 72 \gamma^3 \hbar^7 x_j^4 y_i^4 - 32 \gamma^3 \hbar^8 x_j^5 y_i^5 - 3 \gamma^3 \hbar^9 x_j^6 y_i^6) \epsilon^3 + 0[\epsilon]^4$ },
 E_{{i,k}→{i}}[$\frac{\alpha_k \beta_i}{\hbar}$, $\frac{\eta_i \xi_k}{\hbar}$, 1 + $\frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4 \hbar} + \frac{1}{288 \hbar^2} (36 \gamma^2 \hbar^2 \eta_i^2 \xi_k^2 + 40 \gamma^2 \hbar \eta_i^3 \xi_k^3 + 9 \gamma^2 \eta_i^4 \xi_k^4) \epsilon^2 - \frac{1}{1152 \hbar^3}$
 $(-48 \gamma^3 \hbar^4 \eta_i^2 \xi_k^3 - 192 \gamma^3 \hbar^3 \eta_i^3 \xi_k^3 - 156 \gamma^3 \hbar^2 \eta_i^4 \xi_k^4 - 40 \gamma^3 \hbar \eta_i^5 \xi_k^5 - 3 \gamma^3 \eta_i^6 \xi_k^6) \epsilon^3 + 0[\epsilon]^4$], True}}

as and \overline{aS} are inverses, bs and \overline{bS} are inverses:

In[*]:= **Timing**[HL /@ {($\overline{aS_1}$ // aS₁) ≡ E_{{1}→{1}}[a₁ α₁, x₁ ξ₁, 1], ($\overline{bS_1}$ // bS₁) ≡ E_{{1}→{1}}[b₁ β₁, y₁ η₁, 1]]}]

Out[*]:= {0.484375, {True, True}}

(co)-associativity on both sides

In[*]:= Timing[
 HL /@ { (a $\Delta_{1 \rightarrow 1, 2}$ // a $\Delta_{2 \rightarrow 2, 3}$) \equiv (a $\Delta_{1 \rightarrow 1, 3}$ // a $\Delta_{1 \rightarrow 1, 2}$), (b $\Delta_{1 \rightarrow 1, 2}$ // b $\Delta_{2 \rightarrow 2, 3}$) \equiv (b $\Delta_{1 \rightarrow 1, 3}$ // b $\Delta_{1 \rightarrow 1, 2}$),
 (am $_{1, 2 \rightarrow 1}$ // am $_{1, 3 \rightarrow 1}$) \equiv (am $_{2, 3 \rightarrow 2}$ // am $_{1, 2 \rightarrow 1}$), (bm $_{1, 2 \rightarrow 1}$ // bm $_{1, 3 \rightarrow 1}$) \equiv (bm $_{2, 3 \rightarrow 2}$ // bm $_{1, 2 \rightarrow 1}$) }]

Out[*]:= {0.4375, {True, True, True, True}}

Δ is an algebra morphism

In[*]:= Timing[HL /@ { (am $_{1, 2 \rightarrow 1}$ // a $\Delta_{1 \rightarrow 1, 2}$) \equiv ((a $\Delta_{1 \rightarrow 1, 3}$ a $\Delta_{2 \rightarrow 2, 4}$) // (am $_{3, 4 \rightarrow 2}$ am $_{1, 2 \rightarrow 1}$)),
 (bm $_{1, 2 \rightarrow 1}$ // b $\Delta_{1 \rightarrow 1, 2}$) \equiv ((b $\Delta_{1 \rightarrow 1, 3}$ b $\Delta_{2 \rightarrow 2, 4}$) // (bm $_{3, 4 \rightarrow 2}$ bm $_{1, 2 \rightarrow 1}$)) }]

Out[*]:= {0.640625, {True, True}}

An explicit formula for aS;

In[*]:= Timing@Block[{\$k = 4}, HL[aS $_i$ \equiv (E $_{\{i\} \rightarrow \{i, j\}}$ [- α_i a $_j$, - ξ_i x $_i$,
 Sum[Expand[$\frac{e^{\xi_i x_i} (-\hbar \gamma \epsilon)^k}{2^k k!}$ Nest[Expand[x $_i^2$ $\partial_{\{x_i, 2\}}$ #] &, e $^{-\xi_i e^{\hbar \epsilon a_i} x_i}$, k]], {k, 0, \$k}]]] \$k //
 am $_{i, j \rightarrow i}$)]]

Out[*]:= {4.60938, True}

S is convolution inverse of id

In[*]:= Timing[HL[# \equiv E $_{\{1\} \rightarrow \{1\}}$ [0, 0, 1]] & /@ {
 (a $\Delta_{1 \rightarrow 1, 2} \sim B_1 \sim aS_1$) $\sim B_{1, 2} \sim am_{1, 2 \rightarrow 1}$, (a $\Delta_{1 \rightarrow 1, 2} \sim B_2 \sim aS_2$) $\sim B_{1, 2} \sim am_{1, 2 \rightarrow 1}$,
 (b $\Delta_{1 \rightarrow 1, 2} \sim B_1 \sim bS_1$) $\sim B_{1, 2} \sim bm_{1, 2 \rightarrow 1}$, (b $\Delta_{1 \rightarrow 1, 2} \sim B_2 \sim bS_2$) $\sim B_{1, 2} \sim bm_{1, 2 \rightarrow 1}$ }]

Out[*]:= {0.65625, {True, True, True, True}}

But not with the opposite product:

In[*]:= Timing[Short[# \equiv E $_{\{1\} \rightarrow \{1\}}$ [0, 0, 1]] & /@ {
 (a $\Delta_{1 \rightarrow 1, 2} \sim B_1 \sim aS_1$) $\sim B_{1, 2} \sim am_{2, 1 \rightarrow 1}$, (a $\Delta_{1 \rightarrow 1, 2} \sim B_2 \sim aS_2$) $\sim B_{1, 2} \sim am_{2, 1 \rightarrow 1}$,
 (b $\Delta_{1 \rightarrow 1, 2} \sim B_1 \sim bS_1$) $\sim B_{1, 2} \sim bm_{2, 1 \rightarrow 1}$, (b $\Delta_{1 \rightarrow 1, 2} \sim B_2 \sim bS_2$) $\sim B_{1, 2} \sim bm_{2, 1 \rightarrow 1}$ }]

Out[*]:= {0.703125, { $\frac{1}{2} (-2 \gamma \epsilon \hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2 \epsilon^2 \hbar^2 \langle\langle 1 \rangle\rangle \mathcal{A}_1 \xi_1 - \langle\langle 1 \rangle\rangle + 2 \gamma^2 \epsilon^2 \hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2) = 0$,

$$\frac{1}{2} (-2 \gamma \epsilon \hbar x_1 \xi_1 - \gamma^2 \epsilon^2 \hbar^2 x_1 \xi_1 + 2 \gamma^2 \epsilon^2 \hbar^2 x_1^2 \xi_1^2) = 0,$$

$$\frac{1}{2} (-2 \gamma \epsilon \hbar y_1 \eta_1 - \gamma^2 \epsilon^2 \hbar^2 y_1 \eta_1 + 2 \gamma^2 \epsilon^2 \hbar^2 y_1^2 \eta_1^2) = 0,$$

$$\frac{1}{2 B_1^2} (-2 \gamma \epsilon \hbar B_1 y_1 \eta_1 + \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle + 2 \langle\langle 4 \rangle\rangle \eta_1^2) = 0 \}}$$

S is an algebra anti-(co)morphism

In[*]:= Timing[HL /@ { am $_{1, 2 \rightarrow 1} \sim B_1 \sim aS_1$ \equiv (aS $_1$ aS $_2$) $\sim B_{1, 2} \sim am_{2, 1 \rightarrow 1}$, bm $_{1, 2 \rightarrow 1} \sim B_1 \sim bS_1$ \equiv (bS $_1$ bS $_2$) $\sim B_{1, 2} \sim bm_{2, 1 \rightarrow 1}$,
 aS $_1 \sim B_1 \sim a\Delta_{1 \rightarrow 1, 2}$ \equiv a $\Delta_{1 \rightarrow 2, 1} \sim B_{1, 2} \sim$ (aS $_1$ aS $_2$), bS $_1 \sim B_1 \sim b\Delta_{1 \rightarrow 1, 2}$ \equiv b $\Delta_{1 \rightarrow 2, 1} \sim B_{1, 2} \sim$ (bS $_1$ bS $_2$) }]

Out[*]:= {0.984375, {True, True, True, True}}

Pairing axioms


```
In[ ]:= Timing[HL /@ { (bm1,2→1 E{3}→{3} [α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
  (E{1}→{1} [β1 b1, η1 y1, 1] E{2}→{2} [β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
  (bΔ1→1,2 E{3}→{3} [α3 a3, ξ3 x3, 1] E{4}→{4} [α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
  (E{1}→{1} [β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
```

Out[]:= {0.421875, {True, True}}

```
In[ ]:= Timing[HL /@ { ((bs1 E{2}→{2} [α2 a2, ξ2 x2, 1]) // P1,2) ≡ ((E{1}→{1} [β1 b1, η1 y1, 1] aS2) // P1,2),
  (bs1 E{2}→{2} [α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (E{1}→{1} [β1 b1, η1 y1, 1] aS2) ~ B1,2 ~ P1,2}]
```

Out[]:= {0.34375, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[ ]:= Timing@{ {
  "[a,y]" →
    ((E{1}→{1,2} [0, 0, y2 a1] ~ B1,2 ~ dm1,2→1) [3] - (E{1}→{1,2} [0, 0, y1 a2] ~ B1,2 ~ dm1,2→1) [3]),
  "[b,x]" → ((E{1}→{1,2} [0, 0, x2 b1] ~ B1,2 ~ dm1,2→1) [3] -
    (E{1}→{1,2} [0, 0, x1 b2] ~ B1,2 ~ dm1,2→1) [3]),
  "xy-qyx" → ((E{1}→{1,2} [0, 0, x1 y2] ~ B1,2 ~ dm1,2→1) [3] -
    (1 + ε) (E{1}→{1,2} [0, 0, y1 x2] ~ B1,2 ~ dm1,2→1) [3])
} /. {z-1 → z} // Expand // Factor,
{
  "Δ(a)" → ((E{1}→{1} [0, 0, a1] ~ B1 ~ dΔ1→1,2) [3]),
  "Δ(x)" → ((E{1}→{1} [0, 0, x1] ~ B1 ~ dΔ1→1,2) [3]),
  "Δ(b)" → ((E{1}→{1} [0, 0, b1] ~ B1 ~ dΔ1→1,2) [3]),
  "Δ(y)" → ((E{1}→{1} [0, 0, y1] ~ B1 ~ dΔ1→1,2) [3])
} // Simplify,
{
  "S(a)" → ((E{1}→{1} [0, 0, a1] ~ B1 ~ dS1) [3]),
  "S(x)" → ((E{1}→{1} [0, 0, x1] ~ B1 ~ dS1) [3]),
  "S(b)" → ((E{1}→{1} [0, 0, b1] ~ B1 ~ dS1) [3]),
  "S(y)" → ((E{1}→{1} [0, 0, y1] ~ B1 ~ dS1) [3])
} /. {z-1 → z} // Simplify
}
```

Out[]:= {7.79688, { { [a,y] → -y γ + 0[ε]³, [b,x] → x ε + 0[ε]³,
 xy-qyx → (-x y + $\frac{1 - B + x y \hbar}{\hbar}$) + (a B - x y + x y γ ħ) ε + $\frac{1}{2}$ (-a² B ħ + x y γ² ħ²) ε² + 0[ε]³},
 { Δ(a) → (a₁ + a₂) + 0[ε]³, Δ(x) → (x₁ + x₂) - ħ a₁ x₂ ε + $\frac{1}{2}$ ħ² a₁² x₂ ε² + 0[ε]³,
 Δ(b) → (b₁ + b₂) + 0[ε]³, Δ(y) → (y₁ + B₁ y₂) + 0[ε]³},
 { S(a) → -a + 0[ε]³, S(x) → -x - a x ħ ε - $\frac{1}{2}$ (a² x ħ²) ε² + 0[ε]³,
 S(b) → -b + 0[ε]³, S(y) → - $\frac{y}{B}$ + $\frac{y \gamma \hbar \epsilon}{B}$ - $\frac{(y \gamma^2 \hbar^2) \epsilon^2}{2 B}$ + 0[ε]³ } } }

(co)-associativity

In[]:= Timing[

$$\text{HL} / @ \left\{ \left(d\Delta_{1 \rightarrow 1, 2} // d\Delta_{2 \rightarrow 2, 3} \right) \equiv \left(d\Delta_{1 \rightarrow 1, 3} // d\Delta_{1 \rightarrow 1, 2} \right), \left(dm_{1, 2 \rightarrow 1} // dm_{1, 3 \rightarrow 1} \right) \equiv \left(dm_{2, 3 \rightarrow 2} // dm_{1, 2 \rightarrow 1} \right) \right\}$$

”

$$\left\{ 1.203, \{ \epsilon_{n\$25472[1]}, y_{n\$25472[1]} \}, a_1 \alpha_1 + a_1 \alpha_2 + a_1 \alpha_3 + b_1 \beta_1 + b_1 \beta_2 + b_1 \beta_3, \right.$$

$$\frac{1}{\hbar \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3} \left(\hbar y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_1 + \hbar y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3 \eta_2 + \hbar y_1 \mathcal{A}_3 \eta_3 + \hbar y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_{n\$25472[1]} + \right.$$

$$\hbar x_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_3 \epsilon_1 + \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_2 \epsilon_1 - \mathbf{B}_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_2 \epsilon_1 + \hbar x_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \epsilon_2 +$$

$$\hbar x_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \epsilon_3 + \hbar x_1 \mathcal{A}_1 \mathcal{A}_2 \epsilon_{n\$25472[1]} + \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_3 \epsilon_{n\$25472[1]} - \mathbf{B}_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_3 \epsilon_{n\$25472[1]} \left. \right),$$

$$1 + \frac{1}{4 \hbar \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3} \left(-4 \hbar y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3 \beta_1 \eta_2 - 4 \hbar y_1 \mathcal{A}_3 \beta_1 \eta_3 - 4 \hbar y_1 \mathcal{A}_3 \beta_2 \eta_3 - 4 \hbar x_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_3 \beta_2 \epsilon_1 + \right.$$

$$4 \gamma \hbar^2 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_3 \eta_2 \epsilon_1 + 4 \hbar a_1 \mathbf{B}_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_2 \epsilon_1 + 2 \gamma \hbar y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3 \eta_2^2 \epsilon_1 -$$

$$6 \gamma \hbar \mathbf{B}_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3 \eta_2^2 \epsilon_1 - 8 \gamma \hbar \mathbf{B}_1 y_1 \mathcal{A}_3 \eta_2 \eta_3 \epsilon_1 + 2 \gamma \hbar x_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_3 \eta_2 \epsilon_1^2 -$$

$$6 \gamma \hbar \mathbf{B}_1 x_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_3 \eta_2 \epsilon_1^2 + \gamma \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_2^2 \epsilon_1^2 - 4 \gamma \mathbf{B}_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_2^2 \epsilon_1^2 + 3 \gamma \mathbf{B}_1^2 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_2^2 \epsilon_1^2 -$$

$$4 \hbar x_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \epsilon_{n\$25472[1]} + 4 \gamma \hbar^2 x_1 y_1 \eta_3 \epsilon_{n\$25472[1]} + 4 \hbar a_1 \mathbf{B}_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_3 \epsilon_{n\$25472[1]} + 2 \gamma \hbar y_1 \mathcal{A}_3 \eta_3^2$$

$$\epsilon_{n\$25472[1]} - 6 \gamma \hbar \mathbf{B}_1 y_1 \mathcal{A}_3 \eta_3^2 \epsilon_{n\$25472[1]} + 2 \gamma \hbar x_1 \mathcal{A}_1 \mathcal{A}_2 \eta_3 \epsilon_{n\$25472[1]}^2 - 6 \gamma \hbar \mathbf{B}_1 x_1 \mathcal{A}_1 \mathcal{A}_2 \eta_3 \epsilon_{n\$25472[1]}^2 +$$

$$\gamma \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_3^2 \epsilon_{n\$25472[1]}^2 - 4 \gamma \mathbf{B}_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_3^2 \epsilon_{n\$25472[1]}^2 + 3 \gamma \mathbf{B}_1^2 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \eta_3^2 \epsilon_{n\$25472[1]}^2 \left. \right) +$$

$$\frac{1}{288 \hbar^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2} \left(144 \hbar^2 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1^2 \eta_2 + 144 \hbar^2 y_{n\$25472[1]}^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1^2 \eta_2^2 + 144 \hbar^2 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1^2 \eta_3 + \right.$$

$$288 \hbar^2 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \beta_2 \eta_3 + 144 \hbar^2 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2^2 \eta_3 + 288 \hbar^2 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \beta_1^2 \eta_2 \eta_3 +$$

$$288 \hbar^2 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \beta_2 \eta_2 \eta_3 + 144 \hbar^2 y_1^2 \mathcal{A}_3^2 \beta_1^2 \eta_3^2 + 288 \hbar^2 y_1^2 \mathcal{A}_3^2 \beta_1 \beta_2 \eta_3^2 + 144 \hbar^2 y_1^2 \mathcal{A}_3^2 \beta_2^2 \eta_3^2 +$$

$$144 \hbar^2 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2^2 \epsilon_1 + 144 \gamma^2 \hbar^4 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2 \epsilon_1 - 144 \hbar^3 a_1 \mathbf{B}_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2 \epsilon_1 -$$

$$288 \gamma \hbar^3 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2 \epsilon_1 - 288 \gamma \hbar^3 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2 \epsilon_1 +$$

$$288 \hbar^2 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \beta_2 \eta_2 \epsilon_1 + 144 \gamma^2 \hbar^4 x_{n\$25472[1]} y_{n\$25472[1]}^2 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1 +$$

$$72 \gamma^2 \hbar^3 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1 - 360 \gamma^2 \hbar^3 \mathbf{B}_1 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1 +$$

$$432 \gamma \hbar^3 a_1 \mathbf{B}_1 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1 - 288 \gamma \hbar^3 x_{n\$25472[1]} y_{n\$25472[1]}^2 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \epsilon_1 -$$

$$144 \gamma \hbar^2 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \epsilon_1 + 432 \gamma \hbar^2 \mathbf{B}_1 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \epsilon_1 -$$

$$288 \hbar^2 a_1 \mathbf{B}_1 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \epsilon_1 + 48 \gamma^2 \hbar^3 y_{n\$25472[1]}^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_3^2 \epsilon_1 -$$

$$336 \gamma^2 \hbar^3 \mathbf{B}_1 y_{n\$25472[1]}^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_3^2 \epsilon_1 - 144 \gamma \hbar^2 y_{n\$25472[1]}^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1 \eta_3^2 \epsilon_1 + 432 \gamma \hbar^2 \mathbf{B}_1 y_{n\$25472[1]}^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \beta_1 \eta_3^2 \epsilon_1 +$$

$$288 \hbar^2 x_{n\$25472[1]} y_1 \mathcal{A}_1 \mathcal{A}_3^2 \beta_1 \beta_2 \eta_3 \epsilon_1 + 288 \hbar^2 x_{n\$25472[1]} y_1 \mathcal{A}_1 \mathcal{A}_3^2 \beta_2^2 \eta_3 \epsilon_1 - 576 \gamma^2 \hbar^3 \mathbf{B}_1 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2 \eta_3 \epsilon_1 +$$

$$576 \gamma \hbar^3 a_1 \mathbf{B}_1 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2 \eta_3 \epsilon_1 - 288 \gamma \hbar^3 x_{n\$25472[1]} y_1 y_{n\$25472[1]} \mathcal{A}_3^2 \beta_1 \eta_2 \eta_3 \epsilon_1 +$$

$$576 \gamma \hbar^2 \mathbf{B}_1 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2 \eta_3 \epsilon_1 - 288 \hbar^2 a_1 \mathbf{B}_1 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2 \eta_3 \epsilon_1 -$$

$$288 \gamma \hbar^3 x_{n\$25472[1]} y_1 y_{n\$25472[1]} \mathcal{A}_3^2 \beta_2 \eta_2 \eta_3 \epsilon_1 + 576 \gamma \hbar^2 \mathbf{B}_1 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2 \eta_3 \epsilon_1 -$$

$$288 \hbar^2 a_1 \mathbf{B}_1 y_1 \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2 \eta_3 \epsilon_1 - 864 \gamma^2 \hbar^3 \mathbf{B}_1 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^2 \eta_3 \epsilon_1 -$$

$$144 \gamma \hbar^2 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \eta_3 \epsilon_1 + 1008 \gamma \hbar^2 \mathbf{B}_1 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \eta_3 \epsilon_1 -$$

$$144 \gamma \hbar^2 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2^2 \eta_3 \epsilon_1 + 432 \gamma \hbar^2 \mathbf{B}_1 y_1 y_{n\$25472[1]} \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2^2 \eta_3 \epsilon_1 -$$

$$576 \gamma^2 \hbar^3 \mathbf{B}_1 y_1^2 \mathcal{A}_3^2 \eta_2 \eta_3^2 \epsilon_1 + 576 \gamma \hbar^2 \mathbf{B}_1 y_1^2 \mathcal{A}_3^2 \beta_1 \eta_2 \eta_3^2 \epsilon_1 + 576 \gamma \hbar^2 \mathbf{B}_1 y_1^2 \mathcal{A}_3^2 \beta_2 \eta_2 \eta_3^2 \epsilon_1 +$$

$$144 \hbar^2 x_{n\$25472[1]}^2 \mathcal{A}_1^2 \mathcal{A}_3^2 \beta_2^2 \epsilon_1^2 + 144 \gamma^2 \hbar^4 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_3^2 \eta_2 \epsilon_1^2 + 72 \gamma^2 \hbar^3 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2 \epsilon_1^2 -$$

$$360 \gamma^2 \hbar^3 \mathbf{B}_1 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2 \epsilon_1^2 + 432 \gamma \hbar^3 a_1 \mathbf{B}_1 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2 \epsilon_1^2 -$$

$$288 \gamma \hbar^3 x_{n\$25472[1]}^2 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_3^2 \beta_2 \eta_2 \epsilon_1^2 - 144 \gamma \hbar^2 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2 \epsilon_1^2 +$$

$$432 \gamma \hbar^2 \mathbf{B}_1 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2 \epsilon_1^2 - 288 \hbar^2 a_1 \mathbf{B}_1 x_{n\$25472[1]} \mathcal{A}_1^2 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2 \epsilon_1^2 +$$

$$144 \gamma^2 \hbar^4 x_{n\$25472[1]}^2 y_{n\$25472[1]} \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 + 360 \gamma^2 \hbar^3 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 -$$

$$1512 \gamma^2 \hbar^3 \mathbf{B}_1 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 + 288 \gamma \hbar^3 a_1 \mathbf{B}_1 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 +$$

$$36 \gamma^2 \hbar^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 - 216 \gamma^2 \hbar^2 \mathbf{B}_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 + 288 \gamma \hbar^2 a_1 \mathbf{B}_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 +$$

$$180 \gamma^2 \hbar^2 \mathbf{B}_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 - 432 \gamma \hbar^2 a_1 \mathbf{B}_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 + 144 \hbar^2 a_1^2 \mathbf{B}_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^2 \epsilon_1^2 -$$

$$144 \gamma \hbar^2 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \epsilon_1^2 + 432 \gamma \hbar^2 \mathbf{B}_1 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_1 \eta_2^2 \epsilon_1^2 -$$

$$144 \gamma \hbar^2 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2^2 \epsilon_1^2 + 432 \gamma \hbar^2 \mathbf{B}_1 x_{n\$25472[1]} y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^2 \beta_2 \eta_2^2 \epsilon_1^2 +$$

$$144 \gamma^2 \hbar^3 x_{n\$25472[1]}^2 y_{n\$25472[1]}^2 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^3 \epsilon_1^2 - 432 \gamma^2 \hbar^3 \mathbf{B}_1 x_{n\$25472[1]} y_{n\$25472[1]}^2 \mathcal{A}_2 \mathcal{A}_3^2 \eta_2^3 \epsilon_1^2 +$$

$$120 \gamma^2 \hbar^2 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^3 \epsilon_1^2 - 816 \gamma^2 \hbar^2 \mathbf{B}_1 y_{n\$25472[1]} \mathcal{A}_1 \mathcal{A}_2^2 \mathcal{A}_3^2 \eta_2^3 \epsilon_1^2 +$$

In[*]:= Timing@{R1,2~B1~dS1 ≡ R1,2, HL[R1,2~B2~dS2 ≡ R1,2]}

Out[*]:= {0.75, { $\frac{1}{4 B_1^3} (4 \gamma \in \hbar^2 B_1^2 x_2 y_1 - 2 \gamma^2 \epsilon^2 \hbar^3 B_1^2 x_2 y_1 + 4 \gamma \epsilon^2 \hbar^3 a_2 B_1^2 x_2 y_1 + 8 \gamma^2 \epsilon^2 \hbar^4 B_1 x_2^2 y_1^2 - 4 \gamma \epsilon^2 \hbar^4 a_2 B_1 x_2^2 y_1^2 - 3 \gamma^2 \epsilon^2 \hbar^5 x_2^3 y_1^3) = 0, \text{True}} \}$ }

S is convolution inverse of id

In[*]:= Timing[HL[## ≡ E_{1}→{1} [0, 0, 1]] & /@ { (dA1→1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2→1, (dA1→1,2 ~ B2 ~ dS2) // dm1,2→1 }]

Out[*]:= {9.29688, {True, True}}

S is a (co)-algebra anti-morphism

In[*]:= Timing[HL /@ Expand /@ {dm1,2→1 ~ B1 ~ dS1 ≡ (dS1 dS2) ~ B1,2 ~ dm2,1→1, dS1 ~ B1 ~ dA1→1,2 ≡ dA1→2,1 ~ B1,2 ~ (dS1 dS2)}]

2.453, {E_n\$36157[1], Y_n\$36157[1]}, -a1 α1 - a1 α2 - b1 β1 - b1 β2,

$$\frac{1}{\hbar B_1 \mathcal{A}_1 \mathcal{A}_2} (\hbar B_1 Y_n \mathcal{A}_1 \mathcal{A}_2 \eta_1 + \hbar B_1 Y_n \mathcal{A}_2 \eta_2 - \hbar Y_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n) +$$

$$\hbar B_1 X_n \mathcal{A}_1 \mathcal{A}_2 \eta_1 \xi_1 - \mathcal{A}_1 \mathcal{A}_2 \eta_2 \xi_1 + B_1 \mathcal{A}_1 \mathcal{A}_2 \eta_2 \xi_1 + \hbar B_1 X_n \mathcal{A}_1 \mathcal{A}_2 \xi_2 -$$

$$\hbar B_1 X_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \xi_n + \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n \xi_n - B_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n \xi_n),$$

$$1 + \frac{1}{4 \hbar B_1^2 \mathcal{A}_1 \mathcal{A}_2} (-4 \hbar B_1^2 Y_n \mathcal{A}_2 \beta_1 \eta_2 + 4 \gamma \hbar^2 B_1 Y_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n - 4 \hbar B_1 Y_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \eta_n -$$

$$4 \hbar B_1 Y_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \eta_n - 2 \gamma \hbar^2 Y_1^2 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n - 4 \hbar B_1^2 X_n \mathcal{A}_1 \mathcal{A}_2 \xi_1 + 4 \gamma \hbar^2 B_1^2 X_n \mathcal{A}_1 \mathcal{A}_2 \xi_1 -$$

$$Y_n \mathcal{A}_1 \mathcal{A}_2 \xi_1 - 4 \hbar a_1 B_1 \mathcal{A}_1 \mathcal{A}_2 \eta_2 \xi_1 - 6 \gamma \hbar B_1 Y_n \mathcal{A}_1 \mathcal{A}_2 \eta_2 \xi_1 + 2 \gamma \hbar B_1^2 Y_n \mathcal{A}_1 \mathcal{A}_2 \eta_2 \xi_1 -$$

$$8 \gamma \hbar Y_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2 \xi_1 - 6 \gamma \hbar B_1 X_n \mathcal{A}_1 \mathcal{A}_2 \xi_1^2 + 2 \gamma \hbar B_1^2 X_n \mathcal{A}_1 \mathcal{A}_2 \xi_1^2 + 3 \gamma \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1^2 -$$

$$4 \gamma B_1 \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1^2 + \gamma B_1^2 \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1^2 - 4 \hbar^2 a_1 B_1^2 X_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \xi_n - 4 \hbar B_1^2 X_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \xi_n -$$

$$4 \hbar B_1^2 X_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \xi_n - 4 \gamma \hbar B_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n \xi_n + 4 \hbar a_1 B_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n \xi_n +$$

$$4 \gamma \hbar B_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_n \xi_n - 4 \gamma \hbar^2 B_1 X_1 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \xi_n +$$

$$4 B_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \eta_n \xi_n - 4 B_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \eta_n \xi_n + 4 B_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \eta_n \xi_n -$$

$$4 B_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \eta_n \xi_n + 6 \gamma \hbar Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n -$$

$$2 \gamma \hbar B_1 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n - 8 \gamma \hbar B_1 X_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2 \xi_1 \xi_n +$$

$$8 \gamma \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2 \xi_1 \xi_n - 8 \gamma B_1 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2 \xi_1 \xi_n - 2 \gamma \hbar^2 B_1^2 X_1^2 \mathcal{A}_1^3 \mathcal{A}_2^3 \xi_n^2 +$$

$$6 \gamma \hbar B_1 X_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n - 2 \gamma \hbar B_1^2 X_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n -$$

$$3 \gamma \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n + 4 \gamma B_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n - \gamma B_1^2 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n) \in +$$

$$\frac{1}{288 \hbar^2 B_1^4 \mathcal{A}_1^2 \mathcal{A}_2^2} (144 \hbar^2 B_1^4 Y_n \mathcal{A}_1 \mathcal{A}_2 \beta_1^2 \eta_2 + 144 \hbar^2 B_1^4 Y_n \mathcal{A}_2 \beta_1^2 \eta_2 - 144 \gamma^2 \hbar^4 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_n \xi_n +$$

$$288 \gamma \hbar^3 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \eta_n - 144 \hbar^2 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_n + 288 \gamma \hbar^3 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_n -$$

$$288 \hbar^2 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \beta_2 \eta_n - 144 \hbar^2 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_n - 288 \gamma \hbar^3 B_1^3 Y_1 Y_n \mathcal{A}_1^2 \mathcal{A}_2^3 \beta_1 \eta_n$$

$$\eta_n + 288 \hbar^2 B_1^3 Y_1 Y_n \mathcal{A}_1^2 \mathcal{A}_2^3 \beta_2 \eta_n + 288 \hbar^2 B_1^3 Y_1 Y_n \mathcal{A}_1^2 \mathcal{A}_2^3 \beta_1 \beta_2 \eta_n + 288 \hbar^2 B_1^3 Y_1 Y_n \mathcal{A}_1^2 \mathcal{A}_2^3 \beta_1 \beta_2 \eta_n \eta_n +$$

$$504 \gamma^2 \hbar^4 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_n - 576 \gamma \hbar^3 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_1 \eta_n + 144 \hbar^2 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_2 \eta_n -$$

$$576 \gamma \hbar^3 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_2 \eta_n + 288 \hbar^2 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_1 \beta_2 \eta_n + 144 \hbar^2 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_2 \eta_n +$$

$$144 \gamma \hbar^3 B_1^2 Y_1^2 Y_n \mathcal{A}_1^3 \mathcal{A}_2^4 \beta_1 \eta_n - 288 \gamma^2 \hbar^4 B_1 Y_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n +$$

$$144 \gamma \hbar^3 B_1 Y_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n + 144 \gamma \hbar^3 B_1 Y_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n + 36 \gamma^2 \hbar^4 Y_1^4 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n +$$

$$144 \hbar^2 B_1^4 X_n \mathcal{A}_1 \mathcal{A}_2 \beta_2 \xi_1 + 144 \gamma^2 \hbar^4 B_1^4 X_n Y_n \mathcal{A}_1 \mathcal{A}_2 \xi_1 - 144 \hbar^3 a_1^2 B_1^3 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2 \xi_1 -$$

$$288 \gamma \hbar^3 B_1^4 X_n Y_n \mathcal{A}_1 \mathcal{A}_2 \beta_1 \eta_2 \xi_1 - 288 \gamma \hbar^3 B_1^4 X_n Y_n \mathcal{A}_1 \mathcal{A}_2 \beta_2 \eta_2 \xi_1 +$$

$$288 \hbar^2 B_1^4 X_n Y_n \mathcal{A}_1 \mathcal{A}_2 \beta_1 \beta_2 \eta_2 \xi_1 + 144 \gamma^2 \hbar^4 B_1^4 X_n Y_n \mathcal{A}_2 \eta_2^2 \xi_1 -$$

$$360 \gamma^2 \hbar^3 B_1^3 Y_n \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1 - 432 \gamma \hbar^3 a_1 B_1^3 Y_n \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1 + 72 \gamma^2 \hbar^3 B_1^4 Y_n \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1 -$$

»

$$\begin{aligned}
 & 288 \gamma \hbar^3 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]}^2 \mathcal{A}_2 \beta_1 \eta_2^2 \xi_1 + 432 \gamma \hbar^2 B_1^3 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \beta_1 \eta_2^2 \xi_1 + \\
 & 288 \hbar^2 a_1 B_1^3 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \beta_1 \eta_2^2 \xi_1 - 144 \gamma \hbar^2 B_1^4 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \beta_1 \eta_2^2 \xi_1 - \\
 & 336 \gamma^2 \hbar^3 B_1^3 Y_{n\$36157[1]} \mathcal{A}_2^2 \eta_2^3 \xi_1 + 48 \gamma^2 \hbar^3 B_1^4 Y_{n\$36157[1]} \mathcal{A}_2^2 \eta_2^3 \xi_1 + 432 \gamma \hbar^2 B_1^3 Y_{n\$36157[1]} \mathcal{A}_2^2 \beta_1 \eta_2^3 \xi_1 - \\
 & 144 \gamma \hbar^2 B_1^4 Y_{n\$36157[1]} \mathcal{A}_2^2 \beta_1 \eta_2^3 \xi_1 - 288 \gamma \hbar^3 B_1^3 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_2 \eta_{n\$36157[1]} \xi_1 + \\
 & 288 \hbar^2 B_1^3 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_1 \beta_2 \eta_{n\$36157[1]} \xi_1 + 288 \hbar^2 B_1^3 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_2^2 \eta_{n\$36157[1]} \xi_1 + \\
 & 288 \gamma^2 \hbar^4 B_1^3 X_{n\$36157[1]} Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2 \eta_{n\$36157[1]} \xi_1 + 1152 \gamma^2 \hbar^3 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2 \eta_{n\$36157[1]} \xi_1 - \\
 & 864 \gamma \hbar^3 a_1 B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2 \eta_{n\$36157[1]} \xi_1 - 288 \gamma \hbar^3 B_1^3 X_{n\$36157[1]} Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \eta_2 \eta_{n\$36157[1]} \xi_1 - \\
 & 576 \gamma \hbar^2 B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \eta_2 \eta_{n\$36157[1]} \xi_1 + 288 \hbar^2 a_1 B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \eta_2 \eta_{n\$36157[1]} \xi_1 - \\
 & 288 \gamma \hbar^3 B_1^3 X_{n\$36157[1]} Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \eta_2 \eta_{n\$36157[1]} \xi_1 - 576 \gamma \hbar^2 B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_2 \eta_{n\$36157[1]} \xi_1 + \\
 & 288 \hbar^2 a_1 B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_2 \eta_{n\$36157[1]} \xi_1 - 1296 \gamma^2 \hbar^3 B_1^2 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \eta_{n\$36157[1]} \xi_1 + \\
 & 144 \gamma^2 \hbar^3 B_1^3 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \eta_{n\$36157[1]} \xi_1 + 1008 \gamma \hbar^2 B_1^2 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \eta_2^2 \eta_{n\$36157[1]} \xi_1 - \\
 & 144 \gamma \hbar^2 B_1^3 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_1 \eta_2^2 \eta_{n\$36157[1]} \xi_1 + 432 \gamma \hbar^2 B_1^2 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \eta_2^2 \eta_{n\$36157[1]} \xi_1 - \\
 & 144 \gamma \hbar^2 B_1^3 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \beta_2 \eta_2^2 \eta_{n\$36157[1]} \xi_1 + 144 \gamma \hbar^3 B_1^2 X_{n\$36157[1]} Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_2 \eta_{n\$36157[1]} \xi_1 - \\
 & 144 \gamma^2 \hbar^4 B_1^2 X_{n\$36157[1]} Y_1^2 Y_{n\$36157[1]} \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2 \eta_{n\$36157[1]} \xi_1 - 1728 \gamma^2 \hbar^3 B_1 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2 \eta_{n\$36157[1]} \xi_1 + \\
 & 144 \gamma \hbar^3 a_1 B_1 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2 \eta_{n\$36157[1]} \xi_1 + 576 \gamma \hbar^2 B_1 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_1 \eta_2 \eta_{n\$36157[1]} \xi_1 + \\
 & 576 \gamma \hbar^2 B_1 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \beta_2 \eta_2 \eta_{n\$36157[1]} \xi_1 + 216 \gamma^2 \hbar^3 B_1 Y_1^2 Y_{n\$36157[1]} \mathcal{A}_1^3 \mathcal{A}_2^4 \eta_2^2 \eta_{n\$36157[1]} \xi_1 - \\
 & 72 \gamma^2 \hbar^3 B_1^2 Y_1^2 Y_{n\$36157[1]} \mathcal{A}_1^3 \mathcal{A}_2^4 \eta_2^2 \eta_{n\$36157[1]} \xi_1 + 288 \gamma^2 \hbar^3 Y_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_{n\$36157[1]} \xi_1 + \\
 & 144 \hbar^2 B_1^4 X_{n\$36157[1]} \mathcal{A}_1^2 \beta_2^2 \xi_1^2 + 144 \gamma^2 \hbar^4 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \eta_2 \xi_1^2 - \\
 & 360 \gamma^2 \hbar^3 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2 \xi_1^2 - 432 \gamma \hbar^3 a_1 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2 \xi_1^2 + 72 \gamma^2 \hbar^3 B_1^4 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2 \xi_1^2 - \\
 & 288 \gamma \hbar^3 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \beta_2 \eta_2 \xi_1^2 + 432 \gamma \hbar^2 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \beta_2 \eta_2 \xi_1^2 + \\
 & 288 \hbar^2 a_1 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \beta_2 \eta_2 \xi_1^2 - 144 \gamma \hbar^2 B_1^4 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \beta_2 \eta_2 \xi_1^2 + \\
 & 144 \gamma^2 \hbar^4 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \eta_2^2 \xi_1^2 - 1512 \gamma^2 \hbar^3 B_1^3 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1^2 - \\
 & 288 \gamma \hbar^3 a_1 B_1^3 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1^2 + 360 \gamma^2 \hbar^3 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \eta_2^2 \xi_1^2 + \\
 & 180 \gamma^2 \hbar^2 B_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \xi_1^2 + 432 \gamma \hbar^2 a_1 B_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \xi_1^2 + 144 \hbar^2 a_1^2 B_1^2 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \xi_1^2 - 216 \gamma^2 \hbar^2 B_1^3 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \xi_1^2 - \\
 & 288 \gamma \hbar^2 a_1 B_1^3 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \xi_1^2 + 36 \gamma^2 \hbar^2 B_1^4 \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \xi_1^2 + 432 \gamma \hbar^2 B_1^3 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \beta_1 \eta_2^2 \xi_1^2 - \\
 & 144 \gamma \hbar^2 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \beta_1 \eta_2^2 \xi_1^2 + 432 \gamma \hbar^2 B_1^3 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \beta_2 \eta_2^2 \xi_1^2 - \\
 & 144 \gamma \hbar^2 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2 \beta_2 \eta_2^2 \xi_1^2 - 432 \gamma^2 \hbar^3 B_1^3 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_2 \eta_2^3 \xi_1^2 + \\
 & 144 \gamma^2 \hbar^3 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_2 \eta_2^3 \xi_1^2 + 984 \gamma^2 \hbar^2 B_1^2 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \eta_2^3 \xi_1^2 + \\
 & 432 \gamma \hbar^2 a_1 B_1^2 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \eta_2^3 \xi_1^2 - 816 \gamma^2 \hbar^2 B_1^3 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \eta_2^3 \xi_1^2 - \\
 & 144 \gamma \hbar^2 a_1 B_1^3 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \eta_2^3 \xi_1^2 + 120 \gamma^2 \hbar^2 B_1^4 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \eta_2^3 \xi_1^2 - \\
 & 216 \gamma \hbar B_1^2 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \beta_1 \eta_2^3 \xi_1^2 + 288 \gamma \hbar B_1^3 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \beta_1 \eta_2^3 \xi_1^2 - 72 \gamma \hbar B_1^4 Y_{n\$36157[1]} \mathcal{A}_1 \mathcal{A}_2^2 \beta_1 \eta_2^3 \xi_1^2 + \\
 & 324 \gamma^2 \hbar^2 B_1^2 Y_{n\$36157[1]} \mathcal{A}_2^2 \eta_2^4 \xi_1^2 - 216 \gamma^2 \hbar^2 B_1^3 Y_{n\$36157[1]} \mathcal{A}_2^2 \eta_2^4 \xi_1^2 + 36 \gamma^2 \hbar^2 B_1^4 Y_{n\$36157[1]} \mathcal{A}_2^2 \eta_2^4 \xi_1^2 - \\
 & 1296 \gamma^2 \hbar^3 B_1^2 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \eta_2 \eta_{n\$36157[1]} \xi_1^2 + 144 \gamma^2 \hbar^3 B_1^3 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \eta_2 \eta_{n\$36157[1]} \xi_1^2 + \\
 & 432 \gamma \hbar^2 B_1^2 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_1 \eta_2 \eta_{n\$36157[1]} \xi_1^2 - 144 \gamma \hbar^2 B_1^3 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_1 \eta_2 \eta_{n\$36157[1]} \xi_1^2 + \\
 & 1008 \gamma \hbar^2 B_1^2 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_2 \eta_2 \eta_{n\$36157[1]} \xi_1^2 - 144 \gamma \hbar^2 B_1^3 X_{n\$36157[1]} Y_1 \mathcal{A}_1^3 \mathcal{A}_2^2 \beta_2 \eta_2 \eta_{n\$36157[1]} \xi_1^2 - \\
 & 576 \gamma^2 \hbar^3 B_1^2 X_{n\$36157[1]} Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^2 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + 504 \gamma^2 \hbar^2 B_1 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + \\
 & 576 \gamma \hbar^2 a_1 B_1 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - 864 \gamma^2 \hbar^2 B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + \\
 & 72 \gamma^2 \hbar^2 B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - 216 \gamma \hbar B_1 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + \\
 & 288 \gamma \hbar B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - 72 \gamma \hbar B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_1 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - \\
 & 216 \gamma \hbar B_1 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + 288 \gamma \hbar B_1^2 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - \\
 & 72 \gamma \hbar B_1^3 Y_1 \mathcal{A}_1^3 \mathcal{A}_2^3 \beta_2 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + 864 \gamma^2 \hbar^2 B_1 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^3 \eta_2^3 \eta_{n\$36157[1]} \xi_1^2 - \\
 & 288 \gamma^2 \hbar^2 B_1^2 Y_1 Y_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2^3 \eta_2^3 \eta_{n\$36157[1]} \xi_1^2 + 216 \gamma^2 \hbar^3 B_1 X_{n\$36157[1]} Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2 \eta_{n\$36157[1]} \xi_1^2 - \\
 & 72 \gamma^2 \hbar^3 B_1^2 X_{n\$36157[1]} Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2 \eta_{n\$36157[1]} \xi_1^2 + 468 \gamma^2 \hbar^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 + \\
 & 144 \gamma^2 \hbar^2 B_1 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - 36 \gamma^2 \hbar^2 B_1^2 Y_1^2 \mathcal{A}_1^4 \mathcal{A}_2^4 \eta_2^2 \eta_{n\$36157[1]} \xi_1^2 - 336 \gamma^2 \hbar^3 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \eta_2 \xi_1^3 + \\
 & 48 \gamma^2 \hbar^3 B_1^4 X_{n\$36157[1]} \mathcal{A}_1^2 \eta_2 \xi_1^3 + 432 \gamma \hbar^2 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \beta_2 \eta_2 \xi_1^3 - 144 \gamma \hbar^2 B_1^4 X_{n\$36157[1]} \mathcal{A}_1^2 \beta_2 \eta_2 \xi_1^3 - \\
 & 432 \gamma^2 \hbar^3 B_1^3 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \eta_2^2 \xi_1^3 + 144 \gamma^2 \hbar^3 B_1^4 X_{n\$36157[1]} Y_{n\$36157[1]} \mathcal{A}_1 \eta_2^2 \xi_1^3 + \\
 & 984 \gamma^2 \hbar^2 B_1^2 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2^2 \xi_1^3 + 432 \gamma \hbar^2 a_1 B_1^2 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2^2 \xi_1^3 - \\
 & 816 \gamma^2 \hbar^2 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2^2 \xi_1^3 - 144 \gamma \hbar^2 a_1 B_1^3 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2^2 \xi_1^3 + 120 \gamma^2 \hbar^2 B_1^4 X_{n\$36157[1]} \mathcal{A}_1^2 \mathcal{A}_2 \eta_2^2 \xi_1^3 -
 \end{aligned}$$

$$\begin{aligned}
 & 864 \gamma \hbar B_1^3 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n^2 \mathcal{E}_n^3 + 216 \gamma \hbar B_1^4 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n^2 \mathcal{E}_n^3 + \\
 & 648 \gamma \hbar B_1^2 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^2 \mathcal{E}_n^3 - 864 \gamma \hbar B_1^3 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^2 \mathcal{E}_n^3 + \\
 & 216 \gamma \hbar B_1^4 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^2 \mathcal{E}_n^3 + 544 \gamma^2 \hbar B_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 - \\
 & 216 \gamma \hbar a_1 B_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 - 1104 \gamma^2 \hbar B_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 + \\
 & 288 \gamma \hbar a_1 B_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 + 672 \gamma^2 \hbar B_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 - \\
 & 72 \gamma \hbar a_1 B_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 - 112 \gamma^2 \hbar B_1^4 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_n^3 \mathcal{E}_n^3 + \\
 & 864 \gamma^2 \hbar^2 B_1 x_1 y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^3 - 720 \gamma^2 \hbar^2 B_1^2 x_1 y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^3 + \\
 & 144 \gamma^2 \hbar^2 B_1^3 x_1 y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^3 - 216 \gamma B_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n^3 \mathcal{E}_n^3 + \\
 & 504 \gamma B_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n^3 \mathcal{E}_n^3 - 360 \gamma B_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n^3 \mathcal{E}_n^3 + \\
 & 72 \gamma B_1^4 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_1 \eta_n^3 \mathcal{E}_n^3 - 216 \gamma B_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^3 \mathcal{E}_n^3 + \\
 & 504 \gamma B_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^3 \mathcal{E}_n^3 - 360 \gamma B_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^3 \mathcal{E}_n^3 + \\
 & 72 \gamma B_1^4 \mathcal{A}_1^5 \mathcal{A}_2^5 \beta_2 \eta_n^3 \mathcal{E}_n^3 - 324 \gamma^2 \hbar y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^3 + \\
 & 540 \gamma^2 \hbar B_1 y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^3 - 252 \gamma^2 \hbar B_1^2 y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^3 + \\
 & 36 \gamma^2 \hbar B_1^3 y_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^3 + 288 \gamma^2 \hbar^3 B_1^3 x_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \mathcal{E}_1 \mathcal{E}_n^3 - \\
 & 1152 \gamma^2 \hbar^2 B_1^2 x_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n \mathcal{E}_1 \mathcal{E}_n^3 + 576 \gamma^2 \hbar^2 B_1^3 x_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n \mathcal{E}_1 \mathcal{E}_n^3 + \\
 & 1296 \gamma^2 \hbar B_1 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^2 \mathcal{E}_1 \mathcal{E}_n^3 - 1728 \gamma^2 \hbar B_1^2 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^2 \mathcal{E}_1 \mathcal{E}_n^3 + \\
 & 432 \gamma^2 \hbar B_1^3 x_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^2 \mathcal{E}_1 \mathcal{E}_n^3 - 432 \gamma^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^3 \mathcal{E}_1 \mathcal{E}_n^3 + \\
 & 1008 \gamma^2 B_1 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^3 \mathcal{E}_1 \mathcal{E}_n^3 - 720 \gamma^2 B_1^2 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^3 \mathcal{E}_1 \mathcal{E}_n^3 + \\
 & 144 \gamma^2 B_1^3 \mathcal{A}_1^5 \mathcal{A}_2^5 \eta_2 \eta_n^3 \mathcal{E}_1 \mathcal{E}_n^3 + 36 \gamma^2 \hbar^4 B_1^4 x_1^4 \mathcal{A}_1^6 \mathcal{A}_2^6 \mathcal{E}_n^4 - \\
 & 216 \gamma^2 \hbar^3 B_1^3 x_1^3 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n \mathcal{E}_n^4 + 72 \gamma^2 \hbar^3 B_1^4 x_1^3 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n \mathcal{E}_n^4 + \\
 & 432 \gamma^2 \hbar^2 B_1^2 x_1^2 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^2 \mathcal{E}_n^4 - 360 \gamma^2 \hbar^2 B_1^3 x_1^2 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^2 \mathcal{E}_n^4 + \\
 & 72 \gamma^2 \hbar^2 B_1^4 x_1^2 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^2 \mathcal{E}_n^4 - 324 \gamma^2 \hbar B_1 x_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^4 + \\
 & 540 \gamma^2 \hbar B_1^2 x_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^4 - 252 \gamma^2 \hbar B_1^3 x_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^4 + \\
 & 36 \gamma^2 \hbar B_1^4 x_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^3 \mathcal{E}_n^4 + 81 \gamma^2 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^4 - \\
 & 216 \gamma^2 B_1 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^4 + 198 \gamma^2 B_1^2 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^4 - \\
 & 72 \gamma^2 B_1^3 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^4 + 9 \gamma^2 B_1^4 \mathcal{A}_1^6 \mathcal{A}_2^6 \eta_n^4 \mathcal{E}_n^4 \Big) \in^2 + O[\epsilon]^3 \Big\}
 \end{aligned}$$

Out[*]= {18.6875, {True, True}}

Quasi-triangular axiom 1:

$$In[*]= \text{Timing@HL}[R_{1,2} \sim B_1 \sim dA_{1 \rightarrow 1,3} \equiv (R_{1,4} R_{3,2}) \sim B_{2,4} \sim dm_{2,4 \rightarrow 2}]$$

Out[*]= {0.5, True}

Quasi-triangular axiom 2:

$$In[*]= \text{Timing@HL} \left[\left((dA_{1 \rightarrow 1,2} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}) \right) \equiv \left((dA_{1 \rightarrow 2,1} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1 \rightarrow 1} dm_{4,2 \rightarrow 2}) \right) \right]$$

Out[*]= {6.01563, True}

The Drinfel'd element inverse property, $(u_1 \bar{u}_2) \sim B_{1,2} \sim dm_{1,2 \rightarrow 1} \equiv \mathbb{E}[0, 0, 1]$:

$$In[*]= \text{Timing@HL} \left[\left((R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1 \rightarrow 1}) (R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1 \rightarrow j}) \right) \sim B_{i,j} \sim dm_{i,j \rightarrow i} \equiv \mathbb{E}_{\{i \rightarrow j\}}[0, 0, 1] \right]$$

Out[*]= {2.95313, True}

The ribbon element v satisfies $v^2 = S(u)u$. The spinner $C=uv^{-1}$. It is convenient to compute $z = S(u)u^{-1}$ which is something easy.

In[*]:= **Timing@Block** [{ \$k = 2,
 $((R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1 \rightarrow i}) \sim B_i \sim dS_i) (R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1 \rightarrow j})) \sim B_{i,j} \sim dm_{i,j \rightarrow i}$]

Out[*]:= { 3.67188, $\mathbb{E}_{\{\} \rightarrow \{i\}}$ [$\theta, \theta, \frac{1}{B_i} + \frac{\hbar a_i \epsilon}{B_i} + \frac{\hbar^2 a_i^2 \epsilon^2}{2 B_i} + O[\epsilon^3]$] }

In[*]:= **Timing@Block** [{ \$k = 2, **HL** /@ { $(C_i \bar{C}_j) \sim B_{i,j} \sim dm_{i,j \rightarrow i} \equiv \mathbb{E}_{\{\} \rightarrow \{i\}} [\theta, \theta, 1]$, $(\bar{C}_i \bar{C}_j) \sim B_{i,j} \sim dm_{i,j \rightarrow i} \equiv$
 $((R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1 \rightarrow i}) \sim B_i \sim dS_i) (R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1 \rightarrow j})) \sim B_{i,j} \sim dm_{i,j \rightarrow i}$ }]

Out[*]:= { 4.21875, { **True**, **True** } }

Reidemeister 2:

In[*]:= **Timing** [**HL** [# $\equiv \mathbb{E}_{\{\} \rightarrow \{1,2\}} [\theta, \theta, 1]$] & /@
 $\{ (\bar{R}_{1,2} \bar{R}_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}), (R_{1,2} \bar{R}_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}) \}$]

Out[*]:= { 5.35938, { **True**, **True** } }

Cyclic Reidemeister 2:

In[*]:= **Timing@HL** [$(R_{1,4} \bar{R}_{5,2} \bar{C}_3) \sim B_{2,4} \sim dm_{2,4 \rightarrow 2} \sim B_{1,3} \sim dm_{1,3 \rightarrow 1} \sim B_{1,5} \sim dm_{1,5 \rightarrow 1} \equiv \bar{C}_1 \mathbb{E}_{\{\} \rightarrow \{2\}} [\theta, \theta, 1]$]

Out[*]:= { 2.34375, **True** }

Reidemeister 3:

In[*]:= **Timing@HL** [$((R_{1,2} R_{4,3} R_{5,6}) \sim B_{1,4} \sim dm_{1,4 \rightarrow 1} \sim B_{2,5} \sim dm_{2,5 \rightarrow 2} \sim B_{3,6} \sim dm_{3,6 \rightarrow 3}) \equiv$
 $(R_{1,6} R_{2,3} R_{4,5}) \sim B_{1,4} \sim dm_{1,4 \rightarrow 1} \sim B_{2,5} \sim dm_{2,5 \rightarrow 2} \sim B_{3,6} \sim dm_{3,6 \rightarrow 3}$]

Out[*]:= { 4.98438, **True** }

Relations between the four kinks:

In[*]:= **Timing** [**HL** /@ { **Kink**_i $\equiv (R_{3,1} C_2) \sim B_{1,2} \sim dm_{1,2 \rightarrow 1} \sim B_{1,3} \sim dm_{1,3 \rightarrow 1}$,
 $\bar{\mathbf{Kink}}_j \equiv (\bar{R}_{3,1} \bar{C}_2) \sim B_{1,2} \sim dm_{1,2 \rightarrow 1} \sim B_{1,3} \sim dm_{1,3 \rightarrow 1}$, $(\mathbf{Kink}_i \bar{\mathbf{Kink}}_j) \sim B_{i,j} \sim dm_{i,j \rightarrow 1} \equiv \mathbb{E}_{\{\} \rightarrow \{1\}} [\theta, \theta, 1]$ }]

Out[*]:= { 4.89063, { **True**, **True**, **True** } }

The Trefoil

In[*]:= **Timing@Block** [{ \$k = 1,
 $Z = R_{1,5} R_{6,2} R_{3,7} \bar{C}_4 \bar{\mathbf{Kink}}_8 \bar{\mathbf{Kink}}_9 \bar{\mathbf{Kink}}_{10}$;
Do [$Z = Z \sim B_{1,r} \sim dm_{1,r \rightarrow 1}$, { r, 2, 10 }];
Simplify /@ Z, **Simplify** /@ ($Z \sim B_1 \sim b2t_1 /. T_1 \rightarrow T$)]]

Out[*]:= { 9.17188, $\mathbb{E}_{\{\} \rightarrow \{1\}}$ [$\theta, \theta,$
 $\frac{B_1}{1 - B_1 + B_1^2} - (\hbar B_1 (-a_1 (-1 + B_1 - B_1^3 + B_1^4) + \gamma (B_1 - 2 B_1^2 - 2 B_1^4 + 2 \hbar x_1 y_1 + B_1^3 (3 + 2 \hbar x_1 y_1))) \epsilon) /$
 $(1 - B_1 + B_1^2)^3 + O[\epsilon^2]$, $\mathbb{E}_{\{\} \rightarrow \{1\}}$ [$\theta, \theta,$
 $\frac{T}{1 - T + T^2} + (T \hbar (T (-1 + 2 T - 3 T^2 + 2 T^3) \gamma + 2 (-1 + T - T^3 + T^4) a_1 - 2 (1 + T^3) \gamma \hbar x_1 y_1) \epsilon) /$
 $(1 - T + T^2)^3 + O[\epsilon^2]$]] }

Program

```
In[*]:= Define [kRi,j = Ri,j // (b2ti b2tj) /. ti|j → t,
kR̄i,j = R̄i,j // (b2ti b2tj) /. {ti|j → t, Ti|j → T},
kmi,j→k = (t2bi t2bj) // dmi,j→k // b2tk /. {tk → t, Tk → T, τi|j → 0},
kCi = Ci // b2ti /. Ti → T,
kC̄i = C̄i // b2ti /. Ti → T,
kKinki = Kinki // b2ti /. {ti → t, Ti → T},
kKink̄i = Kink̄i // b2ti /. {ti → t, Ti → T}]
```

```
In[*]:= Timing@Block[{k = 1},
Z = kR1,5 kR6,2 kR3,7 kC4 kKink8 kKink9 kKink10;
Do[Z = Z ~ B1,r ~ km1,r→1, {r, 2, 10}];
Simplify /@ Z]
```

```
Out[*]:= {4.71875, E{}→{1}} [0, 0,
T / (1 - T + T2) + (T ħ (T (-1 + 2 T - 3 T2 + 2 T3) γ + 2 (-1 + T - T3 + T4) a1 - 2 (1 + T3) γ ħ x1 y1) ε) /
(1 - T + T2)3 + O[ε]2]
```

RVK, rot, Z from 2016-09/OneSmidgen.nb. See also local version in this folder.

Program

Some details of the code below are at <http://drorbn.net/bbs/show?shot=Dror-160920-151350.jpg>.

Program

```
In[*]:= RVK::usage =
"RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
xs and a length 2n list of rotation numbers rots. Crossing
sites are indexed 1 through 2n, and rots[[k]] is the rotation
between site k-1 and site k. RVK is also a casting operator
converting to the RVK presentation from other knot presentations.";
```

Program

```
In[*]:= RVK[pd_PD] := PPRVK@Module[{n, xs, x, rots, front = {0}, k},
n = Length@pd; rots = Table[0, {2 n}];
xs = Cases[pd, x_X => {Xp[x[[4]], x[[1]] PositiveQ@x},
{Xm[x[[2]], x[[1]] True}];
For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
front = Flatten[front /. k → (xs /. {
Xp[k + 1, L_] | Xm[L_, k + 1] => {L, k + 1, 1 - L},
Xp[L_, k + 1] | Xm[k + 1, L_] => (++rots[[L]]; {1 - L, k + 1, L})
})],
Cases[front, k | -k] /. {k, -k} => --rots[[k + 1]];
]];
RVK[xs, rots];
RVK[K_] := RVK[PD[K]];
```

```
In[*]:= xs = Cases[pd, x_X => If[PositiveQ@x, Xp[x[[4]], x[[1]], Xm[x[[2]], x[[1]]]]];
```

```
In[ ]:= RVK[Knot[10, 100]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[ ]:= RVK[{Xp[1, 6], Xp[5, 18], Xm[13, 20], Xm[7, 14], Xm[3, 10], Xm[9, 16], Xm[11, 4], Xm[15, 8],
  Xm[19, 12], Xp[17, 2]}, {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}]
```

Program

```
In[ ]:= rot[i_, 0] := E_{i} [0, 0, 1];
rot[i_, n_] := Module[{j},
  rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC_j] // km_{i,j-i};
```

Program

```
In[ ]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
PP_Z@Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List@@rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = E_{i} [0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[Echo@todo != {},
    {cx} = MaximalBy[todo, Length[done ∩ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
    {i, j} = List@@cx;
    ξ1 = Switch[Head[cx],
      Xp, (kR_{i,j} kKink_k) // km_{j,k→j},
      Xm, (kR_{i,j} kKink_k) // km_{j,k→j}
    ];
    ξ1 = (rot[k, rots[[i]] ξ1) // km_{k,i→i}; rots[[i]] = 0;
    ξ1 = (ξ1 rot[k, rots[[i + 1]]) // km_{i,k→i}; rots[[i + 1]] = 0;
    ξ1 = (rot[k, rots[[j]] ξ1) // km_{k,j→j}; rots[[j]] = 0;
    ξ1 = (ξ1 rot[k, rots[[j + 1]]) // km_{j,k→j}; rots[[j + 1]] = 0;
    ξ *= Echo@ξ1;
    If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]];
    If[MemberQ[done, i - 1], ξ = ξ // km_{st[[i],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]];
    If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]];
    If[MemberQ[done, j - 1], ξ = ξ // km_{st[[j],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]];
    done = done ∪ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, cx]
  ];
  Simplify/@ (ξ /. {x_0 → x, y_0 → y, a_0 → a})
]
```


Knot

In[*]:= \$k = 1; Timing@Z@Knot[10, 100]

Knot

Out[*]:= $\left\{ 259.031, \mathbb{E}_{\{\cdot\} \rightarrow \{\emptyset\}} \left[0, 0, T^4 / \left(1 - 4T + 9T^2 - 12T^3 + 13T^4 - 12T^5 + 9T^6 - 4T^7 + T^8 \right) + \right. \right.$
 $\left. \left(T^4 \hbar \left(4a \left(-2 + 14T - 51T^2 + 120T^3 - 203T^4 + 258T^5 - 246T^6 + 152T^7 - \right. \right. \right. \right.$
 $\left. \left. \left. 152T^9 + 246T^{10} - 258T^{11} + 203T^{12} - 120T^{13} + 51T^{14} - 14T^{15} + 2T^{16} \right) + \right.$
 $\left. \gamma \left(-6 + 2T^{16} - 8xy\hbar - 440T^9 \left(-1 + xy\hbar \right) - 4T^{15} \left(3 + 2xy\hbar \right) + 8T^8 \left(-97 + 21xy\hbar \right) + \right. \right.$
 $\left. 8T^7 \left(131 + 21xy\hbar \right) - 20T^6 \left(57 + 22xy\hbar \right) + T^{14} \left(37 + 48xy\hbar \right) + T \left(44 + 48xy\hbar \right) - \right.$
 $\left. 8T^{11} \left(2 + 61xy\hbar \right) + 8T^5 \left(127 + 68xy\hbar \right) - 2T^{13} \left(35 + 78xy\hbar \right) + 4T^{10} \left(-39 + 136xy\hbar \right) - \right.$
 $\left. T^2 \left(167 + 156xy\hbar \right) + T^{12} \left(79 + 324xy\hbar \right) + T^3 \left(410 + 324xy\hbar \right) - T^4 \left(733 + 488xy\hbar \right) \right) \right)$
 $\epsilon) / \left(1 - 4T + 9T^2 - 12T^3 + 13T^4 - 12T^5 + 9T^6 - 4T^7 + T^8 \right)^3 + 0[\epsilon]^2 \}$

In[*]:= EndProfile[];

Profile

In[*]:= PrintProfile[]

Profile

Out[*]:= ProfileRoot is root. Profiled time: 371.875

(1)	0.250/	259.031	above Z
(157)	0.622/	96.717	above B
(147)	0.015/	0.061	above CF
(2)	0/	0	above RVK
(17)	0.047/	3.329	above Boot[1]
(15)	0.062/	8.174	above Boot[2]
(4)	0.047/	0.110	above Boot[3]
(1)	0.015/	4.453	above Boot[4]

Exp: called 251506 times, time in 115.182/182.088

(251506)	115.182/	182.088	under Together
(246655)	34.751/	66.906	above CF

Together: called 251506 times, time in 89.514/271.602

(251506)	89.514/	271.602	under CF
(251506)	115.182/	182.088	above Exp

CF: called 251506 times, time in 64.479/336.081

(246655)	34.751/	66.906	under Exp
(1764)	25.305/	152.191	under LZip
(147)	0.015/	0.061	under ProfileRoot
(2940)	4.408/	116.923	under QZip
(251506)	89.514/	271.602	above Together

LZip: called 294 times, time in 48.969/229.918

(294)	48.969/	229.918	under B
(1764)	25.305/	152.191	above CF
(294)	6.124/	28.758	above Zip

Zip: called 2675 times, time in 37.595/125.753

(294)	6.124/	28.758	under LZip
(294)	4.439/	15.998	under QZip
(2087)	27.032/	80.997	under Zip
(2675)	7.161/	7.161	above Collect
(2087)	27.032/	80.997	above Zip

QZip: called 294 times, time in 7.352/140.273

(294)	7.352/	140.273	under B
(2940)	4.408/	116.923	above CF

```

( 294) 4.439/ 15.998 above Zip
Collect: called 2675 times, time in 7.161/7.161
( 2675) 7.161/ 7.161 under Zip
B: called 294 times, time in 1.06/371.251
( 72) 0.297/ 258.672 under Z
( 157) 0.622/ 96.717 under ProfileRoot
( 33) 0.015/ 3.391 under Boot[1]
( 25) 0.079/ 8.080 under Boot[2]
( 3) 0.016/ 1.079 under Boot[3]
( 4) 0.031/ 3.312 under Boot[4]
( 294) 48.969/ 229.918 above LZip
( 294) 7.352/ 140.273 above QZip
Z: called 1 times, time in 0.25/259.031
( 1) 0.250/ 259.031 under ProfileRoot
( 72) 0.297/ 258.672 above B
( 3) 0/ 0.109 above Boot[1]
Boot[4]: called 6 times, time in 0.109/8.625
( 1) 0.015/ 4.453 under ProfileRoot
( 3) 0.031/ 0.031 under Boot[3]
( 2) 0.063/ 4.141 under Boot[4]
( 4) 0.031/ 3.312 above B
( 1) 0.016/ 1.063 above Boot[3]
( 2) 0.063/ 4.141 above Boot[4]
Boot[2]: called 18 times, time in 0.094/8.691
( 15) 0.062/ 8.174 under ProfileRoot
( 3) 0.032/ 0.517 under Boot[2]
( 25) 0.079/ 8.080 above B
( 3) 0.032/ 0.517 above Boot[2]
Boot[3]: called 5 times, time in 0.063/1.173
( 4) 0.047/ 0.110 under ProfileRoot
( 1) 0.016/ 1.063 under Boot[4]
( 3) 0.016/ 1.079 above B
( 3) 0.031/ 0.031 above Boot[4]
Boot[1]: called 27 times, time in 0.047/4.702
( 3) 0/ 0.109 under Z
( 17) 0.047/ 3.329 under ProfileRoot
( 7) 0/ 1.264 under Boot[1]
( 33) 0.015/ 3.391 above B
( 3) 0/ 0 above Boot[0]
( 7) 0/ 1.264 above Boot[1]
Boot[0]: called 3 times, time in 0./0.
( 3) 0/ 0 under Boot[1]
RVK: called 2 times, time in 0./0.
( 2) 0/ 0 under ProfileRoot

```