Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

External Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Internal Utilities

Canonical Form:

In[∘]:=
```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. ⅇ^x_ ⅇ^y_ :→ ⅇ^(x+y) /. ⅇ^x_ :→ ⅇ^CF[x]];
```

The Kronecker δ:

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Normal@P + O[ϵ]^($k+1)];
```

Zip and Bind

Variables and their duals:

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})* := (u*)_i;
```

Finite Zips: (* Perhaps switch Expand to Collect[__, ζ]? *)

In[∘]:=
```
expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ℰ_] := Expand[ℰ];
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] := (expand[P // Zip_{ζs}] /. f_. ζ^d_. :→ ∂_{ζ*,d} f) /. ζ* → 0
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

```
In[•]:= QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → qt.(zs + ys)];
    Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L, Q2, Det[qt] e^{-Q2} Zip_{ζs}[e^{Q1} (P /. zrule)]]];
QZip_{ζs_List} := QZip_{ζs,CF};
```

Upper to lower and lower to Upper:

```
In[•]:= U2l = {B_{i_}^{p_.} → e^{-p ħ γ b_i}, B^{p_.} → e^{-p ħ γ b}, T_{i_}^{p_.} → e^{p ħ t_i}, T^{p_.} → e^{p ħ t}, 𝒜_{i_}^{p_.} → e^{p γ α_i}, 𝒜^{p_.} → e^{p γ α}};
l2U = {e^{c_. b_{i_}+d_.} :→ B_i^{-c/(ħ γ)} e^d, e^{c_. b+d_.} :→ B^{-c/(ħ γ)} e^d,
    e^{c_. t_{i_}+d_.} :→ T_i^{c/ħ} e^d, e^{c_. t+d_.} :→ T^{c/ħ} e^d,
    e^{c_. α_{i_}+d_.} :→ 𝒜_i^{c/γ} e^d, e^{c_. α+d_.} :→ 𝒜^{c/γ} e^d,
    e^{ℰ_} :→ e^{Expand@ℰ}};
```

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^{Q} as a single"P". Here the z's are b and α and the ζ's are β and a.

```
In[•]:= LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L2, Q2, Det[lt] e^{-L2-Q2} Zip_{ζs}[e^{L1+Q1} (P /. U2l /. zrule)]] //. l2U];
LZip_{ζs_List} := LZip_{ζs,CF};
```

```
In[•]:= Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
    Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Flatten@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} // QZip_{Flatten@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]}];
B_{L_List}[L_, R_] := Bind_L[L, R]; B_{is___}[L_, R_] := Bind_{{is}}[L, R];
```

Booting Up

```
In[•]:= $k = 2;
```

In[∘]:=
```
Boot[$k_] := Once[Block[{i, j, k, m, n, tu = TimeUsed[]},
    am_{i_,j_->k_,$k} = 𝔼[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1 + O[ϵ]^{$k+1}];
    bm_{i_,j_→k_,$k} = 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i}-1) η_j y_k} + O[ϵ]^{$k+1}];
    R_{i_,j_,$k} = 𝔼[ħ a_j b_i, ħ x_j y_i, Series[e^(Sum_{k=2}^{$k+1} (1 - e^{γ ϵ ħ})^k (ħ y_i x_j)^k / (k (1 - e^{k γ ϵ ħ}))), {ϵ, 0, $k}]];
    P_{i_,j_,$k} = If[$k == 0, 𝔼[β_i α_j / ħ, η_i ξ_j / ħ, 1 + O[ϵ]],
      MapAt[
       (# - ϵ^{$k} Coefficient[(R_{n,m} ~ B_{n,m} ~ ((P_{n,j,0})_{$k} (P_{i,m,$k-1})_{$k}))〚3〛, ϵ, $k] + O[ϵ]^{$k+1}) &,
       (P_{i,j,$k-1})_{$k}, 3]];
    (* t==ϵa-γb and b==-t/γ+ϵa/γ: *)
    b2t_{i_,$k} = 𝔼[α_i a_i - β_i t_i / γ, ξ_i x_i + η_i y_i, e^{ϵ β_i a_i / γ} + O[ϵ]^{$k+1}];
    t2b_{i_,$k} = 𝔼[α_i a_j - τ_i γ b_j, ξ_i x_j + η_i y_j, e^{ϵ τ_i a_j} + O[ϵ]^{$k+1}];
    Print["Booted @ $k=", $k, " in ", TimeUsed[] - tu, " sec."];
   ]];
```

In[∘]:=
```
am_{i_,j_→k_} := am_{i,j→k,$k}; am_{i_,j_→k_,$k_} := (Boot[$k]; am_{i,j→k,$k});
bm_{i_,j_→k_} := bm_{i,j→k,$k}; bm_{i_,j_→k_,$k_} := (Boot[$k]; bm_{i,j→k,$k});
R_{i_,j_} := R_{i,j,$k}; R_{i_,j_,$k_} := (Boot[$k]; R_{i,j,$k});
P_{i_,j_} := P_{i,j,$k}; P_{i_,j_,$k_} := (Boot[$k]; P_{i,j,$k});
b2t_{i_} := b2t_{i,$k}; b2t_{i_,$k_} := (Boot[$k]; b2t_{i,$k});
t2b_{i_} := t2b_{i,$k}; t2b_{i_,$k_} := (Boot[$k]; t2b_{i,$k});
```

In[∘]:=
```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, R → R_{i,j}, P → P_{i,j}, b2t → b2t_i, t2b → t2b_i
  }] // Column
```

Out[∘]=

$\text{am} \to \mathbb{E}\left[a_k\,(\alpha_i+\alpha_j),\ x_k\,\left(e^{-\gamma\alpha_j}\xi_i+\xi_j\right),\ 1+O[\epsilon]^2\right]$

$\text{bm} \to \mathbb{E}\left[b_k\,(\beta_i+\beta_j),\ y_k\,(\eta_i+\eta_j),\ 1-y_k\,\beta_i\,\eta_j\,\epsilon+O[\epsilon]^2\right]$

$\text{R} \to \mathbb{E}\left[\hbar\, a_j\, b_i,\ \hbar\, x_j\, y_i,\ 1-\frac{1}{4}\left(\gamma\,\hbar^3\, x_j^2\, y_i^2\right)\epsilon+O[\epsilon]^2\right]$

$\text{P} \to \mathbb{E}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1+\frac{\gamma\,\eta_i^2\,\xi_j^2\,\epsilon}{4\,\hbar}+O[\epsilon]^2\right]$

$\text{b2t} \to \mathbb{E}\left[a_i\,\alpha_i-t_i\,\beta_i,\ y_i\,\eta_i+x_i\,\xi_i,\ 1+a_i\,\beta_i\,\epsilon+O[\epsilon]^2\right]$

$\text{t2b} \to \mathbb{E}\left[a_i\,\alpha_i-b_i\,\tau_i,\ y_i\,\eta_i+x_i\,\xi_i,\ 1+a_i\,\tau_i\,\epsilon+O[\epsilon]^2\right]$

Testing

Associativity of am and bm:

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am_{1,2→1} ~ B_1 ~ am_{1,3→1}) ≡ (am_{2,3→2} ~ B_2 ~ am_{1,2→1}), (bm_{1,2→1} ~ B_1 ~ bm_{1,3→1}) ≡ (bm_{2,3→2} ~ B_2 ~ bm_{1,2→1})}
 ]
```

Booted @ $k=3 in 0.015 sec.

Out[∘]= {0.140625, {True, True}}

R and *P* are inverses:

In[∘]:=
```
Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[R_{i,j} ~ B_i ~ P_{i,k} ≡ 𝔼[a_j α_k, x_j ξ_k, 1]]}]
```

Booted @ $k=3 in 0.094 sec.

Out[∘]= $\{0.140625, \{\mathbb{E}[\hbar a_j b_i, \hbar x_j y_i, 1 - \frac{1}{4}(\gamma \hbar^3 x_j^2 y_i^2)\epsilon + \left(\frac{1}{9}\gamma^2\hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2\hbar^6 x_j^4 y_i^4\right)\epsilon^2 + \frac{1}{1152}(24\gamma^3\hbar^5 x_j^2 y_i^2 - 72\gamma^3\hbar^7 x_j^4 y_i^4 - 32\gamma^3\hbar^8 x_j^5 y_i^5 - 3\gamma^3\hbar^9 x_j^6 y_i^6)\epsilon^3 + O[\epsilon]^4],$
$\mathbb{E}[\frac{\alpha_k \beta_i}{\hbar}, \frac{\eta_i \xi_k}{\hbar}, 1 + \frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4\hbar} + \frac{(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4)\epsilon^2}{288\hbar^2} + \frac{1}{1152\hbar^3}(48\gamma^3\hbar^4\eta_i^2\xi_k^2 + 192\gamma^3\hbar^3\eta_i^3\xi_k^3 + 156\gamma^3\hbar^2\eta_i^4\xi_k^4 + 40\gamma^3\hbar\eta_i^5\xi_k^5 + 3\gamma^3\eta_i^6\xi_k^6)\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$

S

In[∘]:=
```
aS_{i_} := 𝔼[-α_i a_j, -ξ_i x_i, Series[
    e^{ξ_i x_i} Sum[Expand[(-ħ γ ϵ)^k / (2^k k!) Nest[Expand[x_i^2 ∂_{{x_i,2}} #] &, e^{-ξ_i e^{ħ ϵ a_i} x_i}, k]], {k, 0, $k}],
    {ϵ, 0, $k}]] ~ B_{i,j} ~ am_{i,j→i};
aS_i
```

Booted @ $k=0 in 0. sec.

Booted @ $k=1 in 0.063 sec.

Booted @ $k=2 in 0.14 sec.

Out[∘]= $\mathbb{E}[-a_i\alpha_i, -x_i \mathcal{A}_i \xi_i, 1 + \frac{1}{2}(-2\hbar a_i x_i \mathcal{A}_i \xi_i - \gamma\hbar x_i^2 \mathcal{A}_i^2 \xi_i^2)\epsilon + \frac{1}{8}(-4\hbar^2 a_i^2 x_i \mathcal{A}_i \xi_i + 2\gamma^2\hbar^2 x_i^2 \mathcal{A}_i^2 \xi_i^2 - 8\gamma\hbar^2 a_i x_i^2 \mathcal{A}_i^2 \xi_i^2 + 4\hbar^2 a_i^2 x_i^2 \mathcal{A}_i^2 \xi_i^2 - 4\gamma^2\hbar^2 x_i^3 \mathcal{A}_i^3 \xi_i^3 + 4\gamma\hbar^2 a_i x_i^3 \mathcal{A}_i^3 \xi_i^3 + \gamma^2\hbar^2 x_i^4 \mathcal{A}_i^4 \xi_i^4)\epsilon^2 + O[\epsilon]^3]$

In[∘]:=
```
aS̄_{i_,0} := 𝔼[-a_i α_i, -x_i 𝒜_i ξ_i, 1];
aS̄_{i_,k_} :=
  MapAt[(# - ϵ^k Coefficient[(aS̄_{i,0} ~ B_i ~ aS_i ~ B_i ~ aS̄_{i,k-1})⟦3⟧, ϵ, k] + O[ϵ]^{$k+1}) &, aS̄_{i,k-1}, 3];
aS̄_{i_} := aS̄_{i,$k};
aS̄_1
```

Out[∘]= $\mathbb{E}[-a_1\alpha_1, -x_1\mathcal{A}_1\xi_1, 1 + \left(\gamma\hbar x_1\mathcal{A}_1\xi_1 - \hbar a_1 x_1\mathcal{A}_1\xi_1 - \frac{1}{2}\gamma\hbar x_1^2\mathcal{A}_1^2\xi_1^2\right)\epsilon + \frac{1}{8}(-4\gamma^2\hbar^2 x_1\mathcal{A}_1\xi_1 + 8\gamma\hbar^2 a_1 x_1\mathcal{A}_1\xi_1 - 4\hbar^2 a_1^2 x_1\mathcal{A}_1\xi_1 + 14\gamma^2\hbar^2 x_1^2\mathcal{A}_1^2\xi_1^2 - 16\gamma\hbar^2 a_1 x_1^2\mathcal{A}_1^2\xi_1^2 + 4\hbar^2 a_1^2 x_1^2\mathcal{A}_1^2\xi_1^2 - 8\gamma^2\hbar^2 x_1^3\mathcal{A}_1^3\xi_1^3 + 4\gamma\hbar^2 a_1 x_1^3\mathcal{A}_1^3\xi_1^3 + \gamma^2\hbar^2 x_1^4\mathcal{A}_1^4\xi_1^4)\epsilon^2 + O[\epsilon]^3]$

In[∘]:=
```
Timing@HL[aS̄_1 ~ B_1 ~ aS_1 ≡ 𝔼[a_1 α_1, x_1 ξ_1, 1]]
```

Out[∘]= {0.5625, True}

In[∘]:=
```
bS_{i_} := R_{i,n} ~ B_n ~ aS_n ~ B_n ~ P_{i,n};
bS̄_{i_} := R_{i,n} ~ B_n ~ aS̄_n ~ B_n ~ P_{i,n};
```

In[∘]:=
```
Timing@HL[bS̄₁ ~ B₁ ~ bS₁ ≡ 𝔼[b₁ β₁, y₁ η₁, 1]]
```

Out[∘]= {0.984375, True}

Δ

In[∘]:=
```
Block[{i, j, k, l, m, n}, aΔ_{i_→j_,k_} = (R_{n,j} R_{m,k}) ~ B_{n,m} ~ bm_{n,m→l} ~ B_l ~ P_{l,i}];
aΔ_{i→j,k}
```

Out[∘]= $\mathbb{E}\Big[a_j \alpha_i + a_k \alpha_i,\ x_j \xi_i + x_k \xi_i,\ 1 + \frac{1}{2}\left(-2\hbar a_j x_k \xi_i + \gamma\hbar x_j x_k \xi_i^2\right)\epsilon + \frac{1}{24}\left(12\hbar^2 a_j^2 x_k \xi_i + 6\gamma^2\hbar^2 x_j x_k \xi_i^2 - 12\gamma\hbar^2 a_j x_j x_k \xi_i^2 + 12\hbar^2 a_j^2 x_k^2 \xi_i^2 + 4\gamma^2\hbar^2 x_j^2 x_k \xi_i^3 + 4\gamma^2\hbar^2 x_j x_k^2 \xi_i^3 - 12\gamma\hbar^2 a_j x_j x_k^2 \xi_i^3 + 3\gamma^2\hbar^2 x_j^2 x_k^2 \xi_i^4\right)\epsilon^2 + O[\epsilon]^3\Big]$

In[∘]:=
```
Block[{i, j, k, l, m, n}, bΔ_{i_→j_,k_} = (R_{j,n} R_{k,m}) ~ B_{n,m} ~ am_{n,m→l} ~ B_l ~ P_{i,l}];
bΔ_{i→j,k}
```

Out[∘]= $\mathbb{E}\Big[b_j \beta_i + b_k \beta_i,\ B_k y_j \eta_i + y_k \eta_i,\ 1 + \frac{1}{2}\gamma\hbar B_k y_j y_k \eta_i^2 \epsilon + \frac{1}{24}\left(6\gamma^2\hbar^2 B_k y_j y_k \eta_i^2 + 4\gamma^2\hbar^2 B_k^2 y_j^2 y_k \eta_i^3 + 4\gamma^2\hbar^2 B_k y_j y_k^2 \eta_i^3 + 3\gamma^2\hbar^2 B_k^2 y_j^2 y_k^2 \eta_i^4\right)\epsilon^2 + O[\epsilon]^3\Big]$

The two halves

First check that on the generators this agrees with our conventions in the handout:

In[∘]:=
```
Timing@{{"[a,x]" → ((𝔼[0, 0, a₂ x₁] ~ B_{1,2} ~ am_{1,2→1}) ⟦3⟧ - (𝔼[0, 0, a₁ x₂] ~ B_{1,2} ~ am_{1,2→1}) ⟦3⟧),
    "[b,y]" → ((𝔼[0, 0, y₂ b₁] ~ B_{1,2} ~ bm_{1,2→1}) ⟦3⟧ - (𝔼[0, 0, y₁ b₂] ~ B_{1,2} ~ bm_{1,2→1}) ⟦3⟧)} /.
   z_1 → z,
  {"Δ[y]" → Last[𝔼[0, 0, y₁] ~ B₁ ~ bΔ_{1→1,2}],
   "Δ[b]" → Last[𝔼[0, 0, b₁] ~ B₁ ~ bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼[0, 0, a₁] ~ B₁ ~ aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼[0, 0, x₁] ~ B₁ ~ aΔ_{1→1,2}]},
  {
    "S(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ aS₁) ⟦3⟧),
    "S(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ aS₁) ⟦3⟧),
    "S(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ bS₁) ⟦3⟧),
    "S(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ bS₁) ⟦3⟧)
  } /. z_1 → z}
```

Out[∘]= $\Big\{0.875,\ \Big\{\big\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\big\},\ \big\{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\big\},\ \big\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}\left(a^2 x \hbar^2\right)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\big\}\Big\}\Big\}$

Hopf algebra axioms on both sides separately (co)-associativity on both sides

In[◦]:=
$$\text{Timing}\Big[\text{HL} \;/@\; \big\{(a\Delta_{1\to1,2} \sim B_2 \sim a\Delta_{2\to2,3}) \equiv (a\Delta_{1\to1,3} \sim B_1 \sim a\Delta_{1\to1,2}),\ (b\Delta_{1\to1,2} \sim B_2 \sim b\Delta_{2\to2,3}) \equiv (b\Delta_{1\to1,3} \sim B_1 \sim b\Delta_{1\to1,2}),\ (am_{1,2\to1} \sim B_1 \sim am_{1,3\to1}) \equiv (am_{2,3\to2} \sim B_2 \sim am_{1,2\to1}),\ (bm_{1,2\to1} \sim B_1 \sim bm_{1,3\to1}) \equiv (bm_{2,3\to2} \sim B_2 \sim bm_{1,2\to1})\big\}\Big]$$

Out[◦]= {0.40625, {True, True, True, True}}

Δ is an algebra morphism

In[◦]:=
$$\text{Timing}\Big[\text{HL} \;/@\; \big\{am_{1,2\to1} \sim B_1 \sim a\Delta_{1\to1,2} \equiv (a\Delta_{1\to1,3}\ a\Delta_{2\to2,4}) \sim B_{1,2,3,4} \sim (am_{3,4\to2}\ am_{1,2\to1}),\ bm_{1,2\to1} \sim B_1 \sim b\Delta_{1\to1,2} \equiv (b\Delta_{1\to1,3}\ b\Delta_{2\to2,4}) \sim B_{1,2,3,4} \sim (bm_{3,4\to2}\ bm_{1,2\to1})\big\}\Big]$$

Out[◦]= {0.703125, {True, True}}

S is convolution inverse of id

In[◦]:=
$$\text{Timing}\Big[\text{HL}[\# \equiv \mathbb{E}[0, 0, 1]]\ \&\ /@\ \big\{(a\Delta_{1\to1,2} \sim B_1 \sim aS_1) \sim B_{1,2} \sim am_{1,2\to1},\ (a\Delta_{1\to1,2} \sim B_2 \sim aS_2) \sim B_{1,2} \sim am_{1,2\to1},\ (b\Delta_{1\to1,2} \sim B_1 \sim bS_1) \sim B_{1,2} \sim bm_{1,2\to1},\ (b\Delta_{1\to1,2} \sim B_2 \sim bS_2) \sim B_{1,2} \sim bm_{1,2\to1}\big\}\Big]$$

Out[◦]= {1.26563, {True, True, True, True}}

S is an algebra anti-(co)morphism

In[◦]:=
$$\text{Timing}\Big[\text{HL} \;/@\; \big\{am_{1,2\to1} \sim B_1 \sim aS_1 \equiv (aS_1\ aS_2) \sim B_{1,2} \sim am_{2,1\to1},\ bm_{1,2\to1} \sim B_1 \sim bS_1 \equiv (bS_1\ bS_2) \sim B_{1,2} \sim bm_{2,1\to1},\ aS_1 \sim B_1 \sim a\Delta_{1\to1,2} \equiv a\Delta_{1\to2,1} \sim B_{1,2} \sim (aS_1\ aS_2),\ bS_1 \sim B_1 \sim b\Delta_{1\to1,2} \equiv b\Delta_{1\to2,1} \sim B_{1,2} \sim (bS_1\ bS_2)\big\}\Big]$$

Out[◦]= {2.84375, {True, True, True, True}}

Pairing axioms

In[◦]:=
$$\text{Timing}\Big[\text{HL} \;/@\; \big\{(bm_{1,2\to1}\ \mathbb{E}[\alpha_3\, a_3,\ \xi_3\, x_3,\ 1]) \sim B_{1,3} \sim P_{1,3} \equiv (\mathbb{E}[\beta_1\, b_1,\ \eta_1\, y_1,\ 1]\ \mathbb{E}[\beta_2\, b_2,\ \eta_2\, y_2,\ 1]\ a\Delta_{3\to4,5}) \sim B_{1,4} \sim P_{1,4} \sim B_{2,5} \sim P_{2,5},\ (b\Delta_{1\to1,2}\ \mathbb{E}[\alpha_3\, a_3,\ \xi_3\, x_3,\ 1]\ \mathbb{E}[\alpha_4\, a_4,\ \xi_4\, x_4,\ 1]) \sim B_{1,3} \sim P_{1,3} \sim B_{2,4} \sim P_{2,4} \equiv (\mathbb{E}[\beta_1\, b_1,\ \eta_1\, y_1,\ 1]\ am_{3,4\to3}) \sim B_{1,3} \sim P_{1,3}\big\}\Big]$$

Out[◦]= {0.984375, {True, True}}

In[◦]:=
$$\text{Timing}\Big[\text{HL} \;/@\; \big\{(bS_1\ \mathbb{E}[\alpha_2\, a_2,\ \xi_2\, x_2,\ 1]) \sim B_{1,2} \sim P_{1,2} \equiv (\mathbb{E}[\beta_1\, b_1,\ \eta_1\, y_1,\ 1]\ aS_2) \sim B_{1,2} \sim P_{1,2},\ (\overline{bS}_1\ \mathbb{E}[\alpha_2\, a_2,\ \xi_2\, x_2,\ 1]) \sim B_{1,2} \sim P_{1,2} \equiv (\mathbb{E}[\beta_1\, b_1,\ \eta_1\, y_1,\ 1]\ \overline{aS}_2) \sim B_{1,2} \sim P_{1,2}\big\}\Big]$$

Out[◦]= {1.89063, {True, True}}

The Double

The double multiplication (should really bind the a’s and b’s separately)

In[∘]:=

$$\mathbf{Block}\Big[\{i,j,k\},\ dm_{i\_,j\_\to k\_} = \Big(\mathbb{E}[\beta_i\, b_i + \alpha_j\, a_j,\ \eta_i\, y_i + \xi_j\, x_j,\ 1]\ (a\Delta_{i\to h1,h2} \sim B_{h2} \sim a\Delta_{h2\to h2,h3})\ \big(b\Delta_{j\to t1,t2} \sim B_{t2} \sim b\Delta_{t2\to t2,t3}\big)\Big) \sim B_{h3} \sim \overline{aS}_{h3} \sim B_{t1,h3} \sim (P_{t1,h3}) \sim B_{t3,h1} \sim (P_{t3,h1}) \sim B_{h2,j,i,t2} \sim \big(am_{h2,j\to k}\ bm_{i,t2\to k}\big)\Big]$$

Out[∘]=

$$\mathbb{E}\Big[a_k\,\alpha_i + a_k\,\alpha_j + b_k\,\beta_i + b_k\,\beta_j,\ \frac{1}{\hbar\,\mathcal{A}_i\,\mathcal{A}_j}$$
$$(\hbar\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_i + \hbar\, y_k\,\mathcal{A}_j\,\eta_j + \hbar\, x_k\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i - B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i + \hbar\, x_k\,\mathcal{A}_i\,\mathcal{A}_j\,\xi_j),$$
$$1 + \frac{1}{4\,\hbar\,\mathcal{A}_i\,\mathcal{A}_j}\ \big(-4\,\hbar\, y_k\,\mathcal{A}_j\,\beta_i\,\eta_j - 4\,\hbar\, x_k\,\mathcal{A}_i\,\beta_j\,\xi_i + 4\,\gamma\,\hbar^2\, x_k\, y_k\,\eta_j\,\xi_i +$$
$$4\,\hbar\, a_k\, B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i + 2\,\gamma\,\hbar\, y_k\,\mathcal{A}_j\,\eta_j^2\,\xi_i - 6\,\gamma\,\hbar\, B_k\, y_k\,\mathcal{A}_j\,\eta_j^2\,\xi_i + 2\,\gamma\,\hbar\, x_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2 -$$
$$6\,\gamma\,\hbar\, B_k\, x_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2 + \gamma\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 - 4\,\gamma\, B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 + 3\,\gamma\, B_k^2\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2\big)\ \epsilon + \frac{1}{288\,\hbar^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2}$$
$$\big(144\,\hbar^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i^2\,\eta_j + 144\,\hbar^2\, y_k^2\,\mathcal{A}_j^2\,\beta_i^2\,\eta_j^2 + 144\,\hbar^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j^2\,\xi_i + 144\,\gamma^2\,\hbar^4\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i -$$
$$144\,\hbar^3\, a_k^2\, B_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j\,\xi_i - 288\,\gamma\,\hbar^3\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_i\,\eta_j\,\xi_i - 288\,\gamma\,\hbar^3\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_j\,\eta_j\,\xi_i +$$
$$288\,\hbar^2\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_i\,\beta_j\,\eta_j\,\xi_i + 144\,\gamma^2\,\hbar^4\, x_k\, y_k^2\,\mathcal{A}_j\,\eta_j^2\,\xi_i + 72\,\gamma^2\,\hbar^3\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i -$$
$$360\,\gamma^2\,\hbar^3\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i + 432\,\gamma\,\hbar^3\, a_k\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i - 288\,\gamma\,\hbar^3\, x_k\, y_k^2\,\mathcal{A}_j\,\beta_i\,\eta_j^2\,\xi_i -$$
$$144\,\gamma\,\hbar^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i\,\eta_j^2\,\xi_i + 432\,\gamma\,\hbar^2\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i\,\eta_j^2\,\xi_i - 288\,\hbar^2\, a_k\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i\,\eta_j^2\,\xi_i +$$
$$48\,\gamma^2\,\hbar^3\, y_k^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i - 336\,\gamma^2\,\hbar^3\, B_k\, y_k^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i - 144\,\gamma\,\hbar^2\, y_k^2\,\mathcal{A}_j^2\,\beta_i\,\eta_j^3\,\xi_i + 432\,\gamma\,\hbar^2\, B_k\, y_k^2\,\mathcal{A}_j^2\,\beta_i\,\eta_j^3\,\xi_i +$$
$$144\,\hbar^2\, x_k^2\,\mathcal{A}_i^2\,\beta_j^2\,\xi_i^2 + 144\,\gamma^2\,\hbar^4\, x_k^2\, y_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2 + 72\,\gamma^2\,\hbar^3\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j\,\xi_i^2 - 360\,\gamma^2\,\hbar^3\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j\,\xi_i^2 +$$
$$432\,\gamma\,\hbar^3\, a_k\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j\,\xi_i^2 - 288\,\gamma\,\hbar^3\, x_k^2\, y_k\,\mathcal{A}_i\,\beta_j\,\eta_j\,\xi_i^2 - 144\,\gamma\,\hbar^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j\,\eta_j\,\xi_i^2 +$$
$$432\,\gamma\,\hbar^2\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j\,\eta_j\,\xi_i^2 - 288\,\hbar^2\, a_k\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j\,\eta_j\,\xi_i^2 + 144\,\gamma^2\,\hbar^4\, x_k^2\, y_k^2\,\eta_j^2\,\xi_i^2 +$$
$$360\,\gamma^2\,\hbar^3\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 - 1512\,\gamma^2\,\hbar^3\, B_k\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 + 288\,\gamma\,\hbar^3\, a_k\, B_k\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 +$$
$$36\,\gamma^2\,\hbar^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i^2 - 216\,\gamma^2\,\hbar^2\, B_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i^2 + 288\,\gamma\,\hbar^2\, a_k\, B_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i^2 +$$
$$180\,\gamma^2\,\hbar^2\, B_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i^2 - 432\,\gamma\,\hbar^2\, a_k\, B_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i^2 + 144\,\hbar^2\, a_k^2\, B_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^2\,\xi_i^2 -$$
$$144\,\gamma\,\hbar^2\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_i\,\eta_j^2\,\xi_i^2 + 432\,\gamma\,\hbar^2\, B_k\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_i\,\eta_j^2\,\xi_i^2 - 144\,\gamma\,\hbar^2\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_j\,\eta_j^2\,\xi_i^2 +$$
$$432\,\gamma\,\hbar^2\, B_k\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\beta_j\,\eta_j^2\,\xi_i^2 + 144\,\gamma^2\,\hbar^3\, x_k\, y_k^2\,\mathcal{A}_j\,\eta_j^3\,\xi_i^2 - 432\,\gamma^2\,\hbar^3\, B_k\, x_k\, y_k^2\,\mathcal{A}_j\,\eta_j^3\,\xi_i^2 +$$
$$120\,\gamma^2\,\hbar^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^2 - 816\,\gamma^2\,\hbar^2\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^2 + 144\,\gamma\,\hbar^2\, a_k\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^2 +$$
$$984\,\gamma^2\,\hbar^2\, B_k^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^2 - 432\,\gamma\,\hbar^2\, a_k\, B_k^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^2 - 72\,\gamma\,\hbar\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i\,\eta_j^3\,\xi_i^2 +$$
$$288\,\gamma\,\hbar\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i\,\eta_j^3\,\xi_i^2 - 216\,\gamma\,\hbar\, B_k^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\beta_i\,\eta_j^3\,\xi_i^2 + 36\,\gamma^2\,\hbar^2\, y_k^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^2 -$$
$$216\,\gamma^2\,\hbar^2\, B_k\, y_k^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^2 + 324\,\gamma^2\,\hbar^2\, B_k^2\, y_k^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^2 + 48\,\gamma^2\,\hbar^3\, x_k^2\,\mathcal{A}_i^2\,\eta_j\,\xi_i^3 - 336\,\gamma^2\,\hbar^3\, B_k\, x_k^2\,\mathcal{A}_i^2\,\eta_j\,\xi_i^3 -$$
$$144\,\gamma\,\hbar^2\, x_k^2\,\mathcal{A}_i^2\,\beta_j\,\eta_j\,\xi_i^3 + 432\,\gamma\,\hbar^2\, B_k\, x_k^2\,\mathcal{A}_i^2\,\beta_j\,\eta_j\,\xi_i^3 + 144\,\gamma^2\,\hbar^3\, x_k^2\, y_k\,\mathcal{A}_i\,\eta_j^2\,\xi_i^3 -$$
$$432\,\gamma^2\,\hbar^3\, B_k\, x_k^2\, y_k\,\mathcal{A}_i\,\eta_j^2\,\xi_i^3 + 120\,\gamma^2\,\hbar^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^2\,\xi_i^3 - 816\,\gamma^2\,\hbar^2\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^2\,\xi_i^3 +$$
$$144\,\gamma\,\hbar^2\, a_k\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^2\,\xi_i^3 + 984\,\gamma^2\,\hbar^2\, B_k^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^2\,\xi_i^3 - 432\,\gamma\,\hbar^2\, a_k\, B_k^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^2\,\xi_i^3 -$$
$$72\,\gamma\,\hbar\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j\,\eta_j^2\,\xi_i^3 + 288\,\gamma\,\hbar\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j\,\eta_j^2\,\xi_i^3 - 216\,\gamma\,\hbar\, B_k^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\beta_j\,\eta_j^2\,\xi_i^3 +$$
$$144\,\gamma^2\,\hbar^2\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^3\,\xi_i^3 - 720\,\gamma^2\,\hbar^2\, B_k\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^3\,\xi_i^3 + 864\,\gamma^2\,\hbar^2\, B_k^2\, x_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^3\,\xi_i^3 +$$
$$40\,\gamma^2\,\hbar\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 - 312\,\gamma^2\,\hbar\, B_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 + 72\,\gamma\,\hbar\, a_k\, B_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 + 600\,\gamma^2\,\hbar\, B_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 -$$
$$288\,\gamma\,\hbar\, a_k\, B_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 - 328\,\gamma^2\,\hbar\, B_k^3\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 + 216\,\gamma\,\hbar\, a_k\, B_k^3\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^3\,\xi_i^3 + 36\,\gamma^2\,\hbar\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^3 -$$
$$252\,\gamma^2\,\hbar\, B_k\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^3 + 540\,\gamma^2\,\hbar\, B_k^2\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^3 - 324\,\gamma^2\,\hbar\, B_k^3\, y_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^3 +$$
$$36\,\gamma^2\,\hbar^2\, x_k^2\,\mathcal{A}_i^2\,\eta_j^2\,\xi_i^4 - 216\,\gamma^2\,\hbar^2\, B_k\, x_k^2\,\mathcal{A}_i^2\,\eta_j^2\,\xi_i^4 + 324\,\gamma^2\,\hbar^2\, B_k^2\, x_k^2\,\mathcal{A}_i^2\,\eta_j^2\,\xi_i^4 + 36\,\gamma^2\,\hbar\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^3\,\xi_i^4 -$$
$$252\,\gamma^2\,\hbar\, B_k\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^3\,\xi_i^4 + 540\,\gamma^2\,\hbar\, B_k^2\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^3\,\xi_i^4 - 324\,\gamma^2\,\hbar\, B_k^3\, x_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j^3\,\xi_i^4 + 9\,\gamma^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^4 -$$
$$72\,\gamma^2\, B_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^4 + 198\,\gamma^2\, B_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^4 - 216\,\gamma^2\, B_k^3\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^4 + 81\,\gamma^2\, B_k^4\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j^4\,\xi_i^4\big)\ \epsilon^2 + O[\epsilon]^3\Big]$$

In[∘]:=

$$\mathbf{Block}\big[\{i\},\ dS_{i\_} = \mathbb{E}[\beta_i\, b_1 + \alpha_i\, a_2,\ \eta_i\, y_1 + \xi_i\, x_2,\ 1] \sim B_{1,2} \sim \big(\overline{bS}_1\ aS_2\big) \sim B_{1,2} \sim dm_{2,1\to i}\big]$$

Out[ ]=
$$\mathbb{E}\Big[-a_i\,\alpha_i - b_i\,\beta_i,\ \frac{-\hbar\,y_i\,\mathcal{A}_i\,\eta_i - \hbar\,B_i\,x_i\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\eta_i\,\xi_i - B_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{\hbar\,B_i},$$

$$1 + \frac{1}{4\,\hbar\,B_i^2}\,\big(4\,\gamma\,\hbar^2\,B_i\,y_i\,\mathcal{A}_i\,\eta_i - 4\,\hbar\,B_i\,y_i\,\mathcal{A}_i\,\beta_i\,\eta_i - 2\,\gamma\,\hbar^2\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2 - 4\,\hbar^2\,a_i\,B_i^2\,x_i\,\mathcal{A}_i\,\xi_i - 4\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i\,\beta_i\,\xi_i -$$
$$4\,\gamma\,\hbar\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i + 4\,\hbar\,a_i\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i + 4\,\gamma\,\hbar\,B_i^2\,\mathcal{A}_i\,\eta_i\,\xi_i - 4\,\gamma\,\hbar^2\,B_i\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i +$$
$$4\,B_i\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i - 4\,B_i^2\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 6\,\gamma\,\hbar\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 2\,\gamma\,\hbar\,B_i\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 2\,\gamma\,\hbar^2\,B_i^2\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 +$$
$$6\,\gamma\,\hbar\,B_i\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 2\,\gamma\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 3\,\gamma\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 4\,\gamma\,B_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - \gamma\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2\big)\,\epsilon +$$

$$\frac{1}{288\,\hbar^2\,B_i^4}\,\big(-144\,\gamma^2\,\hbar^4\,B_i^3\,y_i\,\mathcal{A}_i\,\eta_i + 288\,\gamma\,\hbar^3\,B_i^3\,y_i\,\mathcal{A}_i\,\beta_i\,\eta_i - 144\,\hbar^2\,B_i^3\,y_i\,\mathcal{A}_i\,\beta_i^2\,\eta_i +$$
$$504\,\gamma^2\,\hbar^4\,B_i^2\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2 - 576\,\gamma\,\hbar^3\,B_i^2\,y_i^2\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2 + 144\,\hbar^2\,B_i^2\,y_i^2\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2 - 288\,\gamma^2\,\hbar^4\,B_i\,y_i^3\,\mathcal{A}_i^3\,\eta_i^3 +$$
$$144\,\gamma\,\hbar^3\,B_i\,y_i^3\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3 + 36\,\gamma^2\,\hbar^4\,y_i^4\,\mathcal{A}_i^4\,\eta_i^4 - 144\,\hbar^4\,a_i^2\,B_i^4\,x_i\,\mathcal{A}_i\,\xi_i - 288\,\hbar^3\,a_i\,B_i^4\,x_i\,\mathcal{A}_i\,\beta_i\,\xi_i -$$
$$144\,\hbar^2\,B_i^4\,x_i\,\mathcal{A}_i\,\beta_i^2\,\xi_i + 144\,\gamma^2\,\hbar^3\,B_i^3\,\mathcal{A}_i\,\eta_i\,\xi_i - 288\,\gamma\,\hbar^3\,a_i\,B_i^3\,\mathcal{A}_i\,\eta_i\,\xi_i + 144\,\hbar^3\,a_i^2\,B_i^3\,\mathcal{A}_i\,\eta_i\,\xi_i -$$
$$144\,\gamma^2\,\hbar^3\,B_i^4\,\mathcal{A}_i\,\eta_i\,\xi_i + 432\,\gamma^2\,\hbar^4\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i - 576\,\gamma\,\hbar^4\,a_i\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i -$$
$$288\,\gamma\,\hbar^2\,B_i^3\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 288\,\hbar^2\,a_i\,B_i^3\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 288\,\gamma\,\hbar^2\,B_i^4\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i -$$
$$864\,\gamma\,\hbar^3\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i + 288\,\hbar^3\,a_i\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i + 144\,\hbar\,B_i^3\,\mathcal{A}_i\,\beta_i^2\,\eta_i\,\xi_i -$$
$$144\,\hbar\,B_i^4\,\mathcal{A}_i\,\beta_i^2\,\eta_i\,\xi_i + 288\,\hbar^2\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i\,\xi_i - 1512\,\gamma^2\,\hbar^3\,B_i^2\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i +$$
$$720\,\gamma\,\hbar^3\,a_i\,B_i^2\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i + 648\,\gamma^2\,\hbar^3\,B_i^3\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 720\,\gamma^2\,\hbar^4\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i +$$
$$144\,\gamma\,\hbar^4\,a_i\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i + 1440\,\gamma\,\hbar^2\,B_i^2\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i - 288\,\hbar^2\,a_i\,B_i^2\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i -$$
$$864\,\gamma\,\hbar^2\,B_i^3\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i + 432\,\gamma\,\hbar^3\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i - 288\,\hbar\,B_i^2\,y_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i +$$
$$288\,\hbar\,B_i^3\,y_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i + 1344\,\gamma^2\,\hbar^3\,B_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i - 144\,\gamma\,\hbar^3\,a_i\,B_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i -$$
$$480\,\gamma^2\,\hbar^3\,B_i^2\,y_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i + 144\,\gamma^2\,\hbar^4\,B_i\,x_i\,y_i^3\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i - 576\,\gamma\,\hbar^2\,B_i\,y_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i +$$
$$288\,\gamma\,\hbar^2\,B_i^2\,y_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i - 216\,\gamma^2\,\hbar^3\,y_i^3\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i + 72\,\gamma^2\,\hbar^3\,B_i\,y_i^3\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i + 72\,\gamma^2\,\hbar^4\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 -$$
$$288\,\gamma\,\hbar^4\,a_i\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 + 144\,\hbar^4\,a_i^2\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 - 288\,\gamma\,\hbar^3\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\beta_i\,\xi_i^2 + 288\,\hbar^3\,a_i\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\beta_i\,\xi_i^2 +$$
$$144\,\hbar^2\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\beta_i^2\,\xi_i^2 - 792\,\gamma^2\,\hbar^3\,B_i^3\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 + 1152\,\gamma\,\hbar^3\,a_i\,B_i^3\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 288\,\hbar^3\,a_i^2\,B_i^3\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 +$$
$$216\,\gamma^2\,\hbar^3\,B_i^4\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 432\,\gamma\,\hbar^3\,a_i\,B_i^4\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 576\,\gamma^2\,\hbar^4\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i\,\xi_i^2 +$$
$$288\,\gamma\,\hbar^4\,a_i\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i\,\xi_i^2 + 1152\,\gamma\,\hbar^2\,B_i^3\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 - 576\,\hbar^2\,a_i\,B_i^3\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 -$$
$$576\,\gamma\,\hbar^2\,B_i^4\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 + 288\,\hbar^2\,a_i\,B_i^4\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 + 432\,\gamma\,\hbar^3\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i\,\xi_i^2 -$$
$$288\,\hbar\,B_i^3\,x_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i\,\xi_i^2 + 288\,\hbar\,B_i^4\,x_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i\,\xi_i^2 + 756\,\gamma^2\,\hbar^2\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - 720\,\gamma\,\hbar^2\,a_i\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 +$$
$$144\,\hbar^2\,a_i^2\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - 1080\,\gamma^2\,\hbar^2\,B_i^3\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 576\,\gamma\,\hbar^2\,a_i\,B_i^3\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 324\,\gamma^2\,\hbar^2\,B_i^4\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 +$$
$$2232\,\gamma^2\,\hbar^3\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 - 720\,\gamma\,\hbar^3\,a_i\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 - 792\,\gamma^2\,\hbar^3\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 +$$
$$144\,\gamma\,\hbar^3\,a_i\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 + 216\,\gamma^2\,\hbar^4\,B_i^2\,x_i^2\,y_i^2\,\mathcal{A}_i^4\,\eta_i^2\,\xi_i^2 - 720\,\gamma\,\hbar\,B_i^2\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 +$$
$$288\,\hbar\,a_i\,B_i^2\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 + 1152\,\gamma\,\hbar\,B_i^3\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 - 288\,\hbar\,a_i\,B_i^3\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 - 432\,\gamma\,\hbar\,B_i^4\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 -$$
$$1152\,\gamma\,\hbar^2\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^2 + 576\,\gamma\,\hbar^2\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^2 + 144\,B_i^2\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i^2 -$$
$$288\,B_i^3\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i^2 + 144\,B_i^4\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i^2 - 1632\,\gamma^2\,\hbar^2\,B_i\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 + 432\,\gamma\,\hbar^2\,a_i\,B_i\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 +$$
$$1680\,\gamma^2\,\hbar^2\,B_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 - 144\,\gamma\,\hbar^2\,a_i\,B_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 - 336\,\gamma^2\,\hbar^2\,B_i^3\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 -$$
$$648\,\gamma^2\,\hbar^3\,B_i\,x_i\,y_i^2\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^2 + 216\,\gamma^2\,\hbar^3\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^2 + 648\,\gamma\,\hbar\,B_i\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^2 -$$
$$864\,\gamma\,\hbar\,B_i^2\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^2 + 216\,\gamma\,\hbar\,B_i^3\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^2 + 432\,\gamma^2\,\hbar^2\,y_i^2\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^2 - 360\,\gamma^2\,\hbar^2\,B_i\,y_i^2\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^2 +$$
$$72\,\gamma^2\,\hbar^2\,B_i^2\,y_i^2\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^2 - 144\,\gamma^2\,\hbar^4\,B_i^4\,x_i^3\,\mathcal{A}_i^3\,\xi_i^3 + 144\,\gamma\,\hbar^4\,a_i\,B_i^4\,x_i^3\,\mathcal{A}_i^3\,\xi_i^3 + 144\,\gamma\,\hbar^3\,B_i^4\,x_i^3\,\mathcal{A}_i^3\,\beta_i\,\xi_i^3 +$$
$$912\,\gamma^2\,\hbar^3\,B_i^3\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 - 576\,\gamma\,\hbar^3\,a_i\,B_i^3\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 - 336\,\gamma^2\,\hbar^3\,B_i^4\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 +$$
$$144\,\gamma\,\hbar^3\,a_i\,B_i^4\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 + 144\,\gamma^2\,\hbar^4\,B_i^3\,x_i^3\,y_i\,\mathcal{A}_i^4\,\eta_i\,\xi_i^3 - 576\,\gamma\,\hbar^2\,B_i^3\,x_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i\,\xi_i^3 +$$
$$288\,\gamma\,\hbar^2\,B_i^4\,x_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i\,\xi_i^3 - 1416\,\gamma^2\,\hbar^2\,B_i^2\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 + 648\,\gamma\,\hbar^2\,a_i\,B_i^2\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 +$$
$$1392\,\gamma^2\,\hbar^2\,B_i^3\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 - 432\,\gamma\,\hbar^2\,a_i\,B_i^3\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 - 264\,\gamma^2\,\hbar^2\,B_i^4\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 +$$
$$72\,\gamma\,\hbar^2\,a_i\,B_i^4\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 - 648\,\gamma^2\,\hbar^3\,B_i^2\,x_i^2\,y_i\,\mathcal{A}_i^4\,\eta_i^2\,\xi_i^3 + 216\,\gamma^2\,\hbar^3\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^4\,\eta_i^2\,\xi_i^3 +$$
$$648\,\gamma\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^3 - 864\,\gamma\,\hbar\,B_i^3\,x_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^3 + 216\,\gamma\,\hbar\,B_i^4\,x_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^3 + 544\,\gamma^2\,\hbar\,B_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 -$$
$$216\,\gamma\,\hbar\,a_i\,B_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 - 1104\,\gamma^2\,\hbar\,B_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 + 288\,\gamma\,\hbar\,a_i\,B_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 + 672\,\gamma^2\,\hbar\,B_i^3\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 -$$
$$72\,\gamma\,\hbar\,a_i\,B_i^3\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 - 112\,\gamma^2\,\hbar\,B_i^4\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 + 864\,\gamma^2\,\hbar^2\,B_i\,x_i\,y_i\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^3 - 720\,\gamma^2\,\hbar^2\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^3 +$$
$$144\,\gamma^2\,\hbar^2\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^3 - 216\,\gamma\,B_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 + 504\,\gamma\,B_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 - 360\,\gamma\,B_i^3\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 +$$
$$72\,\gamma\,B_i^4\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 - 324\,\gamma^2\,\hbar\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 + 540\,\gamma^2\,\hbar\,B_i\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 - 252\,\gamma^2\,\hbar\,B_i^2\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 +$$
$$36\,\gamma^2\,\hbar\,B_i^3\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 + 36\,\gamma^2\,\hbar^4\,B_i^4\,x_i^4\,\mathcal{A}_i^4\,\xi_i^4 - 216\,\gamma^2\,\hbar^3\,B_i^3\,x_i^3\,\mathcal{A}_i^4\,\eta_i\,\xi_i^4 + 72\,\gamma^2\,\hbar^3\,B_i^4\,x_i^3\,\mathcal{A}_i^4\,\eta_i\,\xi_i^4 +$$

$432\gamma^2\hbar^2 B_i^2 x_i^2 \mathcal{A}_i^4 \eta_i^2 \xi_i^4 - 360\gamma^2\hbar^2 B_i^3 x_i^2 \mathcal{A}_i^4 \eta_i^2 \xi_i^4 + 72\gamma^2\hbar^2 B_i^4 x_i^2 \mathcal{A}_i^4 \eta_i^2 \xi_i^4 - 324\gamma^2\hbar B_i x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 + 540\gamma^2\hbar B_i^2 x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 - 252\gamma^2\hbar B_i^3 x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 + 36\gamma^2\hbar B_i^4 x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 + 81\gamma^2 \mathcal{A}_i^4 \eta_i^4 \xi_i^4 - 216\gamma^2 B_i \mathcal{A}_i^4 \eta_i^4 \xi_i^4 + 198\gamma^2 B_i^2 \mathcal{A}_i^4 \eta_i^4 \xi_i^4 - 72\gamma^2 B_i^3 \mathcal{A}_i^4 \eta_i^4 \xi_i^4 + 9\gamma^2 B_i^4 \mathcal{A}_i^4 \eta_i^4 \xi_i^4)\,\epsilon^2 + O[\epsilon]^3\Big]$

In[◦]:=
```
Block[{i, j, k}, dΔ_{i_→j_,k_} = (bΔ_{i→3,1} aΔ_{i→2,4}) ~B_{1,2,3,4}~ (dm_{3,4→k} dm_{1,2→j})]
```

Out[◦]=
$$\mathbb{E}\Big[a_j\alpha_i + a_k\alpha_i + b_j\beta_i + b_k\beta_i,\ y_j\eta_i + B_j y_k\eta_i + x_j\xi_i + x_k\xi_i,\ 1 + \frac{1}{2}\left(\gamma\hbar B_j y_j y_k \eta_i^2 - 2\hbar a_j x_k \xi_i + \gamma\hbar x_j x_k \xi_i^2\right)\epsilon + \frac{1}{24}\left(6\gamma^2\hbar^2 B_j y_j y_k \eta_i^2 + 4\gamma^2\hbar^2 B_j y_j^2 y_k \eta_i^3 + 4\gamma^2\hbar^2 B_j^2 y_j y_k^2 \eta_i^3 + 3\gamma^2\hbar^2 B_j^2 y_j^2 y_k^2 \eta_i^4 + 12\hbar^2 a_j^2 x_k \xi_i - 12\gamma\hbar^2 a_j B_j x_k y_j y_k \eta_i^2 \xi_i + 6\gamma^2\hbar^2 x_j x_k \xi_i^2 - 12\gamma\hbar^2 a_j x_j x_k \xi_i^2 + 12\hbar^2 a_j^2 x_k^2 \xi_i^2 + 6\gamma^2\hbar^2 B_j x_j x_k y_j y_k \eta_i^2 \xi_i^2 + 4\gamma^2\hbar^2 x_j^2 x_k \xi_i^3 + 4\gamma^2\hbar^2 x_j x_k^2 \xi_i^3 - 12\gamma\hbar^2 a_j x_j x_k^2 \xi_i^3 + 3\gamma^2\hbar^2 x_j^2 x_k^2 \xi_i^4\right)\epsilon^2 + O[\epsilon]^3\Big]$$

First check the double formulas on the generators agree with SL2Portfolio.pdf:

In[◦]:=
```
Timing@{{
    "[a,y]" → ((𝔼[0, 0, y2 a1] ~B1,2~ dm1,2→1)[[3]] - (𝔼[0, 0, y1 a2] ~B1,2~ dm1,2→1)[[3]]),
    "[b,x]"  → ((𝔼[0, 0, x2 b1] ~B1,2~ dm1,2→1)[[3]] - (𝔼[0, 0, x1 b2] ~B1,2~ dm1,2→1)[[3]]),
    "xy-qyx" → ((𝔼[0, 0, x1 y2] ~B1,2~ dm1,2→1)[[3]] - (1 + ϵ) (𝔼[0, 0, y1 x2] ~B1,2~ dm1,2→1)[[3]])
   } /. {z_1 → z} // Expand // Factor,
  {
   "Δ(a)" → ((𝔼[0, 0, a1] ~B1~ dΔ1→1,2)[[3]]),
   "Δ(x)" → ((𝔼[0, 0, x1] ~B1~ dΔ1→1,2)[[3]]),
   "Δ(b)" → ((𝔼[0, 0, b1] ~B1~ dΔ1→1,2)[[3]]),
   "Δ(y)" → ((𝔼[0, 0, y1] ~B1~ dΔ1→1,2)[[3]])
  } // Simplify,
  {
   "S(a)" → ((𝔼[0, 0, a1] ~B1~ dS1)[[3]]),
   "S(x)" → ((𝔼[0, 0, x1] ~B1~ dS1)[[3]]),
   "S(b)" → ((𝔼[0, 0, b1] ~B1~ dS1)[[3]]),
   "S(y)" → ((𝔼[0, 0, y1] ~B1~ dS1)[[3]])
  } /. {z_1 → z} // Simplify
 }
```

Out[◦]=
$$\Big\{3.5,\ \Big\{\Big\{[a,y] \to -y\gamma + O[\epsilon]^3,\ [b,x] \to x\epsilon + O[\epsilon]^3,\ xy{-}qyx \to \left(-x y + \frac{1 - B + x y\hbar}{\hbar}\right) + (a B - x y + x y\gamma\hbar)\,\epsilon + \frac{1}{2}\left(-a^2 B\hbar + x y\gamma^2\hbar^2\right)\epsilon^2 + O[\epsilon]^3\Big\},$$
$$\Big\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar a_1 x_2\,\epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2\,\epsilon^2 + O[\epsilon]^3,\ \Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1 y_2) + O[\epsilon]^3\Big\},$$
$$\Big\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x\hbar\epsilon - \frac{1}{2}\left(a^2 x\hbar^2\right)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\gamma\hbar\epsilon}{B} - \frac{\left(y\gamma^2\hbar^2\right)\epsilon^2}{2B} + O[\epsilon]^3\Big\}\Big\}\Big\}$$

Hopf algebra axioms on double

(co)-associativity

In[∘]:= $\text{Timing}[\text{HL} /@ \{(d\Delta_{1\to 1,2} \sim B_2 \sim d\Delta_{2\to 2,3}) \equiv (d\Delta_{1\to 1,3} \sim B_1 \sim d\Delta_{1\to 1,2}),\ (dm_{1,2\to 1} \sim B_1 \sim dm_{1,3\to 1}) \equiv (dm_{2,3\to 2} \sim B_2 \sim dm_{1,2\to 1})\}]$

Out[∘]= {7.67188, {True, True}}

Δ is an algebra morphism

In[∘]:= $\text{Timing@HL}[dm_{1,2\to 1} \sim B_1 \sim d\Delta_{1\to 1,2} \equiv (d\Delta_{1\to 1,3}\ d\Delta_{2\to 2,4}) \sim B_{1,2,3,4} \sim (dm_{3,4\to 2}\ dm_{1,2\to 1})]$

Out[∘]= {14.9688, True}

S is convolution inverse of id

In[∘]:= $\text{Timing}[\text{HL}[\# \equiv \mathbb{E}[0, 0, 1]]\ \&\ /@ \{(d\Delta_{1\to 1,2} \sim B_1 \sim dS_1) \sim B_{1,2} \sim dm_{1,2\to 1},\ (d\Delta_{1\to 1,2} \sim B_2 \sim dS_2) \sim B_{1,2} \sim dm_{1,2\to 1}\}]$

Out[∘]= {12.4219, {True, True}}

S is a (co)-algebra anti-morphism

In[∘]:= $\text{Timing}[\text{HL} /@ \text{Expand} /@ \{dm_{1,2\to 1} \sim B_1 \sim dS_1 \equiv (dS_1\ dS_2) \sim B_{1,2} \sim dm_{2,1\to 1},\ dS_1 \sim B_1 \sim d\Delta_{1\to 1,2} \equiv d\Delta_{1\to 2,1} \sim B_{1,2} \sim (dS_1\ dS_2)\}]$

Out[∘]= {27.9844, {True, True}}

Quasi-triangular axiom 1:

In[∘]:= $\text{Timing@HL}[R_{1,2} \sim B_1 \sim d\Delta_{1\to 1,3} \equiv (R_{1,4}\ R_{3,2}) \sim B_{2,4} \sim dm_{2,4\to 2}]$

Out[∘]= {0.765625, True}

Quasi-triangular axiom 2:

In[∘]:= $\text{Timing@HL}[((d\Delta_{1\to 1,2}\ R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to 1}\ dm_{2,4\to 2})) \equiv ((d\Delta_{1\to 2,1}\ R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1\to 1}\ dm_{4,2\to 2}))]$

Out[∘]= {12.8594, True}

Reidemeister 3:

In[∘]:= $\text{Timing@HL}[((R_{1,2}\ R_{4,3}\ R_{5,6}) \sim B_{1,4} \sim dm_{1,4\to 1} \sim B_{2,5} \sim dm_{2,5\to 2} \sim B_{3,6} \sim dm_{3,6\to 3}) \equiv ((R_{1,6}\ R_{2,3}\ R_{4,5}) \sim B_{1,4} \sim dm_{1,4\to 1} \sim B_{2,5} \sim dm_{2,5\to 2} \sim B_{3,6} \sim dm_{3,6\to 3})]$

Out[∘]= {6.4375, True}

In[∘]:= $\text{Block}[\{i, j\},\ \overline{R}_{i\_,j\_} = \text{Expand} /@ R_{i,j} \sim B_j \sim dS_j]$

Out[∘]= $\mathbb{E}\Big[-\hbar a_j b_i,\ -\frac{\hbar x_j y_i}{B_i},\ 1 + \frac{(-4\hbar^2 a_j B_i x_j y_i - 3\gamma\hbar^3 x_j^2 y_i^2)\,\epsilon}{4 B_i^2} + \frac{1}{288 B_i^4}(-144\hbar^3 a_j^2 B_i^3 x_j y_i + 144\gamma^2\hbar^4 B_i^2 x_j^2 y_i^2 - 432\gamma\hbar^4 a_j B_i^2 x_j^2 y_i^2 + 144\hbar^4 a_j^2 B_i^2 x_j^2 y_i^2 - 320\gamma^2\hbar^5 B_i x_j^3 y_i^3 + 216\gamma\hbar^5 a_j B_i x_j^3 y_i^3 + 81\gamma^2\hbar^6 x_j^4 y_i^4)\,\epsilon^2 + O[\epsilon]^3\Big]$

Reidemeister 2

In[∘]:= $\text{Timing}[\text{HL} /@ \{(\overline{R}_{1,2}\ R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to 1}\ dm_{2,4\to 2}),\ (R_{1,2}\ \overline{R}_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to 1}\ dm_{2,4\to 2})\}]$

Out[∘]= $\{8.42188, \{\mathbb{E}[0, 0, 1 + O[\epsilon]^3], \mathbb{E}[0, 0, 1 + O[\epsilon]^3]\}\}$

Deriving the Drinfeld element u and its inverse $\overline{u}$

In[◦]:=
```
Block[{i}, {
  u_i_ = R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i},
  ū_i_ = R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→i}
 }]
```

Out[◦]= $\left\{\mathbb{E}\left[-\hbar a_i b_i, -\frac{\hbar x_i y_i}{B_i}, B_i + \frac{1}{4 B_i}\left(-4\hbar a_i B_i^2 - 4\gamma\hbar^2 B_i x_i y_i - 4\hbar^2 a_i B_i x_i y_i - 3\gamma\hbar^3 x_i^2 y_i^2\right)\epsilon + \frac{1}{288 B_i^3}\left(144\hbar^2 a_i^2 B_i^4 - 144\gamma^2\hbar^3 B_i^3 x_i y_i + 144\hbar^3 a_i^2 B_i^3 x_i y_i - 144\gamma^2\hbar^4 B_i^2 x_i^2 y_i^2 + 72\gamma\hbar^4 a_i B_i^2 x_i^2 y_i^2 + 144\hbar^4 a_i^2 B_i^2 x_i^2 y_i^2 - 104\gamma^2\hbar^5 B_i x_i^3 y_i^3 + 216\gamma\hbar^5 a_i B_i x_i^3 y_i^3 + 81\gamma^2\hbar^6 x_i^4 y_i^4\right)\epsilon^2 + O[\epsilon]^3\right], \mathbb{E}\left[\hbar a_i b_i, \hbar x_i y_i, \frac{1}{B_i} + \frac{\left(4\hbar a_i - 4\gamma\hbar^2 x_i y_i - \gamma\hbar^3 x_i^2 y_i^2\right)\epsilon}{4 B_i} + \frac{1}{288 B_i}\left(144\hbar^2 a_i^2 + 144\gamma^2\hbar^3 x_i y_i - 288\gamma\hbar^3 a_i x_i y_i + 288\gamma^2\hbar^4 x_i^2 y_i^2 - 72\gamma\hbar^4 a_i x_i^2 y_i^2 + 104\gamma^2\hbar^5 x_i^3 y_i^3 + 9\gamma^2\hbar^6 x_i^4 y_i^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$

u and $\overline{u}$ are inverses

In[◦]:=
```
Timing@HL[(u_1 ū_2) ~ B_{1,2} ~ dm_{1,2→1} ≡ 𝔼[0, 0, 1]]
```

Out[◦]= {1.53125, True}

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner $C=uv^{-1}$.
It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

In[◦]:=
```
Block[{$k = 3}, ((u_1 ~ B_1 ~ dS_1) ū_2) ~ B_{1,2} ~ dm_{1,2→1}]
```

Out[◦]= $\mathbb{E}\left[0, 0, \frac{1}{B_1} + \frac{\hbar a_1 \epsilon}{B_1} + \frac{\hbar^2 a_1^2 \epsilon^2}{2 B_1} + O[\epsilon]^3\right]$

(* Needs fixing! *) So in our case S(u) = u z so S(u)u = $u^2 z$ and v = $uz^{\frac{1}{2}}$ and finally C = uv^{-1} = $z^{-\frac{1}{2}} = e^{t_1/2}(1 - \epsilon a_1)$.

In[◦]:=
```
Block[{i},
 {CC_i_ = 𝔼[0, 0, B_i^{1/2} e^{-ϵ a_i/2} + O[ϵ]^2],
  C̄C̄_i_ = 𝔼[0, 0, B_i^{-1/2} e^{ϵ a_i/2} + O[ϵ]^2]
 }]
```

Out[◦]= $\left\{\mathbb{E}\left[0, 0, \sqrt{B_i} - \frac{1}{2}\left(a_i\sqrt{B_i}\right)\epsilon + O[\epsilon]^2\right], \mathbb{E}\left[0, 0, \frac{1}{\sqrt{B_i}} + \frac{a_i \epsilon}{2\sqrt{B_i}} + O[\epsilon]^2\right]\right\}$

In[∘]:=
```
Block[{i, j}, {
  Kink_i_ = (R_{1,3} CC̄_2) ~B_{1,2}~ dm_{1,2→1} ~B_{1,3}~ dm_{1,3→i},
  K̄ink_j_ = (R̄_{1,3} CC_2) ~B_{1,2}~ dm_{1,2→1} ~B_{1,3}~ dm_{1,3→j}
 }]
```

Out[∘]=
$$\left\{\mathbb{E}\left[\hbar\, a_i b_i,\ \hbar\, x_i y_i,\ \frac{1}{\sqrt{B_i}}+\frac{\left(2 a_i-\gamma \hbar^3 x_i^2 y_i^2\right)\epsilon}{4\sqrt{B_i}}+O[\epsilon]^2\right],\right.$$
$$\left.\mathbb{E}\left[-\hbar\, a_j b_j,\ -\frac{\hbar\, x_j y_j}{B_j},\ \sqrt{B_j}+\frac{\left(-2 a_j B_j^2-4\hbar^2 a_j B_j x_j y_j-3\gamma\hbar^3 x_j^2 y_j^2\right)\epsilon}{4 B_j^{3/2}}+O[\epsilon]^2\right]\right\}$$

In[∘]:=
```
k2 = (R_{3,1} CC_2) ~B_{1,2}~ dm_{1,2→1} ~B_{1,3}~ dm_{1,3→i} /. e → E;
k4 = (R̄_{3,1} CC̄_2) ~B_{1,2}~ dm_{1,2→1} ~B_{1,3}~ dm_{1,3→j} /. e → E;
Simplify@{Kink_i ≡ k2, K̄ink_j ≡ k4, (Kink_i K̄ink_j) ~B_{i,j}~ dm_{i,j→1}}
```

Out[∘]= $\{\text{True},\ \text{True},\ \mathbb{E}\left[0,\ 0,\ 1+O[\epsilon]^2\right]\}$

Reidemeister 2:

In[∘]:=
```
(R_{1,2} R̄_{3,4}) ~B_{1,3}~ dm_{1,3→1} ~B_{2,4}~ dm_{2,4→2}
```

Out[∘]= $\mathbb{E}\left[0,\ 0,\ 1+O[\epsilon]^2\right]$

Cyclic Reidemeister 2:

In[∘]:=
```
(R_{1,4} R̄_{5,2} CC̄_3) ~B_{2,4}~ dm_{2,4→2} ~B_{1,3}~ dm_{1,3→1} ~B_{1,5}~ dm_{1,5→1} ≡ CC̄_1
```

Out[∘]= True

Trefoil

In[∘]:=
```
Z = R_{1,5} R_{6,2} R_{3,7} CC̄_4 K̄ink_8 K̄ink_9 K̄ink_10;
Do[Z = Z~B_{1,r}~dm_{1,r→1}, {r, 2, 10}];
Simplify /@ Z
```

Out[∘]=
$$\mathbb{E}\left[0,\ 0,\ \frac{B_1}{1-B_1+B_1^2}+\right.$$
$$\left.\left(B_1\left(-B_1+2B_1^2+2B_1^4+a_1\left(-1+B_1-B_1^3+B_1^4\right)-2x_1y_1-B_1^3\left(3+2x_1y_1\right)\right)\epsilon\right)\Big/\left(1-B_1+B_1^2\right)^3+O[\epsilon]^2\right]$$

```
Timing[
 Z = R_{1,5} R_{6,2} R_{3,7} CC̄_4 K̄ink_8 K̄ink_9 K̄ink_10;
 Do[Z = Z~B_{1,r}~dm_{1,r→1}, {r, 2, 10}];
 Simplify /@ Z ]
```

In[∘]:=
```
b2t_i_ := 𝔼[α_i a_i - β_i t_i, ξ_i x_i + η_i y_i, 1 + ϵ β_i a_i + O[ϵ]^2]
t2b_i_ := 𝔼[α_i a_i - τ_i b_i, ξ_i x_i + η_i y_i, 1 + ϵ τ_i a_i + O[ϵ]^2]
```

In[∘]:= $R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC_4}\, \overline{Kink_8}\, \overline{Kink_9}\, \overline{Kink_{10}}$

Out[∘]= $$\mathbb{E}\Big[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},\; x_5 y_1 + x_7 y_3 + x_2 y_6 - \frac{x_8 y_8}{B_8} - \frac{x_9 y_9}{B_9} - \frac{x_{10} y_{10}}{B_{10}},\; \frac{\sqrt{B_8}\,\sqrt{B_9}\,\sqrt{B_{10}}}{\sqrt{B_4}} + \left(\sqrt{B_{10}}\left(\sqrt{B_9}\left(\sqrt{B_8}\left(\frac{a_4}{2\sqrt{B_4}} - \frac{x_5^2 y_1^2}{4\sqrt{B_4}} - \frac{x_7^2 y_3^2}{4\sqrt{B_4}} - \frac{x_2^2 y_6^2}{4\sqrt{B_4}}\right) + \frac{-2 a_8 B_8^2 - 4 a_8 B_8 x_8 y_8 - 3 x_8^2 y_8^2}{4\sqrt{B_4}\, B_8^{3/2}}\right) + \frac{\sqrt{B_8}\left(-2 a_9 B_9^2 - 4 a_9 B_9 x_9 y_9 - 3 x_9^2 y_9^2\right)}{4\sqrt{B_4}\, B_9^{3/2}}\right) + \frac{1}{4\sqrt{B_4}\, B_{10}^{3/2}}\sqrt{B_8}\,\sqrt{B_9}\left(-2 a_{10} B_{10}^2 - 4 a_{10} B_{10} x_{10} y_{10} - 3 x_{10}^2 y_{10}^2\right)\right)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $\left(R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC_4}\, \overline{Kink_8}\, \overline{Kink_9}\, \overline{Kink_{10}}\right) \sim B_{Range[10]} \sim \text{Product}[b2t_i, \{i, 10\}]$

Out[∘]= $$\mathbb{E}\Big[-a_5 t_1 - a_7 t_3 - a_2 t_6 + a_8 t_8 + a_9 t_9 + a_{10} t_{10},\; \frac{1}{T_8 T_9 T_{10}}\left(T_8 T_9 T_{10} x_5 y_1 + T_8 T_9 T_{10} x_7 y_3 + T_8 T_9 T_{10} x_2 y_6 - T_9 T_{10} x_8 y_8 - T_8 T_{10} x_9 y_9 - T_8 T_9 x_{10} y_{10}\right),\; \frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_4}} + \frac{1}{4\sqrt{T_4}\, T_8^{3/2} T_9^{3/2} T_{10}^{3/2}}\left(4 a_4 T_8^2 T_9^2 T_{10}^2 + 4 a_1 a_5 T_8^2 T_9^2 T_{10}^2 + 4 a_2 a_6 T_8^2 T_9^2 T_{10}^2 + 4 a_3 a_7 T_8^2 T_9^2 T_{10}^2 - 4 a_8 T_8^2 T_9^2 T_{10}^2 - 4 a_8^2 T_8^2 T_9^2 T_{10}^2 - 4 a_9 T_8^2 T_9^2 T_{10}^2 - 4 a_9^2 T_8^2 T_9^2 T_{10}^2 - 4 a_{10} T_8^2 T_9^2 T_{10}^2 - 4 a_{10}^2 T_8^2 T_9^2 T_{10}^2 - T_8^2 T_9^2 T_{10}^2 x_5^2 y_1^2 - T_8^2 T_9^2 T_{10}^2 x_7^2 y_3^2 - T_8^2 T_9^2 T_{10}^2 x_2^2 y_6^2 - 8 a_8 T_8 T_9^2 T_{10}^2 x_8 y_8 - 3 T_9^2 T_{10}^2 x_8^2 y_8^2 - 8 a_9 T_8^2 T_9 T_{10}^2 x_9 y_9 - 3 T_8^2 T_{10}^2 x_9^2 y_9^2 - 8 a_{10} T_8^2 T_9^2 T_{10} x_{10} y_{10} - 3 T_8^2 T_9^2 x_{10}^2 y_{10}^2\right)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $Z = \left(\left(\left(R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC_4}\, \overline{Kink_8}\, \overline{Kink_9}\, \overline{Kink_{10}}\right) \sim B_{Range[10]} \sim \text{Product}[b2t_i, \{i, 10\}]\right) \;/.\; T\_ \to T_1\right) \sim B_{Range[10]} \sim \text{Product}[t2b_i, \{i, 10\}]$

Out[∘]= $$\mathbb{E}\Big[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},\; \frac{1}{B_1}\left(B_1 x_5 y_1 + B_1 x_7 y_3 + B_1 x_2 y_6 - x_8 y_8 - x_9 y_9 - x_{10} y_{10}\right),\; B_1 + \frac{1}{4 B_1}\left(4 a_1 B_1^2 + 4 a_4 B_1^2 - 4 a_8 B_1^2 - 4 a_9 B_1^2 - 4 a_{10} B_1^2 - B_1^2 x_5^2 y_1^2 - B_1^2 x_7^2 y_3^2 - B_1^2 x_2^2 y_6^2 + 4 a_1 B_1 x_8 y_8 - 8 a_8 B_1 x_8 y_8 - 3 x_8^2 y_8^2 + 4 a_1 B_1 x_9 y_9 - 8 a_9 B_1 x_9 y_9 - 3 x_9^2 y_9^2 + 4 a_1 B_1 x_{10} y_{10} - 8 a_{10} B_1 x_{10} y_{10} - 3 x_{10}^2 y_{10}^2\right)\epsilon + O[\epsilon]^2\Big]$$

```
Timing[
 Do[Z = Z~B_{1,r}~dm_{1,r→1}, {r, 2, 10}];
 Simplify@Z[[3]] ]
```

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

Out[ ]= $\left\{5.39063, \frac{B_1}{1 - B_1 + B_1^2} + \left(B_1 \left(-B_1 + 2 B_1^2 + 2 B_1^4 + a_1 \left(-1 + B_1 - B_1^3 + B_1^4\right) - 2 x_1 y_1 - B_1^3 \left(3 + 2 x_1 y_1\right)\right) \epsilon\right) / \left(1 - B_1 + B_1^2\right)^3 + O[\epsilon]^2\right\}$

In[ ]:=
```
Timing[
 Z = R1,5 R6,2 R3,7 CC̅4 Kink̅8 Kink̅9 Kink̅10 /. B_ → B1;
 Do[Print["doing ", r]; Z = Z~B1,r~dm1,r→1 /. B_ → B1, {r, 2, 10}];
 Simplify@Z⟦3⟧ ]
```

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

Out[ ]= $\left\{5.3125, \frac{B_1}{1 - B_1 + B_1^2} + \frac{1}{2 B_1 \left(1 - B_1 + B_1^2\right)^3} \left(2 a_1 B_1^2 \left(-1 + B_1 - B_1^3 + B_1^4\right) - 6 x_1^2 y_1^2 + 4 B_1^7 x_1^2 y_1^2 - 2 B_1^8 x_1^2 y_1^2 + B_1^2 x_1 y_1 (5 - 6 x_1 y_1) + 3 B_1 x_1 y_1 \left(-1 + 2 x_1 y_1\right) + B_1^6 \left(3 + 3 x_1 y_1 - 6 x_1^2 y_1^2\right) - B_1^5 \left(4 + 13 x_1 y_1 + 2 x_1^2 y_1^2\right) + B_1^4 \left(2 + 15 x_1 y_1 + 4 x_1^2 y_1^2\right) - B_1^3 \left(1 + 15 x_1 y_1 + 6 x_1^2 y_1^2\right)\right) \epsilon + O[\epsilon]^2\right\}$