Reading Peter: Ker \delta in the group case

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**Assumptions.** $A$ an augmented unital algebra over $\mathbb{Q}$, $I_A = \ker\epsilon$ the augmentation ideal, $F$ a free algebra over $\mathbb{Q}$ with a surjection $F \to A$, $I_F$ the augmentation ideal of $F$; so we have

$$0 \to I_A \to A \xrightarrow{\epsilon} \mathbb{Q} \to 0.$$

Let $M \subseteq I_F \subset F$ be the 2-sided ideal such that

$$\left.\begin{array}{l} 0 \to M \to F \to A \to 0 \\ 0 \to M \to I_F \to I_A \to 0 \end{array}\right\} \text{ are exact.}$$

Aside for the retarded:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, "\cong"] & I_F \arrow[r] \arrow[d] & I_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Finally, assume that $M \subset I_F^2$; "$A$ has a presentation in which every relation has every generator appearing at zero multiplicity.

**Constructions.** $\mu_A : I_A^{\otimes_A m} \to I_A^{\otimes_A (m-1)}$ is the multiplication map, a.k.a "$\delta$".

$R_m^A = \bigoplus I_A^{\otimes j} \otimes R^A \otimes I_A^{\otimes(m-j-2)}$ where

$R^A := \ker(\mu_A : I_A \otimes_A I_A \to I_A^2)$ is the analogue of "$T4T$ relators".

$\partial_A : R_m^A \to I_A^{\otimes_A m}$, the obvious sum over summands, is the {relators} $\to$ {relations} map.

**Theorem.**

$$\partial_A(R_m^A) = \ker \mu_A$$

I.e., "TFT generates $\ker \delta$".

The inclusion $\operatorname{im}\partial_A \subset \ker\mu_A$ is obvious.

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$$R_m^A \xrightarrow{\partial_A} I_A^{\otimes_A m} \xrightarrow{\mu_A} I_A^{\otimes_A(m-1)}$$

$a{:}b{:}c{:}d \longrightarrow a{:}b{:}c{:}d$ -

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Claim: $\mu : I_F \otimes_F I_F \to I_F^2$ is an isomorphism.

<u>Claim</u> If $f, g$ are injective, then so is $f \otimes_R g$.

(depends on $R$?)

$$I_F \otimes_F I_F \xrightarrow{\quad} F \otimes_F I_F \xrightarrow{\sim} I_F \hookrightarrow F$$

injective? Needs flatness of $I_F$.

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Getting closer....

$$\begin{tikzcd}
 & M_m \arrow[r] & I_F^{\otimes m} \arrow[rr] & & I_A^{\otimes m} \\
M_{m+1} \arrow[dr] & & & & \\
 & I_F^{\otimes m+1} \arrow[r, "\mu", "\sim"'] \arrow[drr, two heads] & I_F \cdot I_F^{\otimes m} \arrow[uu, hook] \arrow[rr] & & I_A \cdot I_A^{\otimes m} \arrow[uu, hook] \\
 & & & I_A^{\otimes m+1} \arrow[ur, "\mu"] &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & M_{m+1} \arrow[d] & M_m \arrow[d] \\
 & I_F^{\otimes m+1} \arrow[r, "\sim", "\mu_F"'] \arrow[d] & I_F \cdot I_F^{\otimes m} \arrow[d] \\
R^A \arrow[r] & I_A^{\otimes m+1} \arrow[r, "\mu_A"'] \arrow[d] & I_A \cdot I_A^{\otimes m} \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

Need to find

$$\mu_F^{-1}(M_m) / M_{m+1}$$

$$\cong$$

$$\ker(\mu_A)$$

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d] & C \arrow[d] \\
A/B \arrow[r] & C/D
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
A/B \arrow[rr, "\mu"] & & A/C
\end{tikzcd}$$

$\ker \mu = C/B$