Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

External Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

Program

Canonical Form:

Program

In[∘]:=
```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CF[x]];
```

Program

The Kronecker δ:

Program

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_{$k_} := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Program

Finite Zips: (* Perhaps switch Expand to Collect[__, ζ]? *)

Program

```
In[•]:= expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ℰ_] := Expand[ℰ];
Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] := (expand[P // Zip{ζs}] /. f_. ζ^d_. :> ∂{ζ*,d} f) /. ζ* → 0
```

Program

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

Program

```
In[•]:= QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → qt.(zs + ys)];
    Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L, Q2, Det[qt] e^-Q2 Zip_ζs[e^Q1 (P /. zrule)]]];
QZip_{ζs_List} := QZip_{ζs,CF};
```

Program

Upper to lower and lower to Upper:

Program

```
In[•]:= U2l = {B_i_^p_. → e^(-p ħ ɣ b_i), B^p_. → e^(-p ħ ɣ b), T_i_^p_. → e^(p ħ t_i), T^p_. → e^(p ħ t), 𝒜_i_^p_. → e^(p ɣ α_i), 𝒜^p_. → e^(p ɣ α)};
l2U = {e^(c_. b_i_+d_.) :> B_i^(-c/(ħ ɣ)) e^d, e^(c_. b+d_.) :> B^(-c/(ħ ɣ)) e^d,
    e^(c_. t_i_+d_.) :> T_i^(c/ħ) e^d, e^(c_. t+d_.) :> T^(c/ħ) e^d,
    e^(c_. α_i_+d_.) :> 𝒜_i^(c/ɣ) e^d, e^(c_. α+d_.) :> 𝒜^(c/ɣ) e^d,
    e^ℰ_ :> e^Expand@ℰ};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single"P". Here the z's are b and α and the ζ's are β and a.

Program

```
In[•]:= LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L2, Q2, Det[lt] e^(-L2-Q2) Zip_ζs[e^(L1+Q1) (P /. U2l /. zrule)]] //. l2U];
LZip_{ζs_List} := LZip_{ζs,CF};
```

Program

In[∘]:=
```
Bind{}[L_, R_] := L R;
Bind{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
      L /. Table[(v : b | B | t | T | a | x | y)i → vn@i, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)i → vn@i, {i, {is}}]
     ] // LZipFlatten@Table[{βn@i,τn@i,an@i},{i,{is}}] // QZipFlatten@Table[{ξn@i,yn@i},{i,{is}}] ];
Bl_List[L_, R_] := Bindl[L, R]; Bis___[L_, R_] := Bind{is}[L, R];
```

Program

Booting Up

Program

In[∘]:=
```
$k = 2;
```

Program

In[∘]:=
```
ami_,j_→k_ := ami,j→k,$k; ami_,j_→k_,$k_ := (Boot[$k]; ami,j→k,$k);
bmi_,j_→k_ := bmi,j→k,$k; bmi_,j_→k_,$k_ := (Boot[$k]; bmi,j→k,$k);
dmi_,j_→k_ := dmi,j→k,$k; dmi_,j_→k_,$k_ := (Boot[$k]; dmi,j→k,$k);
Ri_,j_ := Ri,j,$k; Ri_,j_,$k_ := (Boot[$k]; Ri,j,$k);
R̄i_,j_ := R̄i,j,$k; R̄i_,j_,$k_ := (Boot[$k]; R̄i,j,$k);
Pi_,j_ := Pi,j,$k; Pi_,j_,$k_ := (Boot[$k]; Pi,j,$k);
aSi_ := aSi,$k; aSi_,$k_ := (Boot[$k]; aSi,$k);
aS̄i_ := aS̄i,$k; aS̄i_,$k_ := (Boot[$k]; aS̄i,$k);
bSi_ := bSi,$k; bSi_,$k_ := (Boot[$k]; bSi,$k);
bS̄i_ := bS̄i,$k; bS̄i_,$k_ := (Boot[$k]; bS̄i,$k);
dSi_ := dSi,$k; dSi_,$k_ := (Boot[$k]; dSi,$k);
aΔi_→j_,k_ := aΔi→j,k,$k; aΔi_→j_,k_,$k_ := (Boot[$k]; aΔi→j,k,$k);
bΔi_→j_,k_ := bΔi→j,k,$k; bΔi_→j_,k_,$k_ := (Boot[$k]; bΔi→j,k,$k);
dΔi_→j_,k_ := dΔi→j,k,$k; dΔi_→j_,k_,$k_ := (Boot[$k]; dΔi→j,k,$k);
CCi_ := CCi,$k; CCi_,$k_ := (Boot[$k]; CCi,$k);
C̄C̄i_ := C̄C̄i,$k; C̄C̄i_,$k_ := (Boot[$k]; C̄C̄i,$k);
Kinki_ := Kinki,$k; Kinki_,$k_ := (Boot[$k]; Kinki,$k);
K̄inki_ := K̄inki,$k; K̄inki_,$k_ := (Boot[$k]; K̄inki,$k);
b2ti_ := b2ti,$k; b2ti_,$k_ := (Boot[$k]; b2ti,$k);
t2bi_ := t2bi,$k; t2bi_,$k_ := (Boot[$k]; t2bi,$k);
```

Program

```
In[•]:= Boot[kk_] := (If[kk > 0, Boot[kk - 1]];
    Block[{$k = kk, i, j, k, l, m, n, tu = TimeUsed[]},
     am_{i_,j_->k_,$k} = 𝔼[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k};
     bm_{i_,j_→k_,$k} = 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i} - 1) η_j y_k}]_{$k};
     R_{i_,j_,$k} = 𝔼[ħ a_j b_i, ħ x_j y_i, e^(Sum_{k=2}^{$k+1} ((1 - e^{γ ϵ ħ})^k (ħ y_i x_j)^k)/(k (1 - e^{k γ ϵ ħ})))]_{$k};
     P_{i_,j_,$k} = If[$k == 0, 𝔼[β_i α_j / ħ, η_i ξ_j / ħ, 1]_0,
       MapAt[(# - ϵ^{$k} Coefficient[(R_{n,m} ~ B_{n,m} ~ ((P_{n,j,0})_{$k} (P_{i,m,$k-1})_{$k}))〚3〛, ϵ, $k]) &,
        (P_{i,j,$k-1})_{$k}, 3]];
     aS_{i_,$k} = 𝔼[-α_i a_j, -ξ_i x_i,
        e^{ξ_i x_i} Sum[Expand[(-ħ γ ϵ)^k/(2^k k!) Nest[Expand[x_i^2 ∂_{{x_i,2}} #] &, e^{-ξ_i e^{ħ ϵ a_i} x_i}, k]], {k, 0, $k}]]_{$k};
       B_{i,j} ~ am_{i,j→i};
     aS̄_{i_,$k} = If[$k == 0, 𝔼[-a_i α_i, -x_i 𝒜_i ξ_i, 1]_0,
       MapAt[(# - ϵ^{$k} Coefficient[((aS̄_{i,0})_{$k} ~ B_i ~ aS_i ~ B_i ~ (aS̄_{i,$k-1})_{$k})〚3〛, ϵ, $k]) &,
        (aS̄_{i,$k-1})_{$k}, 3]];
     bS_{i_,$k} = R_{i,n} ~ B_n ~ aS_n ~ B_n ~ P_{i,n};
     bS̄_{i_,$k} = R_{i,n} ~ B_n ~ aS̄_n ~ B_n ~ P_{i,n};
     aΔ_{i_→j_,k_,$k} = (R_{n,j} R_{m,k}) ~ B_{n,m} ~ bm_{n,m→l} ~ B_l ~ P_{l,i};
     bΔ_{i_→j_,k_,$k} = (R_{j,n} R_{k,m}) ~ B_{n,m} ~ am_{n,m→l} ~ B_l ~ P_{i,l};
     dm_{i_,j_→k_,$k} =
      (𝔼[β_i b_i + α_j a_j, η_i y_i + ξ_j x_j, 1] (aΔ_{i→1,2} ~ B_2 ~ aΔ_{2→2,3}) (bΔ_{j→-1,-2} ~ B_{-2} ~ bΔ_{-2→-2,-3})) ~
       B_3 ~ aS̄_3 ~ B_{-1,3} ~ (P_{-1,3}) ~ B_{-3,1} ~ (P_{-3,1}) ~ B_{2,j,i,-2} ~ (am_{2,j→k} bm_{i,-2→k});
     dS_{i_,$k} = 𝔼[β_i b_n + α_i a_m, η_i y_n + ξ_i x_m, 1] ~ B_{n,m} ~ (bS̄_n aS_m) ~ B_{n,m} ~ dm_{m,n→i};
     dΔ_{i_→j_,k_,$k} = (bΔ_{i→3,1} aΔ_{i→2,4}) ~ B_{1,2,3,4} ~ (dm_{3,4→k} dm_{1,2→j});
     R̄_{i_,j_,$k} = Expand /@ R_{i,j} ~ B_j ~ dS_j;
     CC_{i_,$k} = 𝔼[0, 0, B_i^{1/2} e^{-ħ ϵ a_i/2}]_{$k};
     CC̄_{i_,$k} = 𝔼[0, 0, B_i^{-1/2} e^{ħ ϵ a_i/2}]_{$k};
     Kink_{i_,$k} = (R_{1,3} CC̄_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i};
     Kink̄_{i_,$k} = (R̄_{1,3} CC_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i};
     (* t==ϵa-γb and b==-t/γ+ϵa/γ: *)
     b2t_{i_,$k} = 𝔼[α_i a_i - β_i t_i / γ, ξ_i x_i + η_i y_i, e^{ϵ β_i a_i/γ}]_{$k};
     t2b_{i_,$k} = 𝔼[α_i a_j - τ_i γ b_j, ξ_i x_j + η_i y_j, e^{ϵ τ_i a_j}]_{$k};
     Boot[$k] = Print["Booted @ $k=", $k, " in ", TimeUsed[] - tu, " sec."];
    ]);
```

Program

Broken “Define” code

Program

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k, l, m, n},
   ReleaseHold@Echo[Hold[
        op_{nisp,$k_Integer} := Block[{i, j, k, l, m, n}, op_{isp,$k} = ℰ; op_{nis,$k}];
        op_{isp} := op_{{is},$k};
        op_{sis__} := op_{{sis}};
       ] /. {
        isp → {is} /. {i → i_, j → j_, k → k_},
        nis → {is} /. {i → ii, j → jj, k → kk},
        nisp → {is} /. {i → ii_, j → jj_, k → kk_}
       }]
  ]]
```

Testing

In[◦]:=

```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j}, aS → aS_i,
   aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k}, dΔ → dΔ_{i→j,k},
   CC → CC_i, CC̄ → CC̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

```
Booted @ $k=0 in 1.782 sec.

Booted @ $k=1 in 3.109 sec.
```

Out[∘]=

$$\text{am} \to \mathbb{E}\left[a_k (\alpha_i + \alpha_j),\ x_k \left(e^{-\gamma \alpha_j} \xi_i + \xi_j\right),\ 1\right]$$

$$\text{bm} \to \mathbb{E}\left[b_k (\beta_i + \beta_j),\ y_k (\eta_i + \eta_j),\ 1 - y_k \beta_i \eta_j \epsilon + O[\epsilon]^2\right]$$

$$\text{dm} \to \mathbb{E}\Big[a_k \alpha_i + a_k \alpha_j + b_k \beta_i + b_k \beta_j,\ \frac{1}{\hbar \mathcal{A}_i \mathcal{A}_j} (\hbar y_k \mathcal{A}_i \mathcal{A}_j \eta_i + \hbar y_k \mathcal{A}_j \eta_j + \hbar x_k \mathcal{A}_i \xi_i + \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i - B_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + \hbar x_k \mathcal{A}_i \mathcal{A}_j \xi_j),$$
$$1 + \frac{1}{4 \hbar \mathcal{A}_i \mathcal{A}_j} \big(-4 \hbar y_k \mathcal{A}_j \beta_i \eta_j - 4 \hbar x_k \mathcal{A}_i \beta_j \xi_i + 4 \gamma \hbar^2 x_k y_k \eta_j \xi_i + 4 \hbar a_k B_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + 2 \gamma \hbar y_k \mathcal{A}_j \eta_j^2 \xi_i - 6 \gamma \hbar B_k y_k \mathcal{A}_j \eta_j^2 \xi_i + 2 \gamma \hbar x_k \mathcal{A}_i \eta_j \xi_i^2 -$$
$$6 \gamma \hbar B_k x_k \mathcal{A}_i \eta_j \xi_i^2 + \gamma \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 - 4 \gamma B_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 + 3 \gamma B_k^2 \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2\big) \epsilon + O[\epsilon]^2\Big]$$

$$R \to \mathbb{E}\left[\hbar a_j b_i,\ \hbar x_j y_i,\ 1 - \frac{1}{4} \left(\gamma \hbar^3 x_j^2 y_i^2\right) \epsilon + O[\epsilon]^2\right]$$

$$\overline{R} \to \mathbb{E}\left[-\hbar a_j b_i,\ -\frac{\hbar x_j y_i}{B_i},\ 1 + \frac{\left(-4 \hbar^2 a_j B_i x_j y_i - 3 \gamma \hbar^3 x_j^2 y_i^2\right) \epsilon}{4 B_i^2} + O[\epsilon]^2\right]$$

$$P \to \mathbb{E}\left[\frac{\alpha_j \beta_i}{\hbar},\ \frac{\eta_i \xi_j}{\hbar},\ 1 + \frac{\gamma \epsilon \eta_i^2 \xi_j^2}{4 \hbar}\right]$$

$$\text{aS} \to \mathbb{E}\left[-a_i \alpha_i,\ -x_i \mathcal{A}_i \xi_i,\ 1 + \frac{1}{2} \left(-2 \hbar a_i x_i \mathcal{A}_i \xi_i - \gamma \hbar x_i^2 \mathcal{A}_i^2 \xi_i^2\right) \epsilon + O[\epsilon]^2\right]$$

$$\overline{\text{aS}} \to \mathbb{E}\left[-a_i \alpha_i,\ -x_i \mathcal{A}_i \xi_i,\ 1 - \frac{1}{2} \epsilon \left(-2 \gamma \hbar x_i \mathcal{A}_i \xi_i + 2 \hbar a_i x_i \mathcal{A}_i \xi_i + \gamma \hbar x_i^2 \mathcal{A}_i^2 \xi_i^2\right)\right]$$

$$\text{bS} \to \mathbb{E}\left[-b_i \beta_i,\ -\frac{y_i \eta_i}{B_i},\ 1 + \frac{\left(-2 B_i y_i \beta_i \eta_i - \gamma \hbar y_i^2 \eta_i^2\right) \epsilon}{2 B_i^2} + O[\epsilon]^2\right]$$

$$\overline{\text{bS}} \to \mathbb{E}\left[-b_i \beta_i,\ -\frac{y_i \eta_i}{B_i},\ 1 + \frac{\left(2 \gamma \hbar B_i y_i \eta_i - 2 B_i y_i \beta_i \eta_i - \gamma \hbar y_i^2 \eta_i^2\right) \epsilon}{2 B_i^2} + O[\epsilon]^2\right]$$

$$\text{dS} \to \mathbb{E}\Big[-a_i \alpha_i - b_i \beta_i,\ \frac{-\hbar y_i \mathcal{A}_i \eta_i - \hbar B_i x_i \mathcal{A}_i \xi_i + \mathcal{A}_i \eta_i \xi_i - B_i \mathcal{A}_i \eta_i \xi_i}{\hbar B_i},$$
$$1 + \frac{1}{4 \hbar B_i^2} \big(4 \gamma \hbar^2 B_i y_i \mathcal{A}_i \eta_i - 4 \hbar B_i y_i \mathcal{A}_i \beta_i \eta_i - 2 \gamma \hbar^2 y_i^2 \mathcal{A}_i^2 \eta_i^2 - 4 \hbar^2 a_i B_i^2 x_i \mathcal{A}_i \xi_i - 4 \hbar B_i^2 x_i \mathcal{A}_i \beta_i \xi_i -$$
$$4 \gamma \hbar B_i \mathcal{A}_i \eta_i \xi_i + 4 \hbar a_i B_i \mathcal{A}_i \eta_i \xi_i + 4 \gamma \hbar B_i^2 \mathcal{A}_i \eta_i \xi_i - 4 \gamma \hbar^2 B_i x_i y_i \mathcal{A}_i^2 \eta_i \xi_i +$$
$$4 B_i \mathcal{A}_i \beta_i \eta_i \xi_i - 4 B_i^2 \mathcal{A}_i \beta_i \eta_i \xi_i + 6 \gamma \hbar y_i \mathcal{A}_i^2 \eta_i^2 \xi_i - 2 \gamma \hbar B_i y_i \mathcal{A}_i^2 \eta_i^2 \xi_i - 2 \gamma \hbar^2 B_i^2 x_i^2 \mathcal{A}_i^2 \xi_i^2 +$$
$$6 \gamma \hbar B_i x_i \mathcal{A}_i^2 \eta_i \xi_i^2 - 2 \gamma \hbar B_i^2 x_i \mathcal{A}_i^2 \eta_i \xi_i^2 - 3 \gamma \mathcal{A}_i^2 \eta_i^2 \xi_i^2 + 4 \gamma B_i \mathcal{A}_i^2 \eta_i^2 \xi_i^2 - \gamma B_i^2 \mathcal{A}_i^2 \eta_i^2 \xi_i^2\big) \epsilon + O[\epsilon]^2\Big]$$

$$\text{a}\triangle \to \mathbb{E}\left[a_j \alpha_i + a_k \alpha_i,\ x_j \xi_i + x_k \xi_i,\ 1 + \frac{1}{2} \left(-2 \hbar a_j x_k \xi_i + \gamma \hbar x_j x_k \xi_i^2\right) \epsilon + O[\epsilon]^2\right]$$

$$\text{b}\triangle \to \mathbb{E}\left[b_j \beta_i + b_k \beta_i,\ B_k y_j \eta_i + y_k \eta_i,\ 1 + \frac{1}{2} \gamma \hbar B_k y_j y_k \eta_i^2 \epsilon + O[\epsilon]^2\right]$$

$$\text{d}\triangle \to \mathbb{E}\Big[a_j \alpha_i + a_k \alpha_i + b_j \beta_i + b_k \beta_i,\ y_j \eta_i + B_j y_k \eta_i + x_j \xi_i + x_k \xi_i,$$
$$1 + \frac{1}{2} \left(\gamma \hbar B_j y_j y_k \eta_i^2 - 2 \hbar a_j x_k \xi_i + \gamma \hbar x_j x_k \xi_i^2\right) \epsilon + O[\epsilon]^2\Big]$$

$$\text{CC} \to \mathbb{E}\left[0,\ 0,\ \sqrt{B_i} - \frac{1}{2} \left(\hbar a_i \sqrt{B_i}\right) \epsilon + O[\epsilon]^2\right]$$

$$\overline{\text{CC}} \to \mathbb{E}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}} + \frac{\hbar a_i \epsilon}{2 \sqrt{B_i}} + O[\epsilon]^2\right]$$

$$\text{Kink} \to \mathbb{E}\left[\hbar a_i b_i,\ \hbar x_i y_i,\ \frac{1}{\sqrt{B_i}} + \frac{\left(2 \hbar a_i - \gamma \hbar^3 x_i^2 y_i^2\right) \epsilon}{4 \sqrt{B_i}} + O[\epsilon]^2\right]$$

$$\overline{\text{Kink}} \to \mathbb{E}\left[-\hbar a_i b_i,\ -\frac{\hbar x_i y_i}{B_i},\ \sqrt{B_i} + \frac{\left(-2 \hbar a_i B_i^2 - 4 \hbar^2 a_i B_i x_i y_i - 3 \gamma \hbar^3 x_i^2 y_i^2\right) \epsilon}{4 B_i^{3/2}} + O[\epsilon]^2\right]$$

$$\text{b2t} \to \mathbb{E}\left[a_i \alpha_i - \frac{t_i \beta_i}{\gamma},\ y_i \eta_i + x_i \xi_i,\ 1 + \frac{a_i \beta_i \epsilon}{\gamma} + O[\epsilon]^2\right]$$

$$\text{t2b} \to \mathbb{E}\left[a_j \alpha_i - \gamma b_j \tau_i,\ y_j \eta_i + x_j \xi_i,\ 1 + a_j \tau_i \epsilon + O[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

```
In[]:= Timing@{{"[a,x]" → ((𝔼[0, 0, a₂ x₁] ~B₁,₂~ am₁,₂→₁)⟦3⟧ - (𝔼[0, 0, a₁ x₂] ~B₁,₂~ am₁,₂→₁)⟦3⟧),
        "[b,y]" → ((𝔼[0, 0, y₂ b₁] ~B₁,₂~ bm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, y₁ b₂] ~B₁,₂~ bm₁,₂→₁)⟦3⟧)} /.
     z_1 → z,
    {"Δ[y]" → Last[𝔼[0, 0, y₁] ~B₁~ bΔ₁→₁,₂],
     "Δ[b]" → Last[𝔼[0, 0, b₁] ~B₁~ bΔ₁→₁,₂],
     "Δ[a]" → Last[𝔼[0, 0, a₁] ~B₁~ aΔ₁→₁,₂],
     "Δ[x]" → Last[𝔼[0, 0, x₁] ~B₁~ aΔ₁→₁,₂]},
    {
      "S(a)" → ((𝔼[0, 0, a₁] ~B₁~ aS₁)⟦3⟧),
      "S(x)" → ((𝔼[0, 0, x₁] ~B₁~ aS₁)⟦3⟧),
      "S(b)" → ((𝔼[0, 0, b₁] ~B₁~ bS₁)⟦3⟧),
      "S(y)" → ((𝔼[0, 0, y₁] ~B₁~ bS₁)⟦3⟧)
    } /. z_1 → z}
```

Booted @ $k=2 in 9.875 sec.

Out[]= $\{10.0781, \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\,\epsilon + O[\epsilon]^3\}, \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar\, a_1 x_2\,\epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2\,\epsilon^2 + O[\epsilon]^3\}, \{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}(a^2 x\,\hbar^2)\,\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
In[]:= Timing@Block[{$k = 3},
    HL /@ {(am₁,₂→₁ ~B₁~ am₁,₃→₁) ≡ (am₂,₃→₂ ~B₂~ am₁,₂→₁), (bm₁,₂→₁ ~B₁~ bm₁,₃→₁) ≡ (bm₂,₃→₂ ~B₂~ bm₁,₂→₁)}
  ]
```

Booted @ $k=3 in 93.968 sec.

Out[]= {94.0781, {True, True}}

R and *P* are inverses:

```
In[]:= Timing@Block[{$k = 3}, {Rᵢ,ⱼ, Pᵢ,ₖ, HL[Rᵢ,ⱼ ~Bᵢ~ Pᵢ,ₖ ≡ 𝔼[aⱼ αₖ, xⱼ ξₖ, 1]]}]
```

Out[]= $\{0.046875, \{\mathbb{E}[\hbar\, a_j b_i,\ \hbar\, x_j y_i,\ 1 - \frac{1}{4}(\gamma\,\hbar^3 x_j^2 y_i^2)\,\epsilon + (\frac{1}{9}\gamma^2\hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2\hbar^6 x_j^4 y_i^4)\,\epsilon^2 + \frac{1}{1152}(24\gamma^3\hbar^5 x_j^2 y_i^2 - 72\gamma^3\hbar^7 x_j^4 y_i^4 - 32\gamma^3\hbar^8 x_j^5 y_i^5 - 3\gamma^3\hbar^9 x_j^6 y_i^6)\,\epsilon^3 + O[\epsilon]^4],$

$\mathbb{E}[\frac{\alpha_k\beta_i}{\hbar},\ \frac{\eta_i\xi_k}{\hbar},\ 1 + \frac{\gamma\,\eta_i^2\xi_k^2\,\epsilon}{4\hbar} + \frac{(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\,\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4)\,\epsilon^2}{288\hbar^2} + \frac{1}{1152\hbar^3}(48\gamma^3\hbar^4\eta_i^2\xi_k^2 + 192\gamma^3\hbar^3\eta_i^3\xi_k^3 + 156\gamma^3\hbar^2\eta_i^4\xi_k^4 + 40\gamma^3\hbar\,\eta_i^5\xi_k^5 + 3\gamma^3\eta_i^6\xi_k^6)\,\epsilon^3 + O[\epsilon]^4],\ \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
In[]:= Timing[HL /@ {aS̄₁ ~B₁~ aS₁ ≡ 𝔼[a₁ α₁, x₁ ξ₁, 1], bS̄₁ ~B₁~ bS₁ ≡ 𝔼[b₁ β₁, y₁ η₁, 1]}]
```

Out[]= {0.171875, {True, True}}

(co)-associativity on both sides

In[∘]:=
```
Timing[HL /@
  {(aΔ1→1,2 ~ B2 ~ aΔ2→2,3) ≡ (aΔ1→1,3 ~ B1 ~ aΔ1→1,2), (bΔ1→1,2 ~ B2 ~ bΔ2→2,3) ≡ (bΔ1→1,3 ~ B1 ~ bΔ1→1,2),
   (am1,2→1 ~ B1 ~ am1,3→1) ≡ (am2,3→2 ~ B2 ~ am1,2→1), (bm1,2→1 ~ B1 ~ bm1,3→1) ≡ (bm2,3→2 ~ B2 ~ bm1,2→1)}]
```
Out[∘]= {0.328125, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:=
```
Timing[HL /@ {am1,2→1 ~ B1 ~ aΔ1→1,2 ≡ (aΔ1→1,3 aΔ2→2,4) ~ B1,2,3,4 ~ (am3,4→2 am1,2→1),
   bm1,2→1 ~ B1 ~ bΔ1→1,2 ≡ (bΔ1→1,3 bΔ2→2,4) ~ B1,2,3,4 ~ (bm3,4→2 bm1,2→1)}]
```
Out[∘]= {0.625, {True, True}}

S is convolution inverse of id

In[∘]:=
```
Timing[HL[# ≡ 𝔼[0, 0, 1]] & /@ {
   (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
   (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
```
Out[∘]= {0.5625, {True, True, True, True}}

S is an algebra anti-(co)morphism

In[∘]:=
```
Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
   aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
```
Out[∘]= {0.890625, {True, True, True, True}}

Pairing axioms

In[∘]:=
```
Timing[HL /@ {(bm1,2→1 𝔼[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
    (𝔼[β1 b1, η1 y1, 1] 𝔼[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
   (bΔ1→1,2 𝔼[α3 a3, ξ3 x3, 1] 𝔼[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
    (𝔼[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
```
Out[∘]= {0.421875, {True, True}}

In[∘]:=
```
Timing[HL /@ {(bS1 𝔼[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼[β1 b1, η1 y1, 1] aS2) ~ B1,2 ~ P1,2,
   (bS̄1 𝔼[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼[β1 b1, η1 y1, 1] aS̄2) ~ B1,2 ~ P1,2}]
```
Out[∘]= {0.3125, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

In[∘]:=
```
Timing@{{
    "[a,y]" → ((𝔼[0, 0, y₂ a₁] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ − (𝔼[0, 0, y₁ a₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧),
    "[b,x]" → ((𝔼[0, 0, x₂ b₁] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ − (𝔼[0, 0, x₁ b₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧),
    "xy-qyx" → ((𝔼[0, 0, x₁ y₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ − (1 + ϵ) (𝔼[0, 0, y₁ x₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧)
   } /. {z_₁ → z} // Expand // Factor,
  {
   "Δ(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
   "Δ(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
   "Δ(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
   "Δ(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧)
  } // Simplify,
  {
   "S(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ dS₁)⟦3⟧),
   "S(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ dS₁)⟦3⟧),
   "S(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ dS₁)⟦3⟧),
   "S(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ dS₁)⟦3⟧)
  } /. {z_₁ → z} // Simplify
 }
```

Out[∘]= $\{3.1875, \{\{[a,y] \to -y\gamma + O[\epsilon]^3,\ [b,x] \to x\epsilon + O[\epsilon]^3,$
$xy\text{-}qyx \to \left(-x y + \frac{1 - B + x y \hbar}{\hbar}\right) + (a B - x y + x y \gamma \hbar)\,\epsilon + \frac{1}{2}\left(-a^2 B \hbar + x y \gamma^2 \hbar^2\right)\epsilon^2 + O[\epsilon]^3\},$
$\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3,$
$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1 y_2) + O[\epsilon]^3\},$
$\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}\left(a^2 x \hbar^2\right)\epsilon^2 + O[\epsilon]^3,$
$S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\gamma\hbar\epsilon}{B} - \frac{\left(y\gamma^2\hbar^2\right)\epsilon^2}{2B} + O[\epsilon]^3\}\}\}$

(co)-associativity

In[∘]:=
```
Timing[HL /@
  {(dΔ₁→₁,₂ ~ B₂ ~ dΔ₂→₂,₃) ≡ (dΔ₁→₁,₃ ~ B₁ ~ dΔ₁→₁,₂), (dm₁,₂→₁ ~ B₁ ~ dm₁,₃→₁) ≡ (dm₂,₃→₂ ~ B₂ ~ dm₁,₂->₁)}]
```

Out[∘]= {6.53125, {True, True}}

Δ is an algebra morphism

In[∘]:=
```
Timing@HL[dm₁,₂→₁ ~ B₁ ~ dΔ₁→₁,₂ ≡ (dΔ₁→₁,₃ dΔ₂→₂,₄) ~ B₁,₂,₃,₄ ~ (dm₃,₄→₂ dm₁,₂→₁)]
```

Out[∘]= {12.9844, True}

S is convolution inverse of id

In[∘]:=
```
Timing[
 HL[# ≡ 𝔼[0, 0, 1]] & /@ {(dΔ₁→₁,₂ ~ B₁ ~ dS₁) ~ B₁,₂ ~ dm₁,₂→₁, (dΔ₁→₁,₂ ~ B₂ ~ dS₂) ~ B₁,₂ ~ dm₁,₂→₁}]
```

Out[∘]= {10.8281, {True, True}}

S is a (co)-algebra anti-morphism

In[∘]:=
$$\text{Timing}\left[\text{HL} /@ \text{Expand} /@ \left\{dm_{1,2\to 1}\sim B_1\sim dS_1 \equiv (dS_1\, dS_2)\sim B_{1,2}\sim dm_{2,1\to 1},\ dS_1\sim B_1\sim d\Delta_{1\to 1,2} \equiv d\Delta_{1\to 2,1}\sim B_{1,2}\sim (dS_1\, dS_2)\right\}\right]$$

Out[∘]= {24.0313, {True, True}}

Quasi-triangular axiom 1:

In[∘]:=
$$\text{Timing@HL}[R_{1,2}\sim B_1\sim d\Delta_{1\text{->}1,3} \equiv (R_{1,4}\, R_{3,2})\sim B_{2,4}\sim dm_{2,4\to 2}]$$

Out[∘]= {0.671875, True}

Quasi-triangular axiom 2:

In[∘]:=
$$\text{Timing@HL}\left[\left((d\Delta_{1\text{->}1,2}\, R_{3,4})\sim B_{1,2,3,4}\sim (dm_{1,3\to 1}\, dm_{2,4\to 2})\right) \equiv \left((d\Delta_{1\to 2,1}\, R_{3,4})\sim B_{1,2,3,4}\sim (dm_{3,1\to 1}\, dm_{4,2\to 2})\right)\right]$$

Out[∘]= {10.4688, True}

The Drinfel'd element inverse property, $(u_1\,\overline{u}_2)\sim B_{1,2}\sim \mathrm{dm}_{1,2\to 1} \equiv \mathbb{E}[0, 0, 1]$:

In[∘]:=
$$\text{Timing@HL}\left[\left((R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i})\ (R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to j})\right)\sim B_{i,j}\sim dm_{i,j\to i} \equiv \mathbb{E}[0, 0, 1]\right]$$

Out[∘]= {3.79688, True}

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

In[∘]:=
$$\text{Timing@Block}\left[\{\$k = 3\},\ \left(\left((R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i})\sim B_i\sim dS_i\right)\ (R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to j})\right)\sim B_{i,j}\sim dm_{i,j\to i}\right]$$

Out[∘]=
$$\left\{40.625,\ \mathbb{E}\left[0, 0, \frac{1}{B_i} + \frac{\hbar\, a_i\, \epsilon}{B_i} + \frac{\hbar^2\, a_i^2\, \epsilon^2}{2\, B_i} + \frac{\hbar^3\, a_i^3\, \epsilon^3}{6\, B_i} + O[\epsilon]^4\right]\right\}$$

In[∘]:=
$$\text{Timing@Block}\left[\{\$k = 2\},\ \text{HL} /@ \left\{(CC_i\, \overline{CC}_j)\sim B_{i,j}\sim dm_{i,j\to i} \equiv \mathbb{E}[0, 0, 1],\ (\overline{CC}_i\, \overline{CC}_j)\sim B_{i,j}\sim dm_{i,j\to i} \equiv \left(\left((R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i})\sim B_i\sim dS_i\right)\ (R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to j})\right)\sim B_{i,j}\sim dm_{i,j\to i}\right\}\right]$$

Out[∘]= {6.70313, {True, True}}

Reidemeister 2:

In[∘]:=
$$\text{Timing}\left[\text{HL}[\# \equiv \mathbb{E}[0, 0, 1]]\ \&\ /@ \left\{(\overline{R}_{1,2}\, R_{3,4})\sim B_{1,2,3,4}\sim (dm_{1,3\to 1}\, dm_{2,4\to 2}),\ (R_{1,2}\, \overline{R}_{3,4})\sim B_{1,2,3,4}\sim (dm_{1,3\to 1}\, dm_{2,4\to 2})\right\}\right]$$

Out[∘]= {7.375, {True, True}}

Cyclic Reidemeister 2:

In[∘]:=
$$\text{Timing@HL}\left[(R_{1,4}\, \overline{R}_{5,2}\, \overline{CC}_3)\sim B_{2,4}\sim dm_{2,4\to 2}\sim B_{1,3}\sim dm_{1,3\to 1}\sim B_{1,5}\sim dm_{1,5\to 1} \equiv \overline{CC}_1\right]$$

Out[∘]= {5.9375, True}

Reidemeister 3:

In[∘]:=
$$\text{Timing@HL}\left[\left((R_{1,2}\, R_{4,3}\, R_{5,6})\sim B_{1,4}\sim dm_{1,4\to 1}\sim B_{2,5}\sim dm_{2,5\to 2}\sim B_{3,6}\sim dm_{3,6\to 3}\right) \equiv \left((R_{1,6}\, R_{2,3}\, R_{4,5})\sim B_{1,4}\sim dm_{1,4\to 1}\sim B_{2,5}\sim dm_{2,5\to 2}\sim B_{3,6}\sim dm_{3,6\to 3}\right)\right]$$

Out[∘]= {4.96875, True}

Relations between the four kinks:

In[∘]:= $\text{Timing}\left[\text{HL} /@ \left\{\text{Kink}_i \equiv \left(R_{3,1}\, CC_2\right) \sim B_{1,2} \sim dm_{1,2\to 1} \sim B_{1,3} \sim dm_{1,3\to i},\; \overline{\text{Kink}}_j \equiv \left(\overline{R}_{3,1}\, \overline{CC}_2\right) \sim B_{1,2} \sim dm_{1,2\to 1} \sim B_{1,3} \sim dm_{1,3\to j},\; \left(\text{Kink}_i\, \overline{\text{Kink}}_j\right) \sim B_{i,j} \sim dm_{i,j\to 1} \equiv \mathbb{E}[0, 0, 1]\right\}\right]$

Out[∘]= {7.51563, {True, True, True}}

The Trefoil

In[∘]:= $\text{Monitor}\left[\text{Timing}@\text{Block}\left[\{\$k = 1\},\; Z = R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{\text{Kink}}_8\, \overline{\text{Kink}}_9\, \overline{\text{Kink}}_{10};\; \text{Do}[Z = Z \sim B_{1,r} \sim dm_{1,r\to 1}, \{r, 2, 10\}];\; \text{Simplify} /@ Z\right], r\right]$

Out[∘]= $\left\{98.5469, \mathbb{E}\left[0, 0, \frac{B_1}{1 - B_1 + B_1^2} - \left(\hbar B_1 \left(-a_1 \left(-1 + B_1 - B_1^3 + B_1^4\right) + \gamma \left(B_1 - 2 B_1^2 - 2 B_1^4 + 2 \hbar x_1 y_1 + B_1^3 \left(3 + 2 \hbar x_1 y_1\right)\right)\right) \epsilon\right) \Big/ \left(1 - B_1 + B_1^2\right)^3 + O[\epsilon]^2\right]\right\}$