Peter's Handout, Page 1

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CHECKING THE HUTCHINGS CRITERION

"Topological syzygies" include the "Zamolodchikov tetrahedron" which

induces algebraic syzygies via: $\left\{ \begin{array}{l} \text{Zam.} \\ \text{with } \sigma_{ij} \mapsto (\bar{\sigma}_{ij}+1) \end{array} \right.$ keeping only lowest degree in $\bar{\sigma}_{ij}$

<u>Query</u>: Do all algebraic syzygies arise this way?

<u>What is $\ker(\pi\circ)$</u>(syzygies in pvb_n)?

<u>FACT</u>: $A = pvb_n$ is Koszul (BEER, Lee)

<u>HENCE</u>: $\bigoplus_{i<j}(R_{p,i}\cap R_{p,j}) \to \bigoplus_i R_{p,i} \to V^{\otimes p} \to A^p \to 0$

is exact, where

$$\begin{tikzcd} & R_{p,i} \arrow[dr, "+"] & \\ R_{p,i}\cap R_{p,j} \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, dashed] & & V^{\otimes p} \\ & R_{p,j} \arrow[ur, "-"'] & \end{tikzcd}$$

"Trivial syzygies" $R_{p,i}\cap R_{p,j}$ if $|i-j|>1$

Non-trivial: $R_{p,i}\cap R_{p,i+1} \cong V^{\otimes i} \otimes A^{!3} \otimes V^{\otimes p-i-3}$

via: $(\Delta^!_{1,1}\otimes \mathrm{Id})\circ\Delta^!_{2,1} : A^{!3*} \xrightarrow{\sim} R\otimes V \cap V\otimes R$ (*)

where $\Delta^!_{\bullet,1}$ is dual to $m^! : A^{!\bullet}\otimes A^{!1} \to A^{!\bullet+1}$

<u>BASIS</u> $A_n^{!k}$ has basis indexed by unordered partitions of $[n] = \{1,\dots,n\}$ into $(n-k)$ ordered subsets "Lah numbers $L(n,n-k)$"

"Chain gangs"

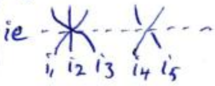
via $\left.\begin{array}{l} i_1\to i_2\to\dots\to i_k \\ j_1\to\dots\to j_\ell \end{array}\right\} \leftrightarrow a_{i_1 i_2}\wedge a_{i_2 i_3}\wedge\dots\wedge a_{i_{k-1}i_k}\wedge a_{j_1 j_2}\wedge\dots$

Ind degree 3 primarily get all chains

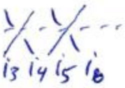
$i_1\to i_2\to i_3\to i_4$

These are easily calculated by (*) and $= F^{Syz}(\text{Zam})$

Also get $\left.\begin{array}{l} i_1\to i_2\to i_3 \\ i_4\to i_5 \end{array}\right\}$ ie



and $\left.\begin{array}{l} i_1\to i_2 \\ i_3\to i_4 \\ i_5\to i_6 \end{array}\right\}$ ie



<u>Proof of Basis</u>

A relations: $[a_{ij},a_{ik}]+[a_{ij},a_{jk}]+[a_{ik},a_{jk}]=0$

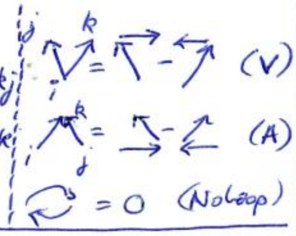
$[a_{ij},a_{k\ell}]=0$

$A^! = \Lambda\{a_{ij}\}$ mod:

$a_{ij}\wedge a_{ik} = a_{ij}\wedge a_{jk} - a_{ik}\wedge a_{kj}$

$a_{ik}\wedge a_{jk} = a_{ij}\wedge a_{jk} - a_{ji}\wedge a_{ik}$

$a_{ij}\wedge a_{ji} = 0$



<u>Idea</u>: $\left.\begin{array}{l} \text{V-join} \\ \text{A-join} \end{array}\right\}$: relations replace A,V-joins by directed segments.

- Define a "Defect" function on (AS) monomials st the maximal term in a relation is preserved by multiplication that produces no loops ("Multiplicative")
- Verify that Defect 0 monomials = Chain Gangs
- Defect 0 monomials generate $A^!$:
  - If a monomial's graph has a loop, it is 0.
  - Multiplicative $\Rightarrow$ forests can be reduced to Defect 0 monomials.
- Prove that Defect 0 monomials are <u>Indep.</u>

<u>Defect</u>
- A pair of vertices is "unordered" if there is no oriented sequence of edges between them.
- Defect of a tree is # unordered pairs of vertices in it.
- Defect of a forest is sum of defects of its trees.
- Clear that Defect 0 monomials $\leftrightarrow$ Chain Gangs.

MAX: In (V) and (A) the join terms have maximal defect.

<u>Multiplicativity</u> Induction on # of vertices in relation

Show: if vertices a, b are ordered in join term, they are ordered in other terms.

Hence: adding edges either increases Defect or preserves it. Now apply "MAX"

3 Cases: Case I: $a\to b$ new edge with a, b new vertices: clear. (connected components are same in all terms)