Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

External Utilities

```
In[∘]:= HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Defaults

```
In[∘]:= $k = 2;
```

Internal Utilities

Canonical Form:

```
In[∘]:= CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CF[x]];
```

The Kronecker δ:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality and multiplication of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

Zip and Bind

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})* := (u*)_i;
```

Finite Zips: (* Perhaps switch Expand to Collect[__, ζ]? *)

```
In[∘]:= expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ℰ_] := Expand[ℰ];
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] := (expand[P // Zip_{ζs}] /. f_. ζ^d_. :→ ∂_{ζ*,d} f) /. ζ* → 0
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = \mathrm{P}e^{L+Q}$. Such zips regard the L variables as scalars.

In[∘]:=
```
QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^{-Q2} Zip_{ζs}[e^{Q1} (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs,CF};
```

Upper to lower and lower to Upper:

In[∘]:=
```
U2l = {B_{i_}^{p_.} → e^{-p ħ γ b_i}, B^{p_.} → e^{-p ħ γ b}, T_{i_}^{p_.} → e^{p ħ t_i}, T^{p_.} → e^{p ħ t}, 𝒜_{i_}^{p_.} → e^{p γ α_i}, 𝒜^{p_.} → e^{p γ α}};
l2U = {e^{c_. b_{i_}+d_.} :> B_i^{-c/(ħ γ)} e^d, e^{c_. b+d_.} :> B^{-c/(ħ γ)} e^d,
   e^{c_. t_{i_}+d_.} :> T_i^{c/ħ} e^d, e^{c_. t+d_.} :> T^{c/ħ} e^d,
   e^{c_. α_{i_}+d_.} :> 𝒜_i^{c/γ} e^d, e^{c_. α+d_.} :> 𝒜^{c/γ} e^d,
   e^{ℰ_} :> e^{Expand@ℰ}};
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = \mathrm{P}e^{L+Q}$. Such zips regard all of $\mathrm{P}e^{Q}$ as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

In[∘]:=
```
LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L2, Q2, Det[lt] e^{-L2-Q2} Zip_{ζs}[e^{L1+Q1} (P /. U2l /. zrule)]] //. l2U ];
LZip_{ζs_List} := LZip_{ζs,CF};
```

In[∘]:=
```
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
   ] // LZip_{Flatten@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Flatten@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{l_List}[L_, R_] := Bind_l[L, R]; B_{is___}[L_, R_] := Bind_{is}[L, R];
```

t and *b* conversions

$t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

In[∘]:=
```
t2b_{i_→j_} := 𝔼[α_i a_j - τ_i γ b_j, ξ_i x_j + η_i y_j, e^{ϵ τ_i a_j} + O[ϵ]^{$k+1}];
b2t_{i_→j_} := 𝔼[α_i a_j - β_i t_j / γ, ξ_i x_j + η_i y_j, e^{e^{ϵ β_i a_j/γ}} + O[ϵ]^{$k+1}];
t2b_{i_} := t2b_{i→i}; b2t_{i_} := b2t_{i→i};
```

m

In[∘]:=
```
am_{i_,j_->k_} := 𝔼[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1];
bm_{i_,j_→k_} := 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i} - 1) η_j y_k} + O[ϵ]^{$k+1}];
```

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am_{1,2→1} ~ B_1 ~ am_{1,3→1}) ≡ (am_{2,3→2} ~ B_2 ~ am_{1,2→1}), (bm_{1,2→1} ~ B_1 ~ bm_{1,3→1}) ≡ (bm_{2,3→2} ~ B_2 ~ bm_{1,2→1})}
 ]
```

Out[∘]= {0.171875, {True, True}}

R

In[∘]:=
```
e_{q_,k_}[x_] := Module[{j}, e^(Sum_{j=1}^{k+1} (1 - q)^j x^j / (j (1 - q^j)))]; e_{q_}[x_] := e_{q,$k}[x]
```

In[∘]:=
```
R_{i_,j_} := 𝔼[ℏ a_j b_i, ℏ x_j y_i, Series[e^{-ℏ y_i x_j} e_{e^{γ ϵ ℏ}}[ℏ y_i x_j], {ϵ, 0, $k}]];
R_{1,2}
```

Out[∘]= $\mathbb{E}\left[\hbar\, a_2\, b_1,\ \hbar\, x_2\, y_1,\ 1 - \frac{1}{4}\left(\gamma\,\hbar^3\, x_2^2\, y_1^2\right)\epsilon + \left(\frac{1}{9}\gamma^2\,\hbar^5\, x_2^3\, y_1^3 + \frac{1}{32}\gamma^2\,\hbar^6\, x_2^4\, y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]$

P

In[∘]:=
```
P_{i_,j_,0} := 𝔼[β_i α_j / ℏ, η_i ξ_j / ℏ, 1];
P_{i_,j_,k_} := Module[{m, n},
   MapAt[(# - ϵ^k Coefficient[(R_{n,m} ~ B_{n,m} ~ (P_{n,j,0} P_{i,m,k-1}))[[3]], ϵ, k] + O[ϵ]^{$k+1}) &, P_{i,j,k-1}, 3]
  ];
P_{i_,j_} := P_{i,j,$k};
```

In[∘]:=
```
Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[R_{i,j} ~ B_i ~ P_{i,k} ≡ 𝔼[a_j α_k, x_j ξ_k, 1]]}]
```

Out[∘]= $\left\{0.671875,\ \left\{\mathbb{E}\left[\hbar\, a_j\, b_i,\ \hbar\, x_j\, y_i,\ 1 - \frac{1}{4}\left(\gamma\,\hbar^3\, x_j^2\, y_i^2\right)\epsilon + \left(\frac{1}{9}\gamma^2\,\hbar^5\, x_j^3\, y_i^3 + \frac{1}{32}\gamma^2\,\hbar^6\, x_j^4\, y_i^4\right)\epsilon^2 + \frac{1}{1152}\left(24\,\gamma^3\,\hbar^5\, x_j^2\, y_i^2 - 72\,\gamma^3\,\hbar^7\, x_j^4\, y_i^4 - 32\,\gamma^3\,\hbar^8\, x_j^5\, y_i^5 - 3\,\gamma^3\,\hbar^9\, x_j^6\, y_i^6\right)\epsilon^3 + O[\epsilon]^4\right],\right.$

$\mathbb{E}\left[\frac{\alpha_k\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_k}{\hbar},\ 1 + \frac{\gamma\,\eta_i^2\,\xi_k^2\,\epsilon}{4\,\hbar} + \frac{\left(36\,\gamma^2\,\hbar^2\,\eta_i^2\,\xi_k^2 + 40\,\gamma^2\,\hbar\,\eta_i^3\,\xi_k^3 + 9\,\gamma^2\,\eta_i^4\,\xi_k^4\right)\epsilon^2}{288\,\hbar^2} + \frac{1}{1152\,\hbar^3}\left(48\,\gamma^3\,\hbar^4\,\eta_i^2\,\xi_k^2 + 192\,\gamma^3\,\hbar^3\,\eta_i^3\,\xi_k^3 + 156\,\gamma^3\,\hbar^2\,\eta_i^4\,\xi_k^4 + 40\,\gamma^3\,\hbar\,\eta_i^5\,\xi_k^5 + 3\,\gamma^3\,\eta_i^6\,\xi_k^6\right)\epsilon^3 + O[\epsilon]^4\right],\ \text{True}\left.\right\}\left.\right\}$

S

In[∘]:=
```
aSi_ := 𝔼[-αi aj, -ξi xi, Series[
    e^(ξi xi) Sum[Expand[(-ħ γ ϵ)^k/(2^k k!) Nest[Expand[xi^2 ∂{xi,2} #] &, e^(-ξi e^(ħ ϵ ai) xi), k]], {k, 0, $k}],
    {ϵ, 0, $k}]] ~ Bi,j ~ ami,j→i;
aSi
```

Out[∘]=
$$\mathbb{E}\Big[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\frac{1}{2}\left(-2\hbar\, a_i x_i \mathcal{A}_i \xi_i-\gamma\hbar\, x_i^2 \mathcal{A}_i^2 \xi_i^2\right)\epsilon+\frac{1}{8}\left(-4\hbar^2 a_i^2 x_i \mathcal{A}_i \xi_i+2\gamma^2\hbar^2 x_i^2\mathcal{A}_i^2\xi_i^2-8\gamma\hbar^2 a_i x_i^2\mathcal{A}_i^2\xi_i^2+4\hbar^2 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2-4\gamma^2\hbar^2 x_i^3\mathcal{A}_i^3\xi_i^3+4\gamma\hbar^2 a_i x_i^3\mathcal{A}_i^3\xi_i^3+\gamma^2\hbar^2 x_i^4\mathcal{A}_i^4\xi_i^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:=
```
aS̄i_,0 := 𝔼[-ai αi, -xi 𝒜i ξi, 1];
aS̄i_,k_ :=
  MapAt[(# - ϵ^k Coefficient[(aS̄i,0 ~ Bi ~ aSi ~ Bi ~ aS̄i,k-1)⟦3⟧, ϵ, k] + O[ϵ]^($k+1)) &, aS̄i,k-1, 3];
aS̄i_ := aS̄i,$k;
aS̄1
```

Out[∘]=
$$\mathbb{E}\Big[-a_1\,\alpha_1,\ -x_1\,\mathcal{A}_1\,\xi_1,\ 1+\left(\gamma\hbar\, x_1\mathcal{A}_1\xi_1-\hbar\, a_1 x_1\mathcal{A}_1\xi_1-\frac{1}{2}\gamma\hbar\, x_1^2\mathcal{A}_1^2\xi_1^2\right)\epsilon+\frac{1}{8}\left(-4\gamma^2\hbar^2 x_1\mathcal{A}_1\xi_1+8\gamma\hbar^2 a_1 x_1\mathcal{A}_1\xi_1-4\hbar^2 a_1^2 x_1\mathcal{A}_1\xi_1+14\gamma^2\hbar^2 x_1^2\mathcal{A}_1^2\xi_1^2-16\gamma\hbar^2 a_1 x_1^2\mathcal{A}_1^2\xi_1^2+4\hbar^2 a_1^2 x_1^2\mathcal{A}_1^2\xi_1^2-8\gamma^2\hbar^2 x_1^3\mathcal{A}_1^3\xi_1^3+4\gamma\hbar^2 a_1 x_1^3\mathcal{A}_1^3\xi_1^3+\gamma^2\hbar^2 x_1^4\mathcal{A}_1^4\xi_1^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:=
```
Timing@HL[aS̄1 ~ B1 ~ aS1 ≡ 𝔼[a1 α1, x1 ξ1, 1]]
```

Out[∘]= {0.5625, True}

In[∘]:=
```
bSi_ := Ri,n ~ Bn ~ aSn ~ Bn ~ Pi,n;
bS̄i_ := Ri,n ~ Bn ~ aS̄n ~ Bn ~ Pi,n;
```

In[∘]:=
```
Timing@HL[bS̄1 ~ B1 ~ bS1 ≡ 𝔼[b1 β1, y1 η1, 1]]
```

Out[∘]= {0.984375, True}

Δ

In[∘]:=
```
Block[{i, j, k, l, m, n}, aΔi_→j_,k_ = (Rn,j Rm,k) ~ Bn,m ~ bmn,m→l ~ Bl ~ Pl,i];
aΔi→j,k
```

Out[∘]=
$$\mathbb{E}\Big[a_j\,\alpha_i+a_k\,\alpha_i,\ x_j\,\xi_i+x_k\,\xi_i,\ 1+\frac{1}{2}\left(-2\hbar\, a_j x_k\xi_i+\gamma\hbar\, x_j x_k\xi_i^2\right)\epsilon+\frac{1}{24}\left(12\hbar^2 a_j^2 x_k\xi_i+6\gamma^2\hbar^2 x_j x_k\xi_i^2-12\gamma\hbar^2 a_j x_j x_k\xi_i^2+12\hbar^2 a_j^2 x_k^2\xi_i^2+4\gamma^2\hbar^2 x_j^2 x_k\xi_i^3+4\gamma^2\hbar^2 x_j x_k^2\xi_i^3-12\gamma\hbar^2 a_j x_j x_k^2\xi_i^3+3\gamma^2\hbar^2 x_j^2 x_k^2\xi_i^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:=
```
Block[{i, j, k, l, m, n}, bΔ_{i_→j_,k_} = (R_{j,n} R_{k,m}) ~ B_{n,m} ~ am_{n,m→l} ~ B_l ~ P_{i,l}];
bΔ_{i→j,k}
```

Out[∘]= $\mathbb{E}\left[b_j \beta_i + b_k \beta_i,\ B_k y_j \eta_i + y_k \eta_i,\ 1 + \frac{1}{2}\gamma\hbar B_k y_j y_k \eta_i^2 \epsilon + \frac{1}{24}\left(6\gamma^2\hbar^2 B_k y_j y_k \eta_i^2 + 4\gamma^2\hbar^2 B_k^2 y_j^2 y_k \eta_i^3 + 4\gamma^2\hbar^2 B_k y_j y_k^2 \eta_i^3 + 3\gamma^2\hbar^2 B_k^2 y_j^2 y_k^2 \eta_i^4\right)\epsilon^2 + O[\epsilon]^3\right]$

The two halves

First check that on the generators this agrees with our conventions in the handout:

In[∘]:=
```
Timing@{{"[a,x]" → ((𝔼[0, 0, a₂ x₁] ~ B_{1,2} ~ am_{1,2→1})⟦3⟧ - (𝔼[0, 0, a₁ x₂] ~ B_{1,2} ~ am_{1,2→1})⟦3⟧),
    "[b,y]" → ((𝔼[0, 0, y₂ b₁] ~ B_{1,2} ~ bm_{1,2→1})⟦3⟧ - (𝔼[0, 0, y₁ b₂] ~ B_{1,2} ~ bm_{1,2→1})⟦3⟧)} /.
   z_1 → z,
  {"Δ[y]" → Last[𝔼[0, 0, y₁] ~ B₁ ~ bΔ_{1→1,2}],
   "Δ[b]" → Last[𝔼[0, 0, b₁] ~ B₁ ~ bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼[0, 0, a₁] ~ B₁ ~ aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼[0, 0, x₁] ~ B₁ ~ aΔ_{1→1,2}]},
  {
    "S(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ aS₁)⟦3⟧),
    "S(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ aS₁)⟦3⟧),
    "S(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ bS₁)⟦3⟧),
    "S(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ bS₁)⟦3⟧)
   } /. z_1 → z}
```

Out[∘]= $\{0.875, \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\},\ \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\},\ \{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x \hbar\epsilon - \frac{1}{2}(a^2 x \hbar^2)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$

Hopf algebra axioms on both sides separately

(co)-associativity on both sides

In[∘]:=
```
Timing[HL /@
  {(aΔ_{1→1,2} ~ B₂ ~ aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} ~ B₁ ~ aΔ_{1→1,2}), (bΔ_{1→1,2} ~ B₂ ~ bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} ~ B₁ ~ bΔ_{1→1,2}),
   (am_{1,2→1} ~ B₁ ~ am_{1,3→1}) ≡ (am_{2,3→2} ~ B₂ ~ am_{1,2→1}), (bm_{1,2→1} ~ B₁ ~ bm_{1,3→1}) ≡ (bm_{2,3→2} ~ B₂ ~ bm_{1,2→1})}]
```

Out[∘]= {0.40625, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:=
```
Timing[HL /@ {am_{1,2→1} ~ B₁ ~ aΔ_{1→1,2} ≡ (aΔ_{1→1,3} aΔ_{2→2,4}) ~ B_{1,2,3,4} ~ (am_{3,4→2} am_{1,2→1}),
   bm_{1,2→1} ~ B₁ ~ bΔ_{1→1,2} ≡ (bΔ_{1→1,3} bΔ_{2→2,4}) ~ B_{1,2,3,4} ~ (bm_{3,4→2} bm_{1,2→1})}]
```

Out[∘]= {0.703125, {True, True}}

S is convolution inverse of id

```
In[∘]:= Timing[HL[# ≡ 𝔼[0, 0, 1]] & /@ {
      (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
      (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
Out[∘]= {1.26563, {True, True, True, True}}
```

S is an algebra anti-(co)morphism

```
In[∘]:= Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
      aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
Out[∘]= {2.67188, {True, True, True, True}}
```

Pairing axioms

```
In[∘]:= Timing[HL /@ {(bm1,2→1 𝔼[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
       (𝔼[β1 b1, η1 y1, 1] 𝔼[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
      (bΔ1→1,2 𝔼[α3 a3, ξ3 x3, 1] 𝔼[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
       (𝔼[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
Out[∘]= {0.984375, {True, True}}
```

```
In[∘]:= Timing[HL /@ {(bS1 𝔼[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼[β1 b1, η1 y1, 1] aS2) ~ B1,2 ~ P1,2,
      (b̄S̄1 𝔼[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼[β1 b1, η1 y1, 1] āS̄2) ~ B1,2 ~ P1,2}]
Out[∘]= {1.89063, {True, True}}
```

The Double

The double multiplication (should really bind the a’s and b’s separately)

In[∘]:=

$$\mathbf{Block}\big[\{i, j, k\}, \mathbf{dm}_{i\_,j\_\to k\_} = \big(\mathbb{E}[\beta_i b_i + \alpha_j a_j, \eta_i y_i + \xi_j x_j, 1]\ (a\Delta_{i\to h1,h2} \sim B_{h2} \sim a\Delta_{h2\to h2,h3})\ (b\Delta_{j\to t1,t2} \sim B_{t2} \sim b\Delta_{t2\to t2,t3})\big) \sim$$
$$B_{h3} \sim \overline{aS}_{h3} \sim B_{t1,h3} \sim (P_{t1,h3}) \sim B_{t3,h1} \sim (P_{t3,h1}) \sim B_{h2,j,i,t2} \sim (am_{h2,j\to k}\ bm_{i,t2\to k})\big]$$

Out[∘]=

$$\mathbb{E}\Big[a_k \alpha_i + a_k \alpha_j + b_k \beta_i + b_k \beta_j, \frac{1}{\hbar \mathcal{A}_i \mathcal{A}_j}$$
$$(\hbar y_k \mathcal{A}_i \mathcal{A}_j \eta_i + \hbar y_k \mathcal{A}_j \eta_j + \hbar x_k \mathcal{A}_i \xi_i + \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i - B_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + \hbar x_k \mathcal{A}_i \mathcal{A}_j \xi_j),$$
$$1 + \frac{1}{4 \hbar \mathcal{A}_i \mathcal{A}_j} \big(-4 \hbar y_k \mathcal{A}_j \beta_i \eta_j - 4 \hbar x_k \mathcal{A}_i \beta_j \xi_i + 4 \gamma \hbar^2 x_k y_k \eta_j \xi_i +$$
$$4 \hbar a_k B_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + 2 \gamma \hbar y_k \mathcal{A}_j \eta_j^2 \xi_i - 6 \gamma \hbar B_k y_k \mathcal{A}_j \eta_j^2 \xi_i + 2 \gamma \hbar x_k \mathcal{A}_i \eta_j \xi_i^2 -$$
$$6 \gamma \hbar B_k x_k \mathcal{A}_i \eta_j \xi_i^2 + \gamma \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 - 4 \gamma B_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 + 3 \gamma B_k^2 \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2\big) \epsilon + \frac{1}{288 \hbar^2 \mathcal{A}_i^2 \mathcal{A}_j^2}$$
$$\big(144 \hbar^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i^2 \eta_j + 144 \hbar^2 y_k^2 \mathcal{A}_j^2 \beta_i^2 \eta_j^2 + 144 \hbar^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j^2 \xi_i + 144 \gamma^2 \hbar^4 x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i -$$
$$144 \hbar^3 a_k^2 B_k \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j \xi_i - 288 \gamma \hbar^3 x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_i \eta_j \xi_i - 288 \gamma \hbar^3 x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_j \eta_j \xi_i +$$
$$288 \hbar^2 x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_i \beta_j \eta_j \xi_i + 144 \gamma^2 \hbar^4 x_k y_k^2 \mathcal{A}_j \eta_j^2 \xi_i + 72 \gamma^2 \hbar^3 y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^2 \xi_i -$$
$$360 \gamma^2 \hbar^3 B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^2 \xi_i + 432 \gamma \hbar^3 a_k B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^2 \xi_i - 288 \gamma \hbar^3 x_k y_k^2 \mathcal{A}_j \beta_i \eta_j^2 \xi_i -$$
$$144 \gamma \hbar^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i \eta_j^2 \xi_i + 432 \gamma \hbar^2 B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i \eta_j^2 \xi_i - 288 \hbar^2 a_k B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i \eta_j^2 \xi_i +$$
$$48 \gamma^2 \hbar^3 y_k^2 \mathcal{A}_j^2 \eta_j^3 \xi_i - 336 \gamma^2 \hbar^3 B_k y_k^2 \mathcal{A}_j^2 \eta_j^3 \xi_i - 144 \gamma \hbar^2 y_k^2 \mathcal{A}_j^2 \beta_i \eta_j^3 \xi_i + 432 \gamma \hbar^2 B_k y_k^2 \mathcal{A}_j^2 \beta_i \eta_j^3 \xi_i +$$
$$144 \hbar^2 x_k^2 \mathcal{A}_i^2 \beta_j^2 \xi_i^2 + 144 \gamma^2 \hbar^4 x_k^2 y_k \mathcal{A}_i \eta_j \xi_i^2 + 72 \gamma^2 \hbar^3 x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j \xi_i^2 - 360 \gamma^2 \hbar^3 B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j \xi_i^2 +$$
$$432 \gamma \hbar^3 a_k B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j \xi_i^2 - 288 \gamma \hbar^3 x_k^2 y_k \mathcal{A}_i \beta_j \eta_j \xi_i^2 - 144 \gamma \hbar^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j \eta_j \xi_i^2 +$$
$$432 \gamma \hbar^2 B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j \eta_j \xi_i^2 - 288 \hbar^2 a_k B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j \eta_j \xi_i^2 + 144 \gamma^2 \hbar^4 x_k^2 y_k^2 \eta_j^2 \xi_i^2 +$$
$$360 \gamma^2 \hbar^3 x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 - 1512 \gamma^2 \hbar^3 B_k x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 + 288 \gamma \hbar^3 a_k B_k x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 +$$
$$36 \gamma^2 \hbar^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 - 216 \gamma^2 \hbar^2 B_k \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 + 288 \gamma \hbar^2 a_k B_k \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 +$$
$$180 \gamma^2 \hbar^2 B_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 - 432 \gamma \hbar^2 a_k B_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 + 144 \hbar^2 a_k^2 B_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 -$$
$$144 \gamma \hbar^2 x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_i \eta_j^2 \xi_i^2 + 432 \gamma \hbar^2 B_k x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_i \eta_j^2 \xi_i^2 - 144 \gamma \hbar^2 x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_j \eta_j^2 \xi_i^2 +$$
$$432 \gamma \hbar^2 B_k x_k y_k \mathcal{A}_i \mathcal{A}_j \beta_j \eta_j^2 \xi_i^2 + 144 \gamma^2 \hbar^3 x_k y_k^2 \mathcal{A}_j \eta_j^3 \xi_i^2 - 432 \gamma^2 \hbar^3 B_k x_k y_k^2 \mathcal{A}_j \eta_j^3 \xi_i^2 +$$
$$120 \gamma^2 \hbar^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^3 \xi_i^2 - 816 \gamma^2 \hbar^2 B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^3 \xi_i^2 + 144 \gamma \hbar^2 a_k B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^3 \xi_i^2 +$$
$$984 \gamma^2 \hbar^2 B_k^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^3 \xi_i^2 - 432 \gamma \hbar^2 a_k B_k^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^3 \xi_i^2 - 72 \gamma \hbar y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i \eta_j^3 \xi_i^2 +$$
$$288 \gamma \hbar B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i \eta_j^3 \xi_i^2 - 216 \gamma \hbar B_k^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \beta_i \eta_j^3 \xi_i^2 + 36 \gamma^2 \hbar^2 y_k^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^2 -$$
$$216 \gamma^2 \hbar^2 B_k y_k^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^2 + 324 \gamma^2 \hbar^2 B_k^2 y_k^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^2 + 48 \gamma^2 \hbar^3 x_k^2 \mathcal{A}_i^2 \eta_j \xi_i^3 - 336 \gamma^2 \hbar^3 B_k x_k^2 \mathcal{A}_i^2 \eta_j \xi_i^3 -$$
$$144 \gamma \hbar^2 x_k^2 \mathcal{A}_i^2 \beta_j \eta_j \xi_i^3 + 432 \gamma \hbar^2 B_k x_k^2 \mathcal{A}_i^2 \beta_j \eta_j \xi_i^3 + 144 \gamma^2 \hbar^3 x_k^2 y_k \mathcal{A}_i \eta_j^2 \xi_i^3 -$$
$$432 \gamma^2 \hbar^3 B_k x_k^2 y_k \mathcal{A}_i \eta_j^2 \xi_i^3 + 120 \gamma^2 \hbar^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^2 \xi_i^3 - 816 \gamma^2 \hbar^2 B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^2 \xi_i^3 +$$
$$144 \gamma \hbar^2 a_k B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^2 \xi_i^3 + 984 \gamma^2 \hbar^2 B_k^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^2 \xi_i^3 - 432 \gamma \hbar^2 a_k B_k^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^2 \xi_i^3 -$$
$$72 \gamma \hbar x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j \eta_j^2 \xi_i^3 + 288 \gamma \hbar B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j \eta_j^2 \xi_i^3 - 216 \gamma \hbar B_k^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \beta_j \eta_j^2 \xi_i^3 +$$
$$144 \gamma^2 \hbar^2 x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j^3 \xi_i^3 - 720 \gamma^2 \hbar^2 B_k x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j^3 \xi_i^3 + 864 \gamma^2 \hbar^2 B_k^2 x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_j^3 \xi_i^3 +$$
$$40 \gamma^2 \hbar \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 - 312 \gamma^2 \hbar B_k \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 + 72 \gamma \hbar a_k B_k \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 + 600 \gamma^2 \hbar B_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 -$$
$$288 \gamma \hbar a_k B_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 - 328 \gamma^2 \hbar B_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 + 216 \gamma \hbar a_k B_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^3 \xi_i^3 + 36 \gamma^2 \hbar y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^4 \xi_i^3 -$$
$$252 \gamma^2 \hbar B_k y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^4 \xi_i^3 + 540 \gamma^2 \hbar B_k^2 y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^4 \xi_i^3 - 324 \gamma^2 \hbar B_k^3 y_k \mathcal{A}_i \mathcal{A}_j^2 \eta_j^4 \xi_i^3 +$$
$$36 \gamma^2 \hbar^2 x_k^2 \mathcal{A}_i^2 \eta_j^2 \xi_i^4 - 216 \gamma^2 \hbar^2 B_k x_k^2 \mathcal{A}_i^2 \eta_j^2 \xi_i^4 + 324 \gamma^2 \hbar^2 B_k^2 x_k^2 \mathcal{A}_i^2 \eta_j^2 \xi_i^4 + 36 \gamma^2 \hbar x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^3 \xi_i^4 -$$
$$252 \gamma^2 \hbar B_k x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^3 \xi_i^4 + 540 \gamma^2 \hbar B_k^2 x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^3 \xi_i^4 - 324 \gamma^2 \hbar B_k^3 x_k \mathcal{A}_i^2 \mathcal{A}_j \eta_j^3 \xi_i^4 + 9 \gamma^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^4 -$$
$$72 \gamma^2 B_k \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^4 + 198 \gamma^2 B_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^4 - 216 \gamma^2 B_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^4 + 81 \gamma^2 B_k^4 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^4 \xi_i^4\big) \epsilon^2 + O[\epsilon]^3\Big]$$

In[∘]:=

$$\mathbf{Block}\big[\{i\}, \mathbf{dS}_{i\_} = \mathbb{E}[\beta_i b_1 + \alpha_i a_2, \eta_i y_1 + \xi_i x_2, 1] \sim B_{1,2} \sim \big(\overline{bS}_1\ aS_2\big) \sim B_{1,2} \sim dm_{2,1\to i}\big]$$

Out[∘]= $$\mathbb{E}\Big[-a_i\,\alpha_i - b_i\,\beta_i,\ \frac{-\hbar\,y_i\,\mathcal{A}_i\,\eta_i - \hbar\,B_i\,x_i\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\eta_i\,\xi_i - B_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{\hbar\,B_i},$$

$$1 + \frac{1}{4\,\hbar\,B_i^2}\,\big(4\,\gamma\,\hbar^2\,B_i\,y_i\,\mathcal{A}_i\,\eta_i - 4\,\hbar\,B_i\,y_i\,\mathcal{A}_i\,\beta_i\,\eta_i - 2\,\gamma\,\hbar^2\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2 - 4\,\hbar^2\,a_i\,B_i^2\,x_i\,\mathcal{A}_i\,\xi_i - 4\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i\,\beta_i\,\xi_i -$$
$$4\,\gamma\,\hbar\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i + 4\,\hbar\,a_i\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i + 4\,\gamma\,\hbar\,B_i^2\,\mathcal{A}_i\,\eta_i\,\xi_i - 4\,\gamma\,\hbar^2\,B_i\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i +$$
$$4\,B_i\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i - 4\,B_i^2\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 6\,\gamma\,\hbar\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 2\,\gamma\,\hbar\,B_i\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 2\,\gamma\,\hbar^2\,B_i^2\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 +$$
$$6\,\gamma\,\hbar\,B_i\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 2\,\gamma\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 3\,\gamma\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 4\,\gamma\,B_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - \gamma\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2\big)\,\epsilon +$$

$$\frac{1}{288\,\hbar^2\,B_i^4}\,\big(-144\,\gamma^2\,\hbar^4\,B_i^3\,y_i\,\mathcal{A}_i\,\eta_i + 288\,\gamma\,\hbar^3\,B_i^3\,y_i\,\mathcal{A}_i\,\beta_i\,\eta_i - 144\,\hbar^2\,B_i^3\,y_i\,\mathcal{A}_i\,\beta_i^2\,\eta_i +$$
$$504\,\gamma^2\,\hbar^4\,B_i^2\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2 - 576\,\gamma\,\hbar^3\,B_i^2\,y_i^2\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2 + 144\,\hbar^2\,B_i^2\,y_i^2\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2 - 288\,\gamma^2\,\hbar^4\,B_i\,y_i^3\,\mathcal{A}_i^3\,\eta_i^3 +$$
$$144\,\gamma\,\hbar^3\,B_i\,y_i^3\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3 + 36\,\gamma^2\,\hbar^4\,y_i^4\,\mathcal{A}_i^4\,\eta_i^4 - 144\,\hbar^4\,a_i^2\,B_i^4\,x_i\,\mathcal{A}_i\,\xi_i - 288\,\hbar^3\,a_i\,B_i^4\,x_i\,\mathcal{A}_i\,\beta_i\,\xi_i -$$
$$144\,\hbar^2\,B_i^4\,x_i\,\mathcal{A}_i\,\beta_i^2\,\xi_i + 144\,\gamma^2\,\hbar^3\,B_i^3\,\mathcal{A}_i\,\eta_i\,\xi_i - 288\,\gamma\,\hbar^3\,a_i\,B_i^3\,\mathcal{A}_i\,\eta_i\,\xi_i + 144\,\hbar^3\,a_i^2\,B_i^3\,\mathcal{A}_i\,\eta_i\,\xi_i -$$
$$144\,\gamma^2\,\hbar^3\,B_i^4\,\mathcal{A}_i\,\eta_i\,\xi_i + 432\,\gamma^2\,\hbar^4\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i - 576\,\gamma\,\hbar^4\,a_i\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i -$$
$$288\,\gamma\,\hbar^2\,B_i^3\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 288\,\hbar^2\,a_i\,B_i^3\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 288\,\gamma\,\hbar^2\,B_i^4\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i -$$
$$864\,\gamma\,\hbar^3\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i + 288\,\hbar^3\,a_i\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i + 144\,\hbar\,B_i^3\,\mathcal{A}_i\,\beta_i^2\,\eta_i\,\xi_i -$$
$$144\,\hbar\,B_i^4\,\mathcal{A}_i\,\beta_i^2\,\eta_i\,\xi_i + 288\,\hbar^2\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i\,\xi_i - 1512\,\gamma^2\,\hbar^3\,B_i^2\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i +$$
$$720\,\gamma\,\hbar^3\,a_i\,B_i^2\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i + 648\,\gamma^2\,\hbar^3\,B_i^3\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 720\,\gamma^2\,\hbar^4\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i +$$
$$144\,\gamma\,\hbar^4\,a_i\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i + 1440\,\gamma\,\hbar^2\,B_i^2\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i - 288\,\hbar^2\,a_i\,B_i^2\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i -$$
$$864\,\gamma\,\hbar^2\,B_i^3\,y_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i + 432\,\gamma\,\hbar^3\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i - 288\,\hbar\,B_i^2\,y_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i +$$
$$288\,\hbar\,B_i^3\,y_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i + 1344\,\gamma^2\,\hbar^3\,B_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i - 144\,\gamma\,\hbar^3\,a_i\,B_i\,y_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i -$$
$$480\,\gamma^2\,\hbar^3\,B_i^2\,y_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i + 144\,\gamma^2\,\hbar^4\,B_i\,x_i\,y_i^3\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i - 576\,\gamma\,\hbar^2\,B_i\,y_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i +$$
$$288\,\gamma\,\hbar^2\,B_i^2\,y_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i - 216\,\gamma^2\,\hbar^3\,y_i^3\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i + 72\,\gamma^2\,\hbar^3\,B_i\,y_i^3\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i + 72\,\gamma^2\,\hbar^4\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 -$$
$$288\,\gamma\,\hbar^4\,a_i\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 + 144\,\hbar^4\,a_i^2\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 - 288\,\gamma\,\hbar^3\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\beta_i\,\xi_i^2 + 288\,\hbar^3\,a_i\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\beta_i\,\xi_i^2 +$$
$$144\,\hbar^2\,B_i^4\,x_i^2\,\mathcal{A}_i^2\,\beta_i^2\,\xi_i^2 - 792\,\gamma^2\,\hbar^3\,B_i^3\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 + 1152\,\gamma\,\hbar^3\,a_i\,B_i^3\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 288\,\hbar^3\,a_i^2\,B_i^3\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 +$$
$$216\,\gamma^2\,\hbar^3\,B_i^4\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 432\,\gamma\,\hbar^3\,a_i\,B_i^4\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 576\,\gamma^2\,\hbar^4\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i\,\xi_i^2 +$$
$$288\,\gamma\,\hbar^4\,a_i\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i\,\xi_i^2 + 1152\,\gamma\,\hbar^2\,B_i^3\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 - 576\,\hbar^2\,a_i\,B_i^3\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 -$$
$$576\,\gamma\,\hbar^2\,B_i^4\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 + 288\,\hbar^2\,a_i\,B_i^4\,x_i\,\mathcal{A}_i^2\,\beta_i\,\eta_i\,\xi_i^2 + 432\,\gamma\,\hbar^3\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i\,\xi_i^2 -$$
$$288\,\hbar\,B_i^3\,x_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i\,\xi_i^2 + 288\,\hbar\,B_i^4\,x_i\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i\,\xi_i^2 + 756\,\gamma^2\,\hbar^2\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - 720\,\gamma\,\hbar^2\,a_i\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 +$$
$$144\,\hbar^2\,a_i^2\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - 1080\,\gamma^2\,\hbar^2\,B_i^3\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 576\,\gamma\,\hbar^2\,a_i\,B_i^3\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 324\,\gamma^2\,\hbar^2\,B_i^4\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 +$$
$$2232\,\gamma^2\,\hbar^3\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 - 720\,\gamma\,\hbar^3\,a_i\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 - 792\,\gamma^2\,\hbar^3\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 +$$
$$144\,\gamma\,\hbar^3\,a_i\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^2 + 216\,\gamma^2\,\hbar^4\,B_i^2\,x_i^2\,y_i^2\,\mathcal{A}_i^4\,\eta_i^2\,\xi_i^2 - 720\,\gamma\,\hbar\,B_i^2\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 +$$
$$288\,\hbar\,a_i\,B_i^2\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 + 1152\,\gamma\,\hbar\,B_i^3\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 - 288\,\hbar\,a_i\,B_i^3\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 - 432\,\gamma\,\hbar\,B_i^4\,\mathcal{A}_i^2\,\beta_i\,\eta_i^2\,\xi_i^2 -$$
$$1152\,\gamma\,\hbar^2\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^2 + 576\,\gamma\,\hbar^2\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^2 + 144\,B_i^2\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i^2 -$$
$$288\,B_i^3\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i^2 + 144\,B_i^4\,\mathcal{A}_i^2\,\beta_i^2\,\eta_i^2\,\xi_i^2 - 1632\,\gamma^2\,\hbar^2\,B_i\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 + 432\,\gamma\,\hbar^2\,a_i\,B_i\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 +$$
$$1680\,\gamma^2\,\hbar^2\,B_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 - 144\,\gamma\,\hbar^2\,a_i\,B_i^2\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 - 336\,\gamma^2\,\hbar^2\,B_i^3\,y_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^2 -$$
$$648\,\gamma^2\,\hbar^3\,B_i\,x_i\,y_i^2\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^2 + 216\,\gamma^2\,\hbar^3\,B_i^2\,x_i\,y_i^2\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^2 + 648\,\gamma\,\hbar\,B_i\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^2 -$$
$$864\,\gamma\,\hbar\,B_i^2\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^2 + 216\,\gamma\,\hbar\,B_i^3\,y_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^2 + 432\,\gamma^2\,\hbar^2\,y_i^2\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^2 - 360\,\gamma^2\,\hbar^2\,B_i\,y_i^2\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^2 +$$
$$72\,\gamma^2\,\hbar^2\,B_i^2\,y_i^2\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^2 - 144\,\gamma^2\,\hbar^4\,B_i^4\,x_i^3\,\mathcal{A}_i^3\,\xi_i^3 + 144\,\gamma\,\hbar^4\,a_i\,B_i^4\,x_i^3\,\mathcal{A}_i^3\,\xi_i^3 + 144\,\gamma\,\hbar^3\,B_i^4\,x_i^3\,\mathcal{A}_i^3\,\beta_i\,\xi_i^3 +$$
$$912\,\gamma^2\,\hbar^3\,B_i^3\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 - 576\,\gamma\,\hbar^3\,a_i\,B_i^3\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 - 336\,\gamma^2\,\hbar^3\,B_i^4\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 +$$
$$144\,\gamma\,\hbar^3\,a_i\,B_i^4\,x_i^2\,\mathcal{A}_i^3\,\eta_i\,\xi_i^3 + 144\,\gamma^2\,\hbar^4\,B_i^3\,x_i^3\,y_i\,\mathcal{A}_i^4\,\eta_i\,\xi_i^3 - 576\,\gamma\,\hbar^2\,B_i^3\,x_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i\,\xi_i^3 +$$
$$288\,\gamma\,\hbar^2\,B_i^4\,x_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i\,\xi_i^3 - 1416\,\gamma^2\,\hbar^2\,B_i^2\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 + 648\,\gamma\,\hbar^2\,a_i\,B_i^2\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 +$$
$$1392\,\gamma^2\,\hbar^2\,B_i^3\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 - 432\,\gamma\,\hbar^2\,a_i\,B_i^3\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 - 264\,\gamma^2\,\hbar^2\,B_i^4\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 +$$
$$72\,\gamma\,\hbar^2\,a_i\,B_i^4\,x_i\,\mathcal{A}_i^3\,\eta_i^2\,\xi_i^3 - 648\,\gamma^2\,\hbar^3\,B_i^2\,x_i^2\,y_i\,\mathcal{A}_i^4\,\eta_i^2\,\xi_i^3 + 216\,\gamma^2\,\hbar^3\,B_i^3\,x_i^2\,y_i\,\mathcal{A}_i^4\,\eta_i^2\,\xi_i^3 +$$
$$648\,\gamma\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^3 - 864\,\gamma\,\hbar\,B_i^3\,x_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^3 + 216\,\gamma\,\hbar\,B_i^4\,x_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^2\,\xi_i^3 + 544\,\gamma^2\,\hbar\,B_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 -$$
$$216\,\gamma\,\hbar\,a_i\,B_i\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 - 1104\,\gamma^2\,\hbar\,B_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 + 288\,\gamma\,\hbar\,a_i\,B_i^2\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 + 672\,\gamma^2\,\hbar\,B_i^3\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 -$$
$$72\,\gamma\,\hbar\,a_i\,B_i^3\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 - 112\,\gamma^2\,\hbar\,B_i^4\,\mathcal{A}_i^3\,\eta_i^3\,\xi_i^3 + 864\,\gamma^2\,\hbar^2\,B_i\,x_i\,y_i\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^3 - 720\,\gamma^2\,\hbar^2\,B_i^2\,x_i\,y_i\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^3 +$$
$$144\,\gamma^2\,\hbar^2\,B_i^3\,x_i\,y_i\,\mathcal{A}_i^4\,\eta_i^3\,\xi_i^3 - 216\,\gamma\,B_i\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 + 504\,\gamma\,B_i^2\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 - 360\,\gamma\,B_i^3\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 +$$
$$72\,\gamma\,B_i^4\,\mathcal{A}_i^3\,\beta_i\,\eta_i^3\,\xi_i^3 - 324\,\gamma^2\,\hbar\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 + 540\,\gamma^2\,\hbar\,B_i\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 - 252\,\gamma^2\,\hbar\,B_i^2\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 +$$
$$36\,\gamma^2\,\hbar\,B_i^3\,y_i\,\mathcal{A}_i^4\,\eta_i^4\,\xi_i^3 + 36\,\gamma^2\,\hbar^4\,B_i^4\,x_i^4\,\mathcal{A}_i^4\,\xi_i^4 - 216\,\gamma^2\,\hbar^3\,B_i^3\,x_i^3\,\mathcal{A}_i^4\,\eta_i\,\xi_i^4 + 72\,\gamma^2\,\hbar^3\,B_i^4\,x_i^3\,\mathcal{A}_i^4\,\eta_i\,\xi_i^4 +$$

$$432\,\gamma^2\hbar^2 B_i^2 x_i^2 \mathcal{A}_i^4 \eta_i^2 \xi_i^4 - 360\,\gamma^2\hbar^2 B_i^3 x_i^2 \mathcal{A}_i^4 \eta_i^2 \xi_i^4 + 72\,\gamma^2\hbar^2 B_i^4 x_i^2 \mathcal{A}_i^4 \eta_i^2 \xi_i^4 - 324\,\gamma^2\hbar B_i x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 + 540\,\gamma^2\hbar B_i^2 x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 - 252\,\gamma^2\hbar B_i^3 x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 + 36\,\gamma^2\hbar B_i^4 x_i \mathcal{A}_i^4 \eta_i^3 \xi_i^4 + 81\,\gamma^2 \mathcal{A}_i^4 \eta_i^4 \xi_i^4 - 216\,\gamma^2 B_i \mathcal{A}_i^4 \eta_i^4 \xi_i^4 + 198\,\gamma^2 B_i^2 \mathcal{A}_i^4 \eta_i^4 \xi_i^4 - 72\,\gamma^2 B_i^3 \mathcal{A}_i^4 \eta_i^4 \xi_i^4 + 9\,\gamma^2 B_i^4 \mathcal{A}_i^4 \eta_i^4 \xi_i^4\big)\,\epsilon^2 + O[\epsilon]^3\Big]$$

In[∘]:=
```
Block[{i, j, k}, dΔ_{i_→j_,k_} = (bΔ_{i→3,1} aΔ_{i→2,4}) ~B_{1,2,3,4}~ (dm_{3,4→k} dm_{1,2→j})]
```

Out[∘]=
$$\mathbb{E}\Big[a_j\alpha_i + a_k\alpha_i + b_j\beta_i + b_k\beta_i,\; y_j\eta_i + B_j y_k\eta_i + x_j\xi_i + x_k\xi_i,\; 1 + \frac{1}{2}\left(\gamma\hbar B_j y_j y_k \eta_i^2 - 2\hbar a_j x_k \xi_i + \gamma\hbar x_j x_k \xi_i^2\right)\epsilon + \frac{1}{24}\big(6\gamma^2\hbar^2 B_j y_j y_k \eta_i^2 + 4\gamma^2\hbar^2 B_j y_j^2 y_k \eta_i^3 + 4\gamma^2\hbar^2 B_j^2 y_j y_k^2 \eta_i^3 + 3\gamma^2\hbar^2 B_j^2 y_j^2 y_k^2 \eta_i^4 + 12\hbar^2 a_j^2 x_k \xi_i - 12\gamma\hbar^2 a_j B_j x_k y_j y_k \eta_i^2 \xi_i + 6\gamma^2\hbar^2 x_j x_k \xi_i^2 - 12\gamma\hbar^2 a_j x_j x_k \xi_i^2 + 12\hbar^2 a_j^2 x_k^2 \xi_i^2 + 6\gamma^2\hbar^2 B_j x_j x_k y_j y_k \eta_i^2 \xi_i^2 + 4\gamma^2\hbar^2 x_j^2 x_k \xi_i^3 + 4\gamma^2\hbar^2 x_j x_k^2 \xi_i^3 - 12\gamma\hbar^2 a_j x_j x_k^2 \xi_i^3 + 3\gamma^2\hbar^2 x_j^2 x_k^2 \xi_i^4\big)\epsilon^2 + O[\epsilon]^3\Big]$$

First check the double formulas on the generators agree with SL2Portfolio.pdf:

In[∘]:=
```
Timing@{{
    "[a,y]" → ((𝔼[0, 0, y2 a1] ~B1,2~ dm1,2→1)⟦3⟧ - (𝔼[0, 0, y1 a2] ~B1,2~ dm1,2→1)⟦3⟧),
    "[b,x]"  → ((𝔼[0, 0, x2 b1] ~B1,2~ dm1,2→1)⟦3⟧ - (𝔼[0, 0, x1 b2] ~B1,2~ dm1,2→1)⟦3⟧),
    "xy-qyx" → ((𝔼[0, 0, x1 y2] ~B1,2~ dm1,2→1)⟦3⟧ - (1 + ϵ) (𝔼[0, 0, y1 x2] ~B1,2~ dm1,2→1)⟦3⟧)
   } /. {z_1 → z} // Expand // Factor,
  {
   "Δ(a)" → ((𝔼[0, 0, a1] ~B1~ dΔ1→1,2)⟦3⟧),
   "Δ(x)" → ((𝔼[0, 0, x1] ~B1~ dΔ1→1,2)⟦3⟧),
   "Δ(b)" → ((𝔼[0, 0, b1] ~B1~ dΔ1→1,2)⟦3⟧),
   "Δ(y)" → ((𝔼[0, 0, y1] ~B1~ dΔ1→1,2)⟦3⟧)
  } // Simplify,
  {
   "S(a)" → ((𝔼[0, 0, a1] ~B1~ dS1)⟦3⟧),
   "S(x)" → ((𝔼[0, 0, x1] ~B1~ dS1)⟦3⟧),
   "S(b)" → ((𝔼[0, 0, b1] ~B1~ dS1)⟦3⟧),
   "S(y)" → ((𝔼[0, 0, y1] ~B1~ dS1)⟦3⟧)
  } /. {z_1 → z} // Simplify
 }
```

Out[∘]=
$$\Big\{3.5,\ \Big\{\Big\{[a,y] \to -y\gamma + O[\epsilon]^3,\ [b,x] \to x\epsilon + O[\epsilon]^3,\ \text{xy-qyx} \to \left(-x y + \frac{1 - B + x y\hbar}{\hbar}\right) + (a B - x y + x y\gamma\hbar)\,\epsilon + \frac{1}{2}\left(-a^2 B\hbar + x y\gamma^2\hbar^2\right)\epsilon^2 + O[\epsilon]^3\Big\},$$
$$\Big\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar a_1 x_2\,\epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2\,\epsilon^2 + O[\epsilon]^3,\ \Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1 y_2) + O[\epsilon]^3\Big\},$$
$$\Big\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x\hbar\epsilon - \frac{1}{2}\left(a^2 x\hbar^2\right)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\gamma\hbar\epsilon}{B} - \frac{\left(y\gamma^2\hbar^2\right)\epsilon^2}{2B} + O[\epsilon]^3\Big\}\Big\}\Big\}$$

Hopf algebra axioms on double

(co)-associativity

In[∘]:= Timing[HL /@ $\{(d\Delta_{1\to1,2} \sim B_2 \sim d\Delta_{2\to2,3}) \equiv (d\Delta_{1\to1,3} \sim B_1 \sim d\Delta_{1\to1,2}),\ (dm_{1,2\to1} \sim B_1 \sim dm_{1,3\to1}) \equiv (dm_{2,3\to2} \sim B_2 \sim dm_{1,2\to1})\}$]

Out[∘]= {7.67188, {True, True}}

Δ is an algebra morphism

In[∘]:= Timing@HL[$dm_{1,2\to1} \sim B_1 \sim d\Delta_{1\to1,2} \equiv (d\Delta_{1\to1,3}\, d\Delta_{2\to2,4}) \sim B_{1,2,3,4} \sim (dm_{3,4\to2}\, dm_{1,2\to1})$]

Out[∘]= {14.9688, True}

S is convolution inverse of id

In[∘]:= Timing[HL[# ≡ 𝔼[0, 0, 1]] & /@ $\{(d\Delta_{1\to1,2} \sim B_1 \sim dS_1) \sim B_{1,2} \sim dm_{1,2\to1},\ (d\Delta_{1\to1,2} \sim B_2 \sim dS_2) \sim B_{1,2} \sim dm_{1,2\to1}\}$]

Out[∘]= {12.4219, {True, True}}

S is a (co)-algebra anti-morphism

In[∘]:= Timing[HL /@ Expand /@ $\{dm_{1,2\to1} \sim B_1 \sim dS_1 \equiv (dS_1\, dS_2) \sim B_{1,2} \sim dm_{2,1\to1},\ dS_1 \sim B_1 \sim d\Delta_{1\to1,2} \equiv d\Delta_{1\to2,1} \sim B_{1,2} \sim (dS_1\, dS_2)\}$]

Out[∘]= {27.9844, {True, True}}

Quasi-triangular axiom 1:

In[∘]:= Timing@HL[$R_{1,2} \sim B_1 \sim d\Delta_{1\to1,3} \equiv (R_{1,4}\, R_{3,2}) \sim B_{2,4} \sim dm_{2,4\to2}$]

Out[∘]= {0.765625, True}

Quasi-triangular axiom 2:

In[∘]:= Timing@HL[$((d\Delta_{1\to1,2}\, R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\, dm_{2,4\to2})) \equiv ((d\Delta_{1\to2,1}\, R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1\to1}\, dm_{4,2\to2}))$]

Out[∘]= {12.8594, True}

Reidemeister 3:

In[∘]:= Timing@HL[$((R_{1,2}\, R_{4,3}\, R_{5,6}) \sim B_{1,4} \sim dm_{1,4\to1} \sim B_{2,5} \sim dm_{2,5\to2} \sim B_{3,6} \sim dm_{3,6\to3}) \equiv ((R_{1,6}\, R_{2,3}\, R_{4,5}) \sim B_{1,4} \sim dm_{1,4\to1} \sim B_{2,5} \sim dm_{2,5\to2} \sim B_{3,6} \sim dm_{3,6\to3})$]

Out[∘]= {6.4375, True}

In[∘]:= Block[{i, j}, $\overline{R}_{i\_,j\_}$ = Expand /@ $R_{i,j} \sim B_j \sim dS_j$]

Out[∘]= $\mathbb{E}\Big[-\hbar\, a_j\, b_i,\ -\frac{\hbar\, x_j\, y_i}{B_i},\ 1 + \frac{(-4\hbar^2 a_j B_i x_j y_i - 3\gamma\hbar^3 x_j^2 y_i^2)\,\epsilon}{4 B_i^2} + \frac{1}{288 B_i^4}(-144\hbar^3 a_j^2 B_i^3 x_j y_i + 144\gamma^2\hbar^4 B_i^2 x_j^2 y_i^2 - 432\gamma\hbar^4 a_j B_i^2 x_j^2 y_i^2 + 144\hbar^4 a_j^2 B_i^2 x_j^2 y_i^2 - 320\gamma^2\hbar^5 B_i x_j^3 y_i^3 + 216\gamma\hbar^5 a_j B_i x_j^3 y_i^3 + 81\gamma^2\hbar^6 x_j^4 y_i^4)\,\epsilon^2 + O[\epsilon]^3\Big]$

Reidemeister 2

In[∘]:= Timing[HL /@ $\{(\overline{R}_{1,2}\, R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\, dm_{2,4\to2}),\ (R_{1,2}\, \overline{R}_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3\to1}\, dm_{2,4\to2})\}$]

Out[∘]= {8.42188, {$\mathbb{E}[0, 0, 1 + O[\epsilon]^3]$, $\mathbb{E}[0, 0, 1 + O[\epsilon]^3]$}}

Deriving the Drinfeld element u and its inverse $\overline{u}$

In[•]:=
```
Block[{i}, {
  u_i_ = R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i},
  ū_i_ = R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→i}
  }]
```

Out[•]=
$$\Big\{\mathbb{E}\Big[-\hbar\, a_i\, b_i,\ -\frac{\hbar\, x_i\, y_i}{B_i},\ B_i+\frac{1}{4\,B_i}\left(-4\,\hbar\, a_i\, B_i^2-4\,\gamma\,\hbar^2\, B_i\, x_i\, y_i-4\,\hbar^2\, a_i\, B_i\, x_i\, y_i-3\,\gamma\,\hbar^3\, x_i^2\, y_i^2\right)\epsilon+\frac{1}{288\,B_i^3}\big(144\,\hbar^2\, a_i^2\, B_i^4-144\,\gamma^2\,\hbar^3\, B_i^3\, x_i\, y_i+144\,\hbar^3\, a_i^2\, B_i^3\, x_i\, y_i-144\,\gamma^2\,\hbar^4\, B_i^2\, x_i^2\, y_i^2+72\,\gamma\,\hbar^4\, a_i\, B_i^2\, x_i^2\, y_i^2+144\,\hbar^4\, a_i^2\, B_i^2\, x_i^2\, y_i^2-104\,\gamma^2\,\hbar^5\, B_i\, x_i^3\, y_i^3+216\,\gamma\,\hbar^5\, a_i\, B_i\, x_i^3\, y_i^3+81\,\gamma^2\,\hbar^6\, x_i^4\, y_i^4\big)\epsilon^2+O[\epsilon]^3\Big],$$
$$\mathbb{E}\Big[\hbar\, a_i\, b_i,\ \hbar\, x_i\, y_i,\ \frac{1}{B_i}+\frac{\left(4\,\hbar\, a_i-4\,\gamma\,\hbar^2\, x_i\, y_i-\gamma\,\hbar^3\, x_i^2\, y_i^2\right)\epsilon}{4\,B_i}+\frac{1}{288\,B_i}\big(144\,\hbar^2\, a_i^2+144\,\gamma^2\,\hbar^3\, x_i\, y_i-288\,\gamma\,\hbar^3\, a_i\, x_i\, y_i+288\,\gamma^2\,\hbar^4\, x_i^2\, y_i^2-72\,\gamma\,\hbar^4\, a_i\, x_i^2\, y_i^2+104\,\gamma^2\,\hbar^5\, x_i^3\, y_i^3+9\,\gamma^2\,\hbar^6\, x_i^4\, y_i^4\big)\epsilon^2+O[\epsilon]^3\Big]\Big\}$$

u and $\overline{u}$ are inverses

In[•]:=
```
Timing@HL[(u_1 ū_2) ~ B_{1,2} ~ dm_{1,2→1} ≡ 𝔼[0, 0, 1]]
```

Out[•]= {1.53125, True}

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}.
It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

In[•]:=
```
Block[{$k = 3}, ((u_1 ~ B_1 ~ dS_1) ū_2) ~ B_{1,2} ~ dm_{1,2→1}]
```

Out[•]=
$$\mathbb{E}\Big[0,\ 0,\ \frac{1}{B_1}+\frac{\hbar\, a_1\,\epsilon}{B_1}+\frac{\hbar^2\, a_1^2\,\epsilon^2}{2\,B_1}+O[\epsilon]^3\Big]$$

(* Needs fixing! *) So in our case S(u) = u z so S(u)u = $u^2\,z$ and v = u$z^{\frac{1}{2}}$ and finally C = uv^{-1} = $z^{-\frac{1}{2}} = e^{t_1/2}(1-\epsilon\, a_1)$.

In[•]:=
```
Block[{i},
  {CC_i_ = 𝔼[0, 0, B_i^(1/2) e^(-ϵ a_i/2) + O[ϵ]^2],
   C̄C̄_i_ = 𝔼[0, 0, B_i^(-1/2) e^(ϵ a_i/2) + O[ϵ]^2]
  }]
```

Out[•]=
$$\Big\{\mathbb{E}\Big[0,\ 0,\ \sqrt{B_i}-\frac{1}{2}\left(a_i\sqrt{B_i}\right)\epsilon+O[\epsilon]^2\Big],\ \mathbb{E}\Big[0,\ 0,\ \frac{1}{\sqrt{B_i}}+\frac{a_i\,\epsilon}{2\,\sqrt{B_i}}+O[\epsilon]^2\Big]\Big\}$$

In[∘]:=
```
Block[{i, j}, {
  Kink_i_ = (R_{1,3} \bar{CC}_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i},
  \bar{Kink}_j_ = (\bar{R}_{1,3} CC_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→j}
}]
```

Out[∘]= $\left\{\mathbb{E}\left[a_i b_i,\ x_i y_i,\ \frac{1}{\sqrt{B_i}} + \frac{\left(2 a_i - x_i^2 y_i^2\right) \epsilon}{4 \sqrt{B_i}} + O[\epsilon]^2\right],\ \mathbb{E}\left[-a_j b_j,\ -\frac{x_j y_j}{B_j},\ \sqrt{B_j} + \frac{\left(-2 a_j B_j^2 - 4 a_j B_j x_j y_j - 3 x_j^2 y_j^2\right) \epsilon}{4 B_j^{3/2}} + O[\epsilon]^2\right]\right\}$

In[∘]:=
```
k2 = (R_{3,1} CC_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i} /. ε → E;
k4 = (\bar{R}_{3,1} \bar{CC}_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→j} /. ε → E;
Simplify@{Kink_i ≡ k2, \bar{Kink}_j ≡ k4, (Kink_i \bar{Kink}_j) ~ B_{i,j} ~ dm_{i,j→1}}
```

Out[∘]= $\left\{\text{True},\ \text{True},\ \mathbb{E}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]\right\}$

Reidemeister 2:

In[∘]:=
```
(R_{1,2} \bar{R}_{3,4}) ~ B_{1,3} ~ dm_{1,3→1} ~ B_{2,4} ~ dm_{2,4→2}
```

Out[∘]= $\mathbb{E}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

Cyclic Reidemeister 2:

In[∘]:=
```
(R_{1,4} \bar{R}_{5,2} \bar{CC}_3) ~ B_{2,4} ~ dm_{2,4→2} ~ B_{1,3} ~ dm_{1,3→1} ~ B_{1,5} ~ dm_{1,5→1} ≡ \bar{CC}_1
```

Out[∘]= True

Trefoil

In[∘]:=
```
Z = R_{1,5} R_{6,2} R_{3,7} \bar{CC}_4 \bar{Kink}_8 \bar{Kink}_9 \bar{Kink}_{10};
Do[Z = Z ~ B_{1,r} ~ dm_{1,r→1}, {r, 2, 10}];
Simplify /@ Z
```

Out[∘]= $\mathbb{E}\left[0,\ 0,\ \frac{B_1}{1 - B_1 + B_1^2} + \left(B_1 \left(-B_1 + 2 B_1^2 + 2 B_1^4 + a_1 \left(-1 + B_1 - B_1^3 + B_1^4\right) - 2 x_1 y_1 - B_1^3 \left(3 + 2 x_1 y_1\right)\right) \epsilon\right) \Big/ \left(1 - B_1 + B_1^2\right)^3 + O[\epsilon]^2\right]$

```
Timing[
 Z = R_{1,5} R_{6,2} R_{3,7} \bar{CC}_4 \bar{Kink}_8 \bar{Kink}_9 \bar{Kink}_{10};
 Do[Z = Z ~ B_{1,r} ~ dm_{1,r→1}, {r, 2, 10}];
 Simplify /@ Z ]
```

In[∘]:=
```
b2t_i_ := E[α_i a_i - β_i t_i, ξ_i x_i + η_i y_i, 1 + ε β_i a_i + O[ε]^2]
t2b_i_ := E[α_i a_i - τ_i b_i, ξ_i x_i + η_i y_i, 1 + ε τ_i a_i + O[ε]^2]
```

In[∘]:= $R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10}$

Out[∘]= $$\mathbb{E}\Big[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},\; x_5 y_1 + x_7 y_3 + x_2 y_6 - \frac{x_8 y_8}{B_8} - \frac{x_9 y_9}{B_9} - \frac{x_{10} y_{10}}{B_{10}},\; \frac{\sqrt{B_8}\,\sqrt{B_9}\,\sqrt{B_{10}}}{\sqrt{B_4}} + \Bigg(\sqrt{B_{10}}\Bigg(\sqrt{B_9}\Bigg(\sqrt{B_8}\left(\frac{a_4}{2\sqrt{B_4}} - \frac{x_5^2 y_1^2}{4\sqrt{B_4}} - \frac{x_7^2 y_3^2}{4\sqrt{B_4}} - \frac{x_2^2 y_6^2}{4\sqrt{B_4}}\right) + \frac{-2 a_8 B_8^2 - 4 a_8 B_8 x_8 y_8 - 3 x_8^2 y_8^2}{4\sqrt{B_4}\, B_8^{3/2}}\Bigg) + \frac{\sqrt{B_8}\,\left(-2 a_9 B_9^2 - 4 a_9 B_9 x_9 y_9 - 3 x_9^2 y_9^2\right)}{4\sqrt{B_4}\, B_9^{3/2}}\Bigg) + \frac{1}{4\sqrt{B_4}\, B_{10}^{3/2}}\sqrt{B_8}\,\sqrt{B_9}\,\left(-2 a_{10} B_{10}^2 - 4 a_{10} B_{10} x_{10} y_{10} - 3 x_{10}^2 y_{10}^2\right)\Bigg) \epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $\left(R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10}\right) \sim B_{\text{Range}[10]} \sim \text{Product}[b2t_i, \{i, 10\}]$

Out[∘]= $$\mathbb{E}\Big[-a_5 t_1 - a_7 t_3 - a_2 t_6 + a_8 t_8 + a_9 t_9 + a_{10} t_{10},\; \frac{1}{T_8 T_9 T_{10}}\left(T_8 T_9 T_{10} x_5 y_1 + T_8 T_9 T_{10} x_7 y_3 + T_8 T_9 T_{10} x_2 y_6 - T_9 T_{10} x_8 y_8 - T_8 T_{10} x_9 y_9 - T_8 T_9 x_{10} y_{10}\right),\; \frac{\sqrt{T_8}\,\sqrt{T_9}\,\sqrt{T_{10}}}{\sqrt{T_4}} + \frac{1}{4\sqrt{T_4}\, T_8^{3/2} T_9^{3/2} T_{10}^{3/2}}\big(4 a_4 T_8^2 T_9^2 T_{10}^2 + 4 a_1 a_5 T_8^2 T_9^2 T_{10}^2 + 4 a_2 a_6 T_8^2 T_9^2 T_{10}^2 + 4 a_3 a_7 T_8^2 T_9^2 T_{10}^2 - 4 a_8 T_8^2 T_9^2 T_{10}^2 - 4 a_8^2 T_8^2 T_9^2 T_{10}^2 - 4 a_9 T_8^2 T_9^2 T_{10}^2 - 4 a_9^2 T_8^2 T_9^2 T_{10}^2 - 4 a_{10} T_8^2 T_9^2 T_{10}^2 - 4 a_{10}^2 T_8^2 T_9^2 T_{10}^2 - T_8^2 T_9^2 T_{10}^2 x_5^2 y_1^2 - T_8^2 T_9^2 T_{10}^2 x_7^2 y_3^2 - T_8^2 T_9^2 T_{10}^2 x_2^2 y_6^2 - 8 a_8 T_8 T_9^2 T_{10}^2 x_8 y_8 - 3 T_9^2 T_{10}^2 x_8^2 y_8^2 - 8 a_9 T_8^2 T_9 T_{10}^2 x_9 y_9 - 3 T_8^2 T_{10}^2 x_9^2 y_9^2 - 8 a_{10} T_8^2 T_9^2 T_{10} x_{10} y_{10} - 3 T_8^2 T_9^2 x_{10}^2 y_{10}^2\big) \epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $Z = \left(\left(\left(R_{1,5}\, R_{6,2}\, R_{3,7}\, \overline{CC}_4\, \overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10}\right) \sim B_{\text{Range}[10]} \sim \text{Product}[b2t_i, \{i, 10\}]\right)\ /.\ T\_ \to T_1\right) \sim B_{\text{Range}[10]} \sim \text{Product}[t2b_i, \{i, 10\}]$

Out[∘]= $$\mathbb{E}\Big[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},\; \frac{1}{B_1}\left(B_1 x_5 y_1 + B_1 x_7 y_3 + B_1 x_2 y_6 - x_8 y_8 - x_9 y_9 - x_{10} y_{10}\right),\; B_1 + \frac{1}{4 B_1}\big(4 a_1 B_1^2 + 4 a_4 B_1^2 - 4 a_8 B_1^2 - 4 a_9 B_1^2 - 4 a_{10} B_1^2 - B_1^2 x_5^2 y_1^2 - B_1^2 x_7^2 y_3^2 - B_1^2 x_2^2 y_6^2 + 4 a_1 B_1 x_8 y_8 - 8 a_8 B_1 x_8 y_8 - 3 x_8^2 y_8^2 + 4 a_1 B_1 x_9 y_9 - 8 a_9 B_1 x_9 y_9 - 3 x_9^2 y_9^2 + 4 a_1 B_1 x_{10} y_{10} - 8 a_{10} B_1 x_{10} y_{10} - 3 x_{10}^2 y_{10}^2\big) \epsilon + O[\epsilon]^2\Big]$$

```
Timing[
 Do[Z = Z~B1,r~dm1,r→1, {r, 2, 10}];
 Simplify@Z[[3]] ]
```

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

Out[∘]= $\left\{5.39063, \frac{B_1}{1-B_1+B_1^2}+\left(B_1\left(-B_1+2B_1^2+2B_1^4+a_1\left(-1+B_1-B_1^3+B_1^4\right)-2x_1y_1-B_1^3\left(3+2x_1y_1\right)\right)\epsilon\right)\Big/\left(1-B_1+B_1^2\right)^3+O[\epsilon]^2\right\}$

In[∘]:=
```
Timing[
 Z = R1,5 R6,2 R3,7 CC4‾ Kink8‾ Kink9‾ Kink10‾ /. B_ → B1;
 Do[Print["doing ", r]; Z = Z~B1,r~dm1,r→1 /. B_ → B1, {r, 2, 10}];
 Simplify@Z〚3〛 ]
```

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

Out[∘]= $\left\{5.3125, \frac{B_1}{1-B_1+B_1^2}+\frac{1}{2B_1\left(1-B_1+B_1^2\right)^3}\left(2a_1B_1^2\left(-1+B_1-B_1^3+B_1^4\right)-6x_1^2y_1^2+4B_1^7x_1^2y_1^2-2B_1^8x_1^2y_1^2+B_1^2x_1y_1\left(5-6x_1y_1\right)+3B_1x_1y_1\left(-1+2x_1y_1\right)+B_1^6\left(3+3x_1y_1-6x_1^2y_1^2\right)-B_1^5\left(4+13x_1y_1+2x_1^2y_1^2\right)+B_1^4\left(2+15x_1y_1+4x_1^2y_1^2\right)-B_1^3\left(1+15x_1y_1+6x_1^2y_1^2\right)\right)\epsilon+O[\epsilon]^2\right\}$