

Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

Profiling

```
In[ ]:=
Once[
  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Invariant"];
  << KnotTheory` ;
  << "../Profile/Profile.m";
];
BeginProfile[];
Once@PopupWindow[Button["Show Profile Monitor"],
  Dynamic[PrintProfile[], UpdateInterval -> 3, TrackedSymbols -> {}]]]
```

Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.

Read more at <http://katlas.org/wiki/KnotTheory>.

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: June 2018. Original version: July 1994.

Out[]:=

External Utilities

```
In[ ]:=
HL[ε_] := Style[ε, Background -> Yellow];
```

Program

Program

Internal Utilities

Program

Canonical Form:

Program

```
In[ ]:=
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := PPCF@ExpandDenominator@ExpandNumerator@PPTogether@Together[PPExp[
  Expand[ε] /. ex ey -> ex+y /. ex -> eCF[x]]]];
```

Program

The Kronecker δ :

Program

```
In[ ]:=
Kδ /: Kδi,j := If[i == j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

```
In[*]:=
E /: E[L1_, Q1_, P1_] ≡ E[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
E /: E[L1_, Q1_, P1_] E[L2_, Q2_, P2_] := E[L1 + L2, Q1 + Q2, P1 * P2];
E[L_, Q_, P_]$k_ := E[L, Q, Series[Normal@P, {ε, 0, $k}]]];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

```
In[*]:=
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})* := (u*)_i;
```

Program

Finite Zips:

Program

```
In[*]:=
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ε_, ζ_] := PPCollect@Collect[ε, ζ];
Zip[_][P_] := P; Zip[ζ_, ζs_][P_] := PPZip[
  (collect[P // Zip[ζs], ζ] /. f_ . ζ^{d_} => ∂_{ζ*, d} f) /. ζ* -> 0]
```

Program

QZip implements the “Q-level zips” on $E(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Program

```
In[*]:=
QZip[ζs_List@E[L_, Q_, P_] := PPQZip@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
  zs = Table[ζ*, {ζ, ζs}];
  c = Q /. Alternatives @@ (ζs ∪ zs) -> 0;
  ys = Table[∂_ζ (Q /. Alternatives @@ zs -> 0), {ζ, ζs}];
  ηs = Table[∂_z (Q /. Alternatives @@ ζs -> 0), {z, zs}];
  qt = Inverse@Table[Kδ_{z, ζ*} - ∂_{z, ζ} Q, {ζ, ζs}, {z, zs}];
  zrule = Thread[zs -> qt.(zs + ys)];
  Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs -> 0;
  CF /@ E[L, Q2, Det[qt] e^{-Q2} Zip_ζs[e^{Q1} (P /. zrule)]]];
```

Program

Upper to lower and lower to Upper:

Program

```
In[*]:=
U2l = {B_{i_}^{p_} -> e^{-p ħ γ b_i}, B_{i_}^{p_} -> e^{-p ħ γ b}, T_{i_}^{p_} -> e^{p ħ t_i}, T_{i_}^{p_} -> e^{p ħ t}, A_{i_}^{p_} -> e^{p γ α_i}, A_{i_}^{p_} -> e^{p γ α}};
L2U = {e^{c_ . b_i + d_} -> B_{i_}^{-c / (ħ γ)} e^d, e^{c_ . b + d_} -> B^{-c / (ħ γ)} e^d,
  e^{c_ . t_i + d_} -> T_{i_}^{c / ħ} e^d, e^{c_ . t + d_} -> T^{c / ħ} e^d,
  e^{c_ . α_i + d_} -> A_{i_}^{c / γ} e^d, e^{c_ . α + d_} -> A^{c / γ} e^d,
  e^ε -> e^{Expand@ε}}];
```

Program

LZip implements the “L-level zips” on $E(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single “P”. Here the z ’s are b and α and the ζ ’s are β and a .

Program

```
In[*]:= LZip $\xi_S$ List@E[L_, Q_, P_] := PPLZip@Module[{ $\xi$ , z, zs, c, ys,  $\eta_S$ , lt, zrule, L1, L2, Q1, Q2},
  zs = Table[ $\xi^*$ , { $\xi$ ,  $\xi_S$ }]];
  c = L /. Alternatives@@(z $\xi_S \cup zs$ )  $\rightarrow$  0;
  ys = Table[ $\partial_\xi(L /. Alternatives@@zs \rightarrow 0)$ , { $\xi$ ,  $\xi_S$ }]];
   $\eta_S$  = Table[ $\partial_z(L /. Alternatives@@z\xi_S \rightarrow 0)$ , {z, zs}];
  lt = Inverse@Table[K $\delta_{z,\xi^*} - \partial_{z,\xi}L$ , { $\xi$ ,  $\xi_S$ }, {z, zs}];
  zrule = Thread[zs  $\rightarrow$  lt.(zs + ys)];
  L2 = (L1 = c +  $\eta_S.zs$  /. zrule) /. Alternatives@@zs  $\rightarrow$  0;
  Q2 = (Q1 = Q /. U21 /. zrule) /. Alternatives@@zs  $\rightarrow$  0;
  CF /@ E[L2, Q2, Det[lt] e-L2-Q2 Zip $\xi_S$ [eL1+Q1(P /. U21 /. zrule)]] // . 12U];
```

Program

```
In[*]:= B_{ }[L_, R_] := LR;
B_{is_}[L_ $\mathcal{E}$ , R_ $\mathcal{E}$ ] := PPB@Module[{n},
  Times[
    L /. Table[(v : b | B | t | T | a | x | y)i  $\rightarrow$  vn $\mathcal{E}$ i, {i, {is}}}],
    R /. Table[(v :  $\beta$  |  $\tau$  |  $\alpha$  |  $\mathcal{A}$  |  $\xi$  |  $\eta$ )i  $\rightarrow$  vn $\mathcal{E}$ i, {i, {is}}}]
  ] // LZipJoin@@Table[{ $\beta_{n\mathcal{E}i}$ ,  $\tau_{n\mathcal{E}i}$ ,  $\alpha_{n\mathcal{E}i}$ }, {i, {is}}] // QZipJoin@@Table[{ $\xi_{n\mathcal{E}i}$ ,  $\eta_{n\mathcal{E}i}$ }, {i, {is}}] ];
Bis_[L_, R_] := B_{is}[L, R];
```

Program

E morphisms with domain and range.

Program

```
In[*]:= Bis_List[Ed1 $\rightarrow$ r1[L1_, Q1_, P1_], Ed2 $\rightarrow$ r2[L2_, Q2_, P2_]] :=
  E(d1 $\cup$ Complement[d2, is]) $\rightarrow$ (r2 $\cup$ Complement[r1, is]) @@ Bis[E[L1, Q1, P1], E[L2, Q2, P2]];
Ed1 $\rightarrow$ r1[L1_, Q1_, P1_] // Ed2 $\rightarrow$ r2[L2_, Q2_, P2_] :=
  Br1 $\cap$ d2[Ed1 $\rightarrow$ r1[L1, Q1, P1], Ed2 $\rightarrow$ r2[L2, Q2, P2]];
Ed1 $\rightarrow$ r1[L1_, Q1_, P1_]  $\equiv$  Ed2 $\rightarrow$ r2[L2_, Q2_, P2_] ^:=
  (d1 == d2)  $\wedge$  (r1 == r2)  $\wedge$  (E[L1, Q1, P1]  $\equiv$  E[L2, Q2, P2]);
Ed1 $\rightarrow$ r1[L1_, Q1_, P1_] Ed2 $\rightarrow$ r2[L2_, Q2_, P2_] ^:=
  E(d1 $\cup$ d2) $\rightarrow$ (r1 $\cup$ r2) @@ (E[L1, Q1, P1] E[L2, Q2, P2]);
Ed $\rightarrow$ r[L_, Q_, P_] $k := Ed $\rightarrow$ r @@ E[L, Q, P] $k;
E_[ $\mathcal{E}$ ___][i_] := { $\mathcal{E}$ }[i];
```

Program

“Define” code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of \$k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

```
In[*]:=
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ = ε_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $k_Integer, PPBoot@$k@Block[{i, j, k}, op_isp, $k = ε; op_nis, $k]];
    SD[op_isp, op_{is}, $k]; SD[op_sis__, op_{sis}];
  ] /. {SD -> SetDelayed,
    isp -> {is} /. {i -> i_, j -> j_, k -> k_},
    nis -> {is} /. {i -> ii, j -> jj, k -> kk},
    nisp -> {is} /. {i -> ii_, j -> jj_, k -> kk_}
  } ] ]]
```

Program

Booting Up

Program

```
In[*]:= $k = 2; (*ħ=γ=1;*)
```

Program

```
In[*]:=
Define[am_{i,j} -> k = E_{i,j} -> {k} [(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k},
  bm_{i,j} -> k = E_{i,j} -> {k} [(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ε β_i} - 1) η_j y_k}]_{$k}]
```

Program

```
In[*]:=
Define[R_{i,j} = E_{i} -> {i,j} [ħ a_j b_i, ħ x_j y_i, e^{∑_{k=2}^{k+1} \frac{(1 - e^{γ ε ħ})^k (ħ y_i x_j)^k}{k (1 - e^{k γ ε ħ})}}]_{$k},
  R_{i,j} = E_{i} -> {i,j} [-ħ a_j b_i, -ħ x_j y_i / B_i, 1 + If[$k == 0, 0, (R_{i,j}, $k-1)_{$k} [3] -
    ((R_{i,j}, 0)_{$k} R_{1,2} (R_{3,4}, $k-1)_{$k}) // (bm_{i,1} -> i am_{j,2} -> j) // (bm_{i,3} -> i am_{j,4} -> j) [3] ]],
  P_{i,j} = E_{i,j} -> {i} [β_i α_j / ħ, η_i ξ_j / ħ, 1 + If[$k == 0, 0, (P_{i,j}, $k-1)_{$k} [3] -
    (R_{1,2} // ((P_{1,j}, 0)_{$k} (P_{i,2}, $k-1)_{$k})) [3] ] ]]
```

Program

```
In[*]:=
Define[aS_j = R_{i,j} ~ B_i ~ P_{i,j},
  aS_i = E_{i} -> {i} [-a_i α_i, -x_i a_i ξ_i, 1 + If[$k == 0, 0, (aS_{i}, $k-1)_{$k} [3] -
    ((aS_{i}, 0)_{$k} ~ B_i ~ aS_i ~ B_i ~ (aS_{i}, $k-1)_{$k}) [3] ] ]]
```

Program

```
In[*]:=
Define[bS_i = R_{i,1} ~ B_1 ~ aS_1 ~ B_1 ~ P_{i,1},
  bS_i = R_{i,1} ~ B_1 ~ aS_1 ~ B_1 ~ P_{i,1},
  aΔ_{i,j,k} = (R_{1,j} R_{2,k}) // bm_{1,2} -> 3 // P_{3,i},
  bΔ_{i,j,k} = (R_{j,1} R_{k,2}) // am_{1,2} -> 3 // P_{i,3}]
```

Program

```
In[*]:= Define [dmi,j→k = (E{i,j}→{i,j} [βi bi + αj aj, ηi yi + ξj xj, 1]
      (aΔi→1,2 // aΔ2→2,3 // aS3) (bΔj→-1,-2 // bΔ-2→-2,-3) // (P-1,3 P-3,1 am2,j→k bmi,-2→k),
dSi = E{i}→{1,2} [βi b1 + αi a2, ηi y1 + ξi x2, 1] // (bS1 aS2) // dm2,1→i,
dΔi→j,k = (bΔi→3,1 aΔi→2,4) // (dm3,4→k dm1,2→j) ]
```

Program

```
In[*]:= Define [Ci = E{i}→{i} [0, 0, Bi1/2 e-ħ e ai/2] $k,
C̄i = E{i}→{i} [0, 0, Bi-1/2 eħ e ai/2] $k,
Kinki = (R1,3 C2) // dm1,2→1 // dm1,3→i,
K̄inki = (R̄1,3 C2) // dm1,2→1 // dm1,3→i ]
```

Program

Note. $t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

Program

```
In[*]:= Define [b2ti = E{i}→{i} [αi ai - βi ti / γ, ξi xi + ηi yi, eε βi ai/γ] $k,
t2bi = E{i}→{i} [αi ai - τi γ bi, ξi xi + ηi yi, eε τi ai] $k ]
```

Testing

```

In[ ]:= Block[{$k = 1}, {
  am → ami,j→k, bm → bmi,j→k, dm → dmi,j→k, R → Ri,j, R̄ → R̄i,j, P → Pi,j,
  aS → aSi, aS̄ → aS̄i, bS → bSi, bS̄ → bS̄i, dS → dSi, aΔ → aΔi→j,k, bΔ → bΔi→j,k,
  dΔ → dΔi→j,k, C → Ci, C̄ → C̄i, Kink → Kinki, K̄ink → K̄inki, b2t → b2ti, t2b → t2bi
}] //
Column

am → E{i,j}→{k} [ak (αi + αj), xk (e-γ αj ξi + ξj), 1]
bm → E{i,j}→{k} [bk (βi + βj), yk (ηi + ηj), 1 - yk βi ηj ∈ + O[ε]2]
dm → E{i,j}→{k} [ak αi + ak αj + bk βi + bk βj,  $\frac{1}{\hbar \mathcal{A}_i \mathcal{A}_j}$ 
  (ħ yk Ai Aj ηi + ħ yk Aj ηj + ħ xk Ai ξi + Ai Aj ηj ξi - Bk Ai Aj ηj ξi + ħ xk Ai Aj ξj),
  1 +  $\frac{1}{4 \hbar \mathcal{A}_i \mathcal{A}_j}$  (-4 ħ yk Aj βi ηj - 4 ħ xk Ai βj ξi + 4 γ ħ2 xk yk ηj ξi +
  4 ħ ak Bk Ai Aj ηj ξi + 2 γ ħ yk Aj ηj2 ξi - 6 γ ħ Bk yk Aj ηj2 ξi + 2 γ ħ xk Ai ηj ξi2 -
  6 γ ħ Bk xk Ai ηj ξi2 + γ Ai Aj ηj2 ξi2 - 4 γ Bk Ai Aj ηj2 ξi2 + 3 γ Bk2 Ai Aj ηj2 ξi2) ∈ + O[ε]2]
R → E{i}→{i,j} [ħ aj bi, ħ xj yi, 1 -  $\frac{1}{4}$  (γ ħ3 xj2 yi2) ∈ + O[ε]2]
R̄ → E{i}→{i,j} [-ħ aj bi, - $\frac{\hbar x_j y_i}{B_i}$ , 1 -  $\frac{(4 \hbar^2 a_j B_i x_j y_i + 3 \gamma \hbar^3 x_j^2 y_i^2) \epsilon}{4 B_i^2} \in + O[\epsilon]^2$ ]
P → E{i,j}→{i} [ $\frac{\alpha_j \beta_i}{\hbar}$ ,  $\frac{\eta_i \xi_j}{\hbar}$ , 1 +  $\frac{\gamma \eta_i^2 \xi_j^2 \epsilon}{4 \hbar} + O[\epsilon]^2$ ]
aS → E{i}→{i} [-ai αi, -xi Ai ξi, 1 +  $\frac{1}{2}$  (-2 ħ ai xi Ai ξi - γ ħ xi2 Ai2 ξi2) ∈ + O[ε]2]
aS̄ → E{i}→{i} [-ai αi, -xi Ai ξi, 1 +  $\frac{1}{2}$  (2 γ ħ xi Ai ξi - 2 ħ ai xi Ai ξi - γ ħ xi2 Ai2 ξi2) ∈ + O[ε]2]
bS → E{i}→{i} [-bi βi, - $\frac{y_i \eta_i}{B_i}$ , 1 +  $\frac{(-2 B_i y_i \beta_i \eta_i - \gamma \hbar y_i^2 \eta_i^2) \epsilon}{2 B_i^2} + O[\epsilon]^2$ ]
bS̄ → E{i}→{i} [-bi βi, - $\frac{y_i \eta_i}{B_i}$ , 1 +  $\frac{(2 \gamma \hbar B_i y_i \eta_i - 2 B_i y_i \beta_i \eta_i - \gamma \hbar y_i^2 \eta_i^2) \epsilon}{2 B_i^2} + O[\epsilon]^2$ ]
Out[ ]:= dS → E{i}→{i} [-ai αi - bi βi,  $\frac{-\hbar y_i \mathcal{A}_i \eta_i - \hbar B_i x_i \mathcal{A}_i \xi_i + \mathcal{A}_i \eta_i \xi_i - B_i \mathcal{A}_i \eta_i \xi_i}{\hbar B_i}$ ,
  1 +  $\frac{1}{4 \hbar B_i^2}$  (4 γ ħ2 Bi yi Ai ηi - 4 ħ Bi yi Ai βi ηi - 2 γ ħ2 yi2 Ai2 ηi2 - 4 ħ2 ai Bi2 xi Ai ξi - 4 ħ Bi2 xi Ai βi ξi -
  4 γ ħ Bi Ai ηi ξi + 4 ħ ai Bi Ai ηi ξi + 4 γ ħ Bi2 Ai ηi ξi - 4 γ ħ2 Bi xi yi Ai2 ηi ξi +
  4 Bi Ai βi ηi ξi - 4 Bi2 Ai βi ηi ξi + 6 γ ħ yi Ai2 ηi2 ξi - 2 γ ħ Bi yi Ai2 ηi2 ξi - 2 γ ħ2 Bi2 xi2 Ai2 ξi2 +
  6 γ ħ Bi xi Ai2 ηi ξi2 - 2 γ ħ Bi2 xi Ai2 ηi ξi2 - 3 γ Ai2 ηi2 ξi2 + 4 γ Bi Ai2 ηi2 ξi2 - γ Bi2 Ai2 ηi2 ξi2) ∈ + O[ε]2]
aΔ → E{i}→{j,k} [aj αi + ak αi, xj ξi + xk ξi, 1 +  $\frac{1}{2}$  (-2 ħ aj xk ξi + γ ħ xj xk ξi2) ∈ + O[ε]2]
bΔ → E{i}→{j,k} [bj βi + bk βi, Bk yj ηi + yk ηi, 1 +  $\frac{1}{2}$  γ ħ Bk yj yk ηi2 ∈ + O[ε]2]
dΔ → E{i}→{j,k} [aj αi + ak αi + bj βi + bk βi,
  yj ηi + Bj yk ηi + xj ξi + xk ξi, 1 +  $\frac{1}{2}$  (γ ħ Bj yj yk ηi2 - 2 ħ aj xk ξi + γ ħ xj xk ξi2) ∈ + O[ε]2]
C → E{i}→{i} [0, 0,  $\sqrt{B_i} - \frac{1}{2} (\hbar a_i \sqrt{B_i}) \in + O[\epsilon]^2$ ]
C̄ → E{i}→{i} [0, 0,  $\frac{1}{\sqrt{B_i}} + \frac{\hbar a_i \epsilon}{2 \sqrt{B_i}} + O[\epsilon]^2$ ]
Kink → E{i}→{i} [ħ ai bi, ħ xi yi,  $\frac{1}{\sqrt{B_i}} + \frac{(2 \hbar a_i - \gamma \hbar^3 x_i^2 y_i^2) \epsilon}{4 \sqrt{B_i}} + O[\epsilon]^2$ ]
K̄ink → E{i}→{i} [-ħ ai bi, - $\frac{\hbar x_i y_i}{B_i}$ ,  $\sqrt{B_i} + \frac{(-2 \hbar a_i B_i^2 - 4 \hbar^2 a_i B_i x_i y_i - 3 \gamma \hbar^3 x_i^2 y_i^2) \epsilon}{4 B_i^{3/2}} + O[\epsilon]^2$ ]
b2t → E{i}→{i} [ai αi -  $\frac{t_i \beta_i}{\gamma}$ , yi ηi + xi ξi, 1 +  $\frac{a_i \beta_i \epsilon}{\gamma} + O[\epsilon]^2$ ]
t2b → E{i}→{i} [ai αi - γ bi τi, yi ηi + xi ξi, 1 + ai τi ∈ + O[ε]2]

```

Check that on the generators this agrees with our conventions in the handout:

In[]:= **Timing@**

```
{ {"[a,x]" -> ((E_{i->{1,2}} [0, 0, a_2 x_1] // am_{1,2->1}) [3] - (E_{i->{1,2}} [0, 0, a_1 x_2] // am_{1,2->1}) [3]),
  "[b,y]" -> ((E_{i->{1,2}} [0, 0, y_2 b_1] // bm_{1,2->1}) [3] - (E_{i->{1,2}} [0, 0, y_1 b_2] // bm_{1,2->1}) [3]) } /.
  z_{-1} -> z,
  {"Δ[y]" -> Last[E_{i->{1}} [0, 0, y_1] ~ B_1 ~ bΔ_{1->1,2}],
  "Δ[b]" -> Last[E_{i->{1}} [0, 0, b_1] ~ B_1 ~ bΔ_{1->1,2}],
  "Δ[a]" -> Last[E_{i->{1}} [0, 0, a_1] ~ B_1 ~ aΔ_{1->1,2}],
  "Δ[x]" -> Last[E_{i->{1}} [0, 0, x_1] ~ B_1 ~ aΔ_{1->1,2}],
  {
    "S(a)" -> ((E_{i->{1}} [0, 0, a_1] ~ B_1 ~ aS_1) [3]),
    "S(x)" -> ((E_{i->{1}} [0, 0, x_1] ~ B_1 ~ aS_1) [3]),
    "S(b)" -> ((E_{i->{1}} [0, 0, b_1] ~ B_1 ~ bS_1) [3]),
    "S(y)" -> ((E_{i->{1}} [0, 0, y_1] ~ B_1 ~ bS_1) [3])
  } /. z_{-1} -> z }
```

```
Out[ ]:= {1.09375,
  {{[a,x] -> -x γ, [b,y] -> -y ε + O[ε]^3}, {Δ[y] -> (B_2 y_1 + y_2) + O[ε]^3, Δ[b] -> (b_1 + b_2) + O[ε]^3,
  Δ[a] -> (a_1 + a_2) + O[ε]^3, Δ[x] -> (x_1 + x_2) - ħ a_1 x_2 ε + (1/2) ħ^2 a_1^2 x_2 ε^2 + O[ε]^3}, {S(a) -> -a + O[ε]^3,
  S(x) -> -x - a x ħ ε - (1/2) (a^2 x ħ^2) ε^2 + O[ε]^3, S(b) -> -b + O[ε]^3, S(y) -> -y/B + O[ε]^3}} }
```

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[]:= **Timing@Block** [{ \$k = 3,

```
HL /@ { (am_{1,2->1} // am_{1,3->1}) ≡ (am_{2,3->2} // am_{1,2->1}), (bm_{1,2->1} // bm_{1,3->1}) ≡ (bm_{2,3->2} // bm_{1,2->1}) } ]
```

```
Out[ ]:= {0.140625, {True, True}}
```

R and P are inverses:

In[]:= **Timing@Block** [{ \$k = 3, {R_{i,j}, P_{i,k}, HL [(R_{i,j} // P_{i,k}) ≡ E_{i->{k} -> {j}} [a_j α_k, x_j ξ_k, 1]] }]

```
Out[ ]:= {0.125, {E_{i->{i,j}} [ħ a_j b_i, ħ x_j y_i, 1 - (1/4) (γ ħ^3 x_j^2 y_i^2) ε + ((1/9) γ^2 ħ^5 x_j^3 y_i^3 + (1/32) γ^2 ħ^6 x_j^4 y_i^4) ε^2 +
  (1/1152) (24 γ^3 ħ^5 x_j^2 y_i^2 - 72 γ^3 ħ^7 x_j^4 y_i^4 - 32 γ^3 ħ^8 x_j^5 y_i^5 - 3 γ^3 ħ^9 x_j^6 y_i^6) ε^3 + O[ε]^4],
  E_{i,k->{i}} [ (α_k β_i / ħ, η_i ξ_k / ħ, 1 + (γ η_i^2 ξ_k^2 ε / (4 ħ) + (36 γ^2 ħ^2 η_i^2 ξ_k^2 + 40 γ^2 ħ η_i^3 ξ_k^3 + 9 γ^2 η_i^4 ξ_k^4) ε^2) / (288 ħ^2) - (1/1152) ħ^3)
  (-48 γ^3 ħ^4 η_i^2 ξ_k^2 - 192 γ^3 ħ^3 η_i^3 ξ_k^3 - 156 γ^3 ħ^2 η_i^4 ξ_k^4 - 40 γ^3 ħ η_i^5 ξ_k^5 - 3 γ^3 η_i^6 ξ_k^6) ε^3 + O[ε]^4], True}}
```

as and aS are inverses, bs and bS are inverses:

In[]:= **Timing** [HL /@ { (aS_1 // aS_1) ≡ E_{1->{1}} [a_1 α_1, x_1 ξ_1, 1], (bS_1 // bS_1) ≡ E_{1->{1}} [b_1 β_1, y_1 η_1, 1] }]

```
Out[ ]:= {0.390625, {True, True}}
```

(co)-associativity on both sides

In[*]:= **Timing**[
HL /@ { (a $\Delta_{1\rightarrow 1,2}$ // a $\Delta_{2\rightarrow 2,3}$) \equiv (a $\Delta_{1\rightarrow 1,3}$ // a $\Delta_{1\rightarrow 1,2}$), (b $\Delta_{1\rightarrow 1,2}$ // b $\Delta_{2\rightarrow 2,3}$) \equiv (b $\Delta_{1\rightarrow 1,3}$ // b $\Delta_{1\rightarrow 1,2}$),
(am $_{1,2\rightarrow 1}$ // am $_{1,3\rightarrow 1}$) \equiv (am $_{2,3\rightarrow 2}$ // am $_{1,2\rightarrow 1}$), (bm $_{1,2\rightarrow 1}$ // bm $_{1,3\rightarrow 1}$) \equiv (bm $_{2,3\rightarrow 2}$ // bm $_{1,2\rightarrow 1}$) }]

Out[*]:= {0.4375, {**True**, **True**, **True**, **True**}}

Δ is an algebra morphism

In[*]:= **Timing**[**HL** /@ { (am $_{1,2\rightarrow 1}$ // a $\Delta_{1\rightarrow 1,2}$) \equiv ((a $\Delta_{1\rightarrow 1,3}$ a $\Delta_{2\rightarrow 2,4}$) // (am $_{3,4\rightarrow 2}$ am $_{1,2\rightarrow 1}$)),
(bm $_{1,2\rightarrow 1}$ // b $\Delta_{1\rightarrow 1,2}$) \equiv ((b $\Delta_{1\rightarrow 1,3}$ b $\Delta_{2\rightarrow 2,4}$) // (bm $_{3,4\rightarrow 2}$ bm $_{1,2\rightarrow 1}$)) }]

Out[*]:= {0.65625, {**True**, **True**}}

An explicit formula for aS_i

In[*]:= **Timing**@**Block**[{**\$k** = 4}, **HL**[aS_i \equiv ($\mathbb{E}_{\{i\}\rightarrow\{i,j\}}$ [- α_i a_j, - ξ_i x_i,
Sum[**Expand**[$\frac{e^{\xi_i x_i} (-\hbar \gamma \epsilon)^k}{2^k k!}$ **Nest**[**Expand**[x_i² $\partial_{\{x_i,2\}}$ #] &, e^{- $\xi_i e^{\hbar \epsilon a_i} x_i$} , k]], {k, θ , \$k}]]]_k //
am_{i,j \rightarrow i})]]]

Out[*]:= {3.34375, **True**}

S is convolution inverse of id

In[*]:= **Timing**[**HL** [# \equiv $\mathbb{E}_{\{1\}\rightarrow\{1\}}$ [0, θ , 1]] & /@ {
(a $\Delta_{1\rightarrow 1,2} \sim B_1 \sim aS_1$) $\sim B_{1,2} \sim am_{1,2\rightarrow 1}$, (a $\Delta_{1\rightarrow 1,2} \sim B_2 \sim aS_2$) $\sim B_{1,2} \sim am_{1,2\rightarrow 1}$,
(b $\Delta_{1\rightarrow 1,2} \sim B_1 \sim bS_1$) $\sim B_{1,2} \sim bm_{1,2\rightarrow 1}$, (b $\Delta_{1\rightarrow 1,2} \sim B_2 \sim bS_2$) $\sim B_{1,2} \sim bm_{1,2\rightarrow 1}$ }]

Out[*]:= {0.703125, {**True**, **True**, **True**, **True**}}

But not with the opposite product:

In[*]:= **Timing**[**Short** [# \equiv $\mathbb{E}_{\{1\}\rightarrow\{1\}}$ [0, θ , 1]] & /@ {
(a $\Delta_{1\rightarrow 1,2} \sim B_1 \sim aS_1$) $\sim B_{1,2} \sim am_{2,1\rightarrow 1}$, (a $\Delta_{1\rightarrow 1,2} \sim B_2 \sim aS_2$) $\sim B_{1,2} \sim am_{2,1\rightarrow 1}$,
(b $\Delta_{1\rightarrow 1,2} \sim B_1 \sim bS_1$) $\sim B_{1,2} \sim bm_{2,1\rightarrow 1}$, (b $\Delta_{1\rightarrow 1,2} \sim B_2 \sim bS_2$) $\sim B_{1,2} \sim bm_{2,1\rightarrow 1}$ }]

Out[*]:= {0.734375, { $\frac{1}{2} (-2 \gamma \epsilon \hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2 \epsilon^2 \hbar^2 x_1 \mathcal{A}_1 \xi_1 - 2 \gamma \epsilon^2 \hbar^2 a_1 x_1 \mathcal{A}_1 \xi_1 + 2 \gamma^2 \epsilon^2 \hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2) = \theta$,

$$\frac{1}{2} (-2 \gamma \epsilon \hbar x_1 \xi_1 - \gamma^2 \epsilon^2 \hbar^2 x_1 \xi_1 + 2 \gamma^2 \epsilon^2 \hbar^2 x_1^2 \xi_1^2) = \theta,$$

$$\frac{1}{2} (-2 \gamma \epsilon \hbar y_1 \eta_1 - \gamma^2 \epsilon^2 \hbar^2 y_1 \eta_1 + 2 \gamma^2 \epsilon^2 \hbar^2 y_1^2 \eta_1^2) = \theta,$$

$$\frac{-2 \gamma \epsilon \hbar B_1 y_1 \eta_1 + \ll 3 \gg + 2 \gamma^2 \epsilon^2 \hbar^2 y_1^2 \eta_1^2}{2 B_1^2} = \theta \}}]$$

S is an algebra anti-(co)morphism

In[*]:= **Timing**[**HL** /@ { am $_{1,2\rightarrow 1} \sim B_1 \sim aS_1 \equiv$ (aS₁ aS₂) $\sim B_{1,2} \sim am_{2,1\rightarrow 1}$, bm $_{1,2\rightarrow 1} \sim B_1 \sim bS_1 \equiv$ (bS₁ bS₂) $\sim B_{1,2} \sim bm_{2,1\rightarrow 1}$,
aS₁ $\sim B_1 \sim a\Delta_{1\rightarrow 1,2} \equiv a\Delta_{1\rightarrow 2,1} \sim B_{1,2} \sim$ (aS₁ aS₂), bS₁ $\sim B_1 \sim b\Delta_{1\rightarrow 1,2} \equiv b\Delta_{1\rightarrow 2,1} \sim B_{1,2} \sim$ (bS₁ bS₂) }]

Out[*]:= {1.04688, {**True**, **True**, **True**, **True**}}

Pairing axioms


```
In[ ]:= Timing[HL /@ { (bm1,2→1 E{3}→{3} [α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
  (E{1}→{1} [β1 b1, η1 y1, 1] E{2}→{2} [β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
  (bΔ1→1,2 E{3}→{3} [α3 a3, ξ3 x3, 1] E{4}→{4} [α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
  (E{1}→{1} [β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
```

```
Out[ ]:= {0.46875, {True, True}}
```

```
In[ ]:= Timing[HL /@ { ((bS1 E{2}→{2} [α2 a2, ξ2 x2, 1]) // P1,2) ≡ ((E{1}→{1} [β1 b1, η1 y1, 1] aS2) // P1,2),
  (bS1 E{2}→{2} [α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (E{1}→{1} [β1 b1, η1 y1, 1] aS2) ~ B1,2 ~ P1,2}]
```

```
Out[ ]:= {0.3125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[ ]:= Timing@{ {
  "[a,y]" →
    ((E{1}→{1,2} [0, 0, y2 a1] ~ B1,2 ~ dm1,2→1) [3] - (E{1}→{1,2} [0, 0, y1 a2] ~ B1,2 ~ dm1,2→1) [3]),
  "[b,x]" → ((E{1}→{1,2} [0, 0, x2 b1] ~ B1,2 ~ dm1,2→1) [3] -
    (E{1}→{1,2} [0, 0, x1 b2] ~ B1,2 ~ dm1,2→1) [3]),
  "xy-qyx" → ((E{1}→{1,2} [0, 0, x1 y2] ~ B1,2 ~ dm1,2→1) [3] -
    (1 + ε) (E{1}→{1,2} [0, 0, y1 x2] ~ B1,2 ~ dm1,2→1) [3])
} /. {z-1 → z} // Expand // Factor,
{
  "Δ(a)" → ((E{1}→{1} [0, 0, a1] ~ B1 ~ dΔ1→1,2) [3]),
  "Δ(x)" → ((E{1}→{1} [0, 0, x1] ~ B1 ~ dΔ1→1,2) [3]),
  "Δ(b)" → ((E{1}→{1} [0, 0, b1] ~ B1 ~ dΔ1→1,2) [3]),
  "Δ(y)" → ((E{1}→{1} [0, 0, y1] ~ B1 ~ dΔ1→1,2) [3])
} // Simplify,
{
  "S(a)" → ((E{1}→{1} [0, 0, a1] ~ B1 ~ dS1) [3]),
  "S(x)" → ((E{1}→{1} [0, 0, x1] ~ B1 ~ dS1) [3]),
  "S(b)" → ((E{1}→{1} [0, 0, b1] ~ B1 ~ dS1) [3]),
  "S(y)" → ((E{1}→{1} [0, 0, y1] ~ B1 ~ dS1) [3])
} /. {z-1 → z} // Simplify
}
```

```
Out[ ]:= {8.0625, { { [a,y] → -y γ + 0[ε]3, [b,x] → x ε + 0[ε]3,
  xy-qyx → (-x y +  $\frac{1 - B + x y \hbar}{\hbar}$ ) + (a B - x y + x y γ ħ) ε +  $\frac{1}{2}$  (-a2 B ħ + x y γ2 ħ2) ε2 + 0[ε]3},
  {Δ(a) → (a1 + a2) + 0[ε]3, Δ(x) → (x1 + x2) - ħ a1 x2 ε +  $\frac{1}{2}$  ħ2 a12 x2 ε2 + 0[ε]3,
  Δ(b) → (b1 + b2) + 0[ε]3, Δ(y) → (y1 + B1 y2) + 0[ε]3},
  {S(a) → -a + 0[ε]3, S(x) → -x - a x ħ ε -  $\frac{1}{2}$  (a2 x ħ2) ε2 + 0[ε]3,
  S(b) → -b + 0[ε]3, S(y) → - $\frac{y}{B}$  +  $\frac{y \gamma \hbar \epsilon}{B}$  -  $\frac{(y \gamma^2 \hbar^2) \epsilon^2}{2 B}$  + 0[ε]3 } } }
```

(co)-associativity

In[*]:= **Timing** [
HL /@ { (dΔ_{1→1,2} // dΔ_{2→2,3}) ≡ (dΔ_{1→1,3} // dΔ_{1→1,2}), (dm_{1,2→1} // dm_{1,3→1}) ≡ (dm_{2,3→2} // dm_{1,2→1}) }]
 Out[*]:= { 6.6875, { **True**, **True** } }

Δ is an algebra morphism

In[*]:= **Timing**@**HL** [dm_{1,2→1} ~ B₁ ~ dΔ_{1→1,2} ≡ (dΔ_{1→1,3} dΔ_{2→2,4}) ~ B_{1,2,3,4} ~ (dm_{3,4→2} dm_{1,2→1})]
 Out[*]:= { 7.5625, **True** }

S₂ inverts R, but not S₁:

In[*]:= **Timing**@{ R_{1,2} ~ B₁ ~ dS₁ ≡ R̄_{1,2}, **HL** [R_{1,2} ~ B₂ ~ dS₂ ≡ R̄_{1,2}] }
 Out[*]:= { 0.75, { $\frac{1}{4 B_1^3} (4 \gamma \in \hbar^2 B_1^2 x_2 y_1 - 2 \gamma^2 \in^2 \hbar^3 B_1^2 x_2 y_1 + 4 \gamma \in^2 \hbar^3 a_2 B_1^2 x_2 y_1 + 8 \gamma^2 \in^2 \hbar^4 B_1 x_2^2 y_1^2 - 4 \gamma \in^2 \hbar^4 a_2 B_1 x_2^2 y_1^2 - 3 \gamma^2 \in^2 \hbar^5 x_2^3 y_1^3) = 0$, **True** } }

S is convolution inverse of id

In[*]:= **Timing** [**HL** [# ≡ E_{{1}→{1}} [0, 0, 1]] & /@
 { (dΔ_{1→1,2} ~ B₁ ~ dS₁) ~ B_{1,2} ~ dm_{1,2→1}, (dΔ_{1→1,2} ~ B₂ ~ dS₂) // dm_{1,2→1} }]
 Out[*]:= { 9.57813, { **True**, **True** } }

S is a (co)-algebra anti-morphism

In[*]:= **Timing** [**HL** /@
Expand /@ { dm_{1,2→1} ~ B₁ ~ dS₁ ≡ (dS₁ dS₂) ~ B_{1,2} ~ dm_{2,1→1}, dS₁ ~ B₁ ~ dΔ_{1→1,2} ≡ dΔ_{1→2,1} ~ B_{1,2} ~ (dS₁ dS₂) }]
 Out[*]:= { 18.3906, { **True**, **True** } }

Quasi-triangular axiom 1:

In[*]:= **Timing**@**HL** [R_{1,2} ~ B₁ ~ dΔ_{1→1,3} ≡ (R_{1,4} R_{3,2}) ~ B_{2,4} ~ dm_{2,4→2}]
 Out[*]:= { 0.625, **True** }

Quasi-triangular axiom 2:

In[*]:= **Timing**@**HL** [((dΔ_{1→1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2})) ≡ ((dΔ_{1→2,1} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{3,1→1} dm_{4,2→2}))]
 Out[*]:= { 7.03125, **True** }

The Drinfel'd element inverse property, (u₁ ū₂) ~ B_{1,2} ~ dm_{1,2→1} ≡ E[0, 0, 1]:

In[*]:= **Timing**@**HL** [((R_{1,2} ~ B₁ ~ dS₁ ~ B_{1,2} ~ dm_{2,1→1}) (R_{1,2} ~ B₂ ~ dS₂ ~ B₂ ~ dS₂ ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i} ≡
 E_{{i}→{i}} [0, 0, 1]]
 Out[*]:= { 3.0625, **True** }

The ribbon element v satisfies v² = S(u) u. The spinner C=uv⁻¹. It is convenient to compute z = S(u) u⁻¹ which is something easy.

$$\text{In[*]:= Timing@Block}[\{\$k = 2\},$$

$$\left(\left(\mathbf{R}_{1,2} \sim \mathbf{B}_1 \sim \mathbf{dS}_1 \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{2,1 \rightarrow i} \right) \sim \mathbf{B}_i \sim \mathbf{dS}_i \right) \left(\mathbf{R}_{1,2} \sim \mathbf{B}_2 \sim \mathbf{dS}_2 \sim \mathbf{B}_2 \sim \mathbf{dS}_2 \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{2,1 \rightarrow j} \right) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow i} \Big]$$

$$\text{Out[*]:= } \left\{ 3.67188, \mathbb{E}_{\{\} \rightarrow \{i\}} \left[\mathbf{0}, \mathbf{0}, \frac{1}{\mathbf{B}_i} + \frac{\hbar a_i \in}{\mathbf{B}_i} + \frac{\hbar^2 a_i^2 \in^2}{2 \mathbf{B}_i} + \mathbf{O}[\in]^3 \right] \right\}$$

$$\text{In[*]:= Timing@Block}[\{\$k = 2\}, \text{HL} / @ \left\{ \left(\mathbf{C}_i \overline{\mathbf{C}}_j \right) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow i} \equiv \mathbb{E}_{\{\} \rightarrow \{i\}} [\mathbf{0}, \mathbf{0}, \mathbf{1}], \left(\overline{\mathbf{C}}_i \overline{\mathbf{C}}_j \right) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow i} \equiv \right.$$

$$\left. \left(\left(\mathbf{R}_{1,2} \sim \mathbf{B}_1 \sim \mathbf{dS}_1 \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{2,1 \rightarrow i} \right) \sim \mathbf{B}_i \sim \mathbf{dS}_i \right) \left(\mathbf{R}_{1,2} \sim \mathbf{B}_2 \sim \mathbf{dS}_2 \sim \mathbf{B}_2 \sim \mathbf{dS}_2 \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{2,1 \rightarrow j} \right) \right\} \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow i} \Big]$$

$$\text{Out[*]:= } \{ 4.20313, \{\text{True}, \text{True}\} \}$$

Reidemeister 2:

$$\text{In[*]:= Timing}[\text{HL}[\# \equiv \mathbb{E}_{\{\} \rightarrow \{1,2\}} [\mathbf{0}, \mathbf{0}, \mathbf{1}]] \& / @$$

$$\left\{ \left(\overline{\mathbf{R}}_{1,2} \mathbf{R}_{3,4} \right) \sim \mathbf{B}_{1,2,3,4} \sim \left(\mathbf{dm}_{1,3 \rightarrow 1} \mathbf{dm}_{2,4 \rightarrow 2} \right), \left(\mathbf{R}_{1,2} \overline{\mathbf{R}}_{3,4} \right) \sim \mathbf{B}_{1,2,3,4} \sim \left(\mathbf{dm}_{1,3 \rightarrow 1} \mathbf{dm}_{2,4 \rightarrow 2} \right) \right\} \Big]$$

$$\text{Out[*]:= } \{ 4.90625, \{\text{True}, \text{True}\} \}$$

Cyclic Reidemeister 2:

$$\text{In[*]:= Timing@HL} \left[\left(\mathbf{R}_{1,4} \overline{\mathbf{R}}_{5,2} \overline{\mathbf{C}}_3 \right) \sim \mathbf{B}_{2,4} \sim \mathbf{dm}_{2,4 \rightarrow 2} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow 1} \sim \mathbf{B}_{1,5} \sim \mathbf{dm}_{1,5 \rightarrow 1} \equiv \overline{\mathbf{C}}_1 \mathbb{E}_{\{\} \rightarrow \{2\}} [\mathbf{0}, \mathbf{0}, \mathbf{1}] \right]$$

$$\text{Out[*]:= } \{ 2.14063, \text{True} \}$$

Reidemeister 3:

$$\text{In[*]:= Timing@HL} \left[\left(\left(\mathbf{R}_{1,2} \mathbf{R}_{4,3} \mathbf{R}_{5,6} \right) \sim \mathbf{B}_{1,4} \sim \mathbf{dm}_{1,4 \rightarrow 1} \sim \mathbf{B}_{2,5} \sim \mathbf{dm}_{2,5 \rightarrow 2} \sim \mathbf{B}_{3,6} \sim \mathbf{dm}_{3,6 \rightarrow 3} \right) \equiv \right.$$

$$\left. \left(\mathbf{R}_{1,6} \mathbf{R}_{2,3} \mathbf{R}_{4,5} \right) \sim \mathbf{B}_{1,4} \sim \mathbf{dm}_{1,4 \rightarrow 1} \sim \mathbf{B}_{2,5} \sim \mathbf{dm}_{2,5 \rightarrow 2} \sim \mathbf{B}_{3,6} \sim \mathbf{dm}_{3,6 \rightarrow 3} \right] \Big]$$

$$\text{Out[*]:= } \{ 4.53125, \text{True} \}$$

Relations between the four kinks:

$$\text{In[*]:= Timing}[\text{HL} / @ \left\{ \text{Kink}_i \equiv \left(\mathbf{R}_{3,1} \mathbf{C}_2 \right) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow i}, \right.$$

$$\left. \overline{\text{Kink}}_j \equiv \left(\overline{\mathbf{R}}_{3,1} \overline{\mathbf{C}}_2 \right) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow j}, \left(\text{Kink}_i \overline{\text{Kink}}_j \right) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow i} \equiv \mathbb{E}_{\{\} \rightarrow \{1\}} [\mathbf{0}, \mathbf{0}, \mathbf{1}] \right\} \Big]$$

$$\text{Out[*]:= } \{ 4.60938, \{\text{True}, \text{True}, \text{True}\} \}$$

The Trefoil

$$\text{In[*]:= Timing@Block}[\{\$k = 1\},$$

$$\mathbf{Z} = \mathbf{R}_{1,5} \mathbf{R}_{6,2} \mathbf{R}_{3,7} \overline{\mathbf{C}}_4 \overline{\text{Kink}}_8 \overline{\text{Kink}}_9 \overline{\text{Kink}}_{10};$$

$$\text{Do}[\mathbf{Z} = \mathbf{Z} \sim \mathbf{B}_{1,r} \sim \mathbf{dm}_{1,r \rightarrow 1}, \{\mathbf{r}, 2, 10\}];$$

$$\{\text{Simplify} / @ \mathbf{Z}, \text{Simplify} / @ (\mathbf{Z} \sim \mathbf{B}_1 \sim \mathbf{b}2\mathbf{t}_1 / . \mathbf{T}_1 \rightarrow \mathbf{T}) \Big]$$

$$\text{Out[*]:= } \left\{ 13.5156, \left\{ \mathbb{E}_{\{\} \rightarrow \{1\}} \left[\mathbf{0}, \mathbf{0}, \right. \right.$$

$$\frac{\mathbf{B}_1}{1 - \mathbf{B}_1 + \mathbf{B}_1^2} - \left(\hbar \mathbf{B}_1 \left(-a_1 \left(-1 + \mathbf{B}_1 - \mathbf{B}_1^3 + \mathbf{B}_1^4 \right) + \gamma \left(\mathbf{B}_1 - 2 \mathbf{B}_1^2 - 2 \mathbf{B}_1^4 + 2 \hbar x_1 y_1 + \mathbf{B}_1^3 \left(3 + 2 \hbar x_1 y_1 \right) \right) \right) \in \right) /$$

$$\left(1 - \mathbf{B}_1 + \mathbf{B}_1^2 \right)^3 + \mathbf{O}[\in]^2 \Big], \mathbb{E}_{\{\} \rightarrow \{1\}} \left[\mathbf{0}, \mathbf{0}, \right.$$

$$\frac{\mathbf{T}}{1 - \mathbf{T} + \mathbf{T}^2} + \left(\mathbf{T} \hbar \left(\mathbf{T} \left(-1 + 2 \mathbf{T} - 3 \mathbf{T}^2 + 2 \mathbf{T}^3 \right) \gamma + 2 \left(-1 + \mathbf{T} - \mathbf{T}^3 + \mathbf{T}^4 \right) a_1 - 2 \left(1 + \mathbf{T}^3 \right) \gamma \hbar x_1 y_1 \right) \in \right) /$$

$$\left(1 - \mathbf{T} + \mathbf{T}^2 \right)^3 + \mathbf{O}[\in]^2 \Big] \Big\}$$

Program

```
In[*]:= Define [kRi,j = Ri,j // (b2ti b2tj) /. ti|j → t,
  kR̄i,j = R̄i,j // (b2ti b2tj) /. {ti|j → t, Ti|j → T},
  kmi,j→k = (t2bi t2bj) // dmi,j→k // b2tk /. {tk → t, Tk → T, τi|j → 0},
  kCi = Ci // b2ti /. Ti → T,
  kC̄i = C̄i // b2ti /. Ti → T,
  kKinki = Kinki // b2ti /. {ti → t, Ti → T},
  kKink̄i = Kink̄i // b2ti /. {ti → t, Ti → T}]
```

```
In[*]:= Timing@Block[{ $k = 1,
  Z = kR1,5 kR6,2 kR3,7 kC4 kKink8 kKink9 kKink10;
  Do[Z = Z ~ B1,r ~ km1,r→1, {r, 2, 10}];
  Simplify /@ Z]
```

```
Out[*]:= {5.59375, E{1}→{1} [0, 0,
   $\frac{T}{1 - T + T^2} + (T \hbar (T (-1 + 2 T - 3 T^2 + 2 T^3) \gamma + 2 (-1 + T - T^3 + T^4) a_1 - 2 (1 + T^3) \gamma \hbar x_1 y_1) \epsilon) /$ 
  (1 - T + T^2)3 + O[ε]2]}]
```

RVK, rot, Z from 2016-09/OneSmidgen.nb. See also local version in this folder.

Program

Some details of the code below are at <http://drorbn.net/bbs/show?shot=Dror-160920-151350.jpg>.

Program

```
In[*]:= RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
  xs and a length 2n list of rotation numbers rots. Crossing
  sites are indexed 1 through 2n, and rots[[k]] is the rotation
  between site k-1 and site k. RVK is also a casting operator
  converting to the RVK presentation from other knot presentations.";
```

Program

```
In[*]:= RVK[pd_PD] := PPRVK@Module[{n, xs, x, rots, front = {0}, k},
  n = Length@pd; rots = Table[0, {2 n}];
  xs = List@@pd /. x_X => If[PositiveQ@x, Xp[x[[4]], x[[1]], Xm[x[[2]], x[[1]]];
  For[k = 0, k < 2 n, ++k,
  If[k == 0 ∨ FreeQ[front, -k],
  front = Flatten[front /. k → Catch[xs /. {
    Xp[k + 1, L_] | Xm[L_, k + 1] => Throw[{L, k + 1, 1 - L}],
    Xp[L_, k + 1] | Xm[k + 1, L_] => (++rots[[L]];
    Throw[{1 - L, k + 1, L}]}]]],
  If[MatchQ[front, {___, k, ___, -k, ___}], --rots[[k + 1]]
  ]
  ];
  RVK[xs, rots]
  ];
  RVK[K_] := RVK[PD[K]];
```

In[]:= **RVK[Knot[3, 1]]**

 **KnotTheory**: Loading precomputed data in PD4Knots` 

Out[]:= **RVK[{Xm[4, 1], Xm[6, 3], Xm[2, 5]}, {0, 0, 0, -1, 0, 0}]**

Program

```
In[ ]:= rot[i_, 0] := E_{i} [0, 0, 1];
rot[i_, n_] := Module[{j},
  rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC_j] // km_{i,j-i};
```

Program

```
In[ ]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
PP_2^n @Module[{todo, n, rots, z, done, st, cx, z1, i, j, k, k1, k2, k3},
  {todo, rots} = List@@rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  z = E_{i} [0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[todo != {},
    {cx} = MaximalBy[todo, Length[done ∩ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
    {i, j} = List@@cx;
    z1 = Switch[Head[cx],
      Xp, (kR_{i,j} kKink_k) // km_{j,k-i},
      Xm, (kR_{i,j} kKink_k) // km_{j,k-i}
    ];
    z1 = (rot[k, rots[[i]] z1) // km_{k,i-i}; rots[[i]] = 0;
    z1 = (z1 rot[k, rots[[i + 1]]) // km_{i,k-i}; rots[[i + 1]] = 0;
    z1 = (rot[k, rots[[j]] z1) // km_{k,j-j}; rots[[j]] = 0;
    z1 = (z1 rot[k, rots[[j + 1]]) // km_{j,k-j}; rots[[j + 1]] = 0;
    z *= z1;
    If[MemberQ[done, i], z = z // km_{i,i+1-i}; st = st /. st[[i + 2]] -> st[[i + 1]];
    If[MemberQ[done, i - 1], z = z // km_{st[[i],i-st[[i]]}; st = st /. st[[i + 1]] -> st[[i]];
    If[MemberQ[done, j], z = z // km_{j,j+1-j}; st = st /. st[[j + 2]] -> st[[j + 1]];
    If[MemberQ[done, j - 1], z = z // km_{st[[j],j-st[[j]]}; st = st /. st[[j + 1]] -> st[[j]];
    done = done ∪ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, cx]
  ];
  Simplify /@ (z /. {x_0 -> x, y_0 -> y, a_0 -> a})
]
```

Knot

In[*]:= \$k = 1; Timing@Z@Knot[10, 100]

Knot

Out[*]:= {983.234, $\mathbb{E} \left[0, 0, T^4 / (1 - 4T + 9T^2 - 12T^3 + 13T^4 - 12T^5 + 9T^6 - 4T^7 + T^8) + \right.$
 $(T^4 (-6 + 2T^{16} + 4a (-2 + 14T - 51T^2 + 120T^3 - 203T^4 + 258T^5 - 246T^6 + 152T^7 -$
 $152T^9 + 246T^{10} - 258T^{11} + 203T^{12} - 120T^{13} + 51T^{14} - 14T^{15} + 2T^{16}) - 8xy -$
 $440T^9 (-1 + xy) - 4T^{15} (3 + 2xy) + 8T^8 (-97 + 21xy) + 8T^7 (131 + 21xy) -$
 $20T^6 (57 + 22xy) + T^{14} (37 + 48xy) + T (44 + 48xy) - 8T^{11} (2 + 61xy) +$
 $8T^5 (127 + 68xy) - 2T^{13} (35 + 78xy) + 4T^{10} (-39 + 136xy) - T^2 (167 + 156xy) +$
 $T^{12} (79 + 324xy) + T^3 (410 + 324xy) - T^4 (733 + 488xy) \left. \right) \in \left. / \right.$
 $(1 - 4T + 9T^2 - 12T^3 + 13T^4 - 12T^5 + 9T^6 - 4T^7 + T^8)^3 + 0[\in]^2 \left. \right\}$

In[*]:= EndProfile[];

Profile

In[*]:= PrintProfile[]

Profile

Out[*]:= ProfileRoot is root. Profiled time: 1069.53
 (1) 0.156/ 983.235 above Z
 (163) 0.706/ 73.748 above B
 (147) 0/ 0.047 above CF
 (2) 0/ 0 above RVK
 (17) 0.016/ 2.468 above Boot[1]
 (18) 0.124/ 7.641 above Boot[2]
 (4) 0.015/ 0.047 above Boot[3]
 (1) 0/ 2.344 above Boot[4]
 Exp: called 247969 times, time in 625.943/696.494
 (247969) 625.943/ 696.494 under Together
 (245998) 38.153/ 70.551 above CF
 QZip: called 304 times, time in 209.497/871.153
 (304) 209.497/ 871.153 under B
 (912) 0.657/ 629.416 above CF
 (304) 10.839/ 32.240 above Zip
 Together: called 247969 times, time in 72.842/769.336
 (247969) 72.842/ 769.336 under CF
 (247969) 625.943/ 696.494 above Exp
 CF: called 247969 times, time in 53.551/822.887
 (245998) 38.153/ 70.551 under Exp
 (912) 14.741/ 122.873 under LZip
 (147) 0/ 0.047 under ProfileRoot
 (912) 0.657/ 629.416 under QZip
 (247969) 72.842/ 769.336 above Together
 Zip: called 2760 times, time in 49.408/149.258
 (304) 6.157/ 27.734 under LZip
 (304) 10.839/ 32.240 under QZip
 (2152) 32.412/ 89.284 under Zip
 (2760) 10.566/ 10.566 above Collect
 (2152) 32.412/ 89.284 above Zip
 LZip: called 304 times, time in 46.162/196.769
 (304) 46.162/ 196.769 under B
 (912) 14.741/ 122.873 above CF

```

( 304) 6.157/ 27.734 above Zip
Collect: called 2760 times, time in 10.566/10.566
( 2760) 10.566/ 10.566 under Zip
B: called 304 times, time in 1.157/1069.08
( 72) 0.310/ 982.986 under Z
( 163) 0.706/ 73.748 under ProfileRoot
( 33) 0.031/ 2.545 under Boot[1]
( 29) 0.095/ 7.487 under Boot[2]
( 3) 0/ 0.595 under Boot[3]
( 4) 0.015/ 1.718 under Boot[4]
( 304) 46.162/ 196.769 above LZip
( 304) 209.497/ 871.153 above QZip
Z: called 1 times, time in 0.156/983.235
( 1) 0.156/ 983.235 under ProfileRoot
( 72) 0.310/ 982.986 above B
( 3) 0/ 0.093 above Boot[1]
Boot[2]: called 23 times, time in 0.154/8.203
( 18) 0.124/ 7.641 under ProfileRoot
( 5) 0.030/ 0.562 under Boot[2]
( 29) 0.095/ 7.487 above B
( 5) 0.030/ 0.562 above Boot[2]
Boot[4]: called 6 times, time in 0.063/4.516
( 1) 0/ 2.344 under ProfileRoot
( 3) 0/ 0 under Boot[3]
( 2) 0.063/ 2.172 under Boot[4]
( 4) 0.015/ 1.718 above B
( 1) 0/ 0.563 above Boot[3]
( 2) 0.063/ 2.172 above Boot[4]
Boot[1]: called 27 times, time in 0.016/3.578
( 3) 0/ 0.093 under Z
( 17) 0.016/ 2.468 under ProfileRoot
( 7) 0/ 1.017 under Boot[1]
( 33) 0.031/ 2.545 above B
( 3) 0/ 0 above Boot[0]
( 7) 0/ 1.017 above Boot[1]
Boot[3]: called 5 times, time in 0.015/0.61
( 4) 0.015/ 0.047 under ProfileRoot
( 1) 0/ 0.563 under Boot[4]
( 3) 0/ 0.595 above B
( 3) 0/ 0 above Boot[4]
Boot[0]: called 3 times, time in 0./0.
( 3) 0/ 0 under Boot[1]
RVK: called 2 times, time in 0./0.
( 2) 0/ 0 under ProfileRoot

```