Peter's "Hutchings Alternative"

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Hutchings Alternative

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Let F be the free augmented algebra over $\mathbb{Q}$ generated by a set S, with augmentation ideal I_F. F is filtered by powers of I_F, and is isomorphic as a vector space to the associated graded $grF = \oplus_m I_F^m / I_F^{m+1}$. With this grading, F is generated in degree 1.

Now let $M \subseteq I_F^2$ be a 2-sided ideal of F. We want to find conditions under which

$$F/M \cong q(F/M) \tag{1}$$

where $q(U)$ for any graded algebra U generated in degree 1 means the quadratic approximation to U.

We use the notation $A := F/M$, with augmentation ideal I_A; also $V := I_A/I_A^2$; and $\mathfrak{R} \subseteq V \otimes_{\mathbb{Q}} V$ the vector subspace of degree two relations of $gr_I A$: i.e., $\mathfrak{R} := ker(\mu : I_A/I_A^2 \otimes_{\mathbb{Q}} I_A/I_A^2 \to I_A^2/I_A^3)$, where μ is the multiplication in $gr_I A$ induced from multiplication in I_A. Then we define $qA := q(A) = TV/\langle \mathfrak{R} \rangle$.

Proposition 1. *With the assumptions above (in particular $M \subseteq I_F^2$), we have for all non-negative integers m:*

$$I_F^m / I_F^{m+1} \cong V^{\otimes_{\mathbb{Q}} m} \tag{2}$$

Hence we can write $F \cong TV$.

Now suppose that $\tilde{M} \cong \mathfrak{R}$ – [$\tilde{M}$ to be defined] and $M =< \tilde{M} >$. Then equation (1) involves two algebras with the same generators and the same generators of relations. Hence the algebras will be isomorphic as vector spaces if the algebras have the same syzygies.

[NTD the following still in progress...] Finally, with regard to condition $\tilde{M} \cong \mathfrak{R}$ – [$\tilde{M}$ to be defined] and $M =< \tilde{M} >$. There is clearly a surjection:

$$(M \cap I_F^2 / I_F^3) \twoheadrightarrow \mathfrak{R}$$

Moreover, if $M \cap I_F^3 = 0$, then [[this surjection is injective (To be confirmed).]]

Might it be "morally wrong" to introduce the free algebra so early, given that the Hutchings criterion makes sense even in the abstract?