Peter's "Hexagon Lemma"

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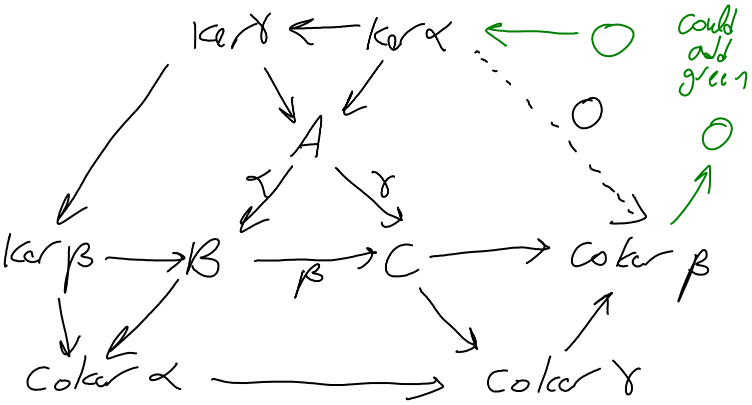
Lemma. Given a commutative triangle

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ B \arrow[rr, "\beta"'] & & C \end{tikzcd}$$

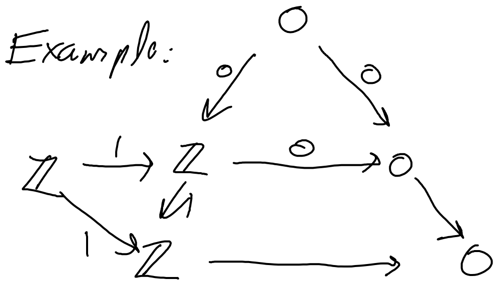
, we get the following commutative hexagon, in which the outer layer starting from $\ker\alpha$ is exact:



That is,

$$\ker\alpha \to \ker\gamma \to \ker\beta \to \operatorname{coker}\alpha \to \operatorname{coker}\gamma$$
$$\to \operatorname{coker}\beta$$

is exact.



This is simple enough that it might have a nice picture.

Possibly, it even has a name....

Question. What's the canonical form of "the composition of two linear transformations"? What's Peter's hexagon in that case? "linear Venn Diagrams"

$$\left.\begin{matrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,0 \\ 0\,1\,1 \end{matrix}\right\}$$

Q. Is there an invariant description for these factors?

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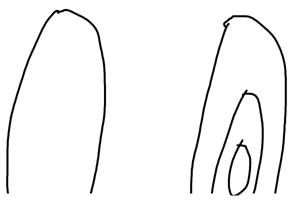
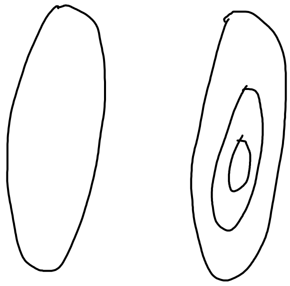


$$\left.\begin{array}{c}100\\010\\001\\110\\011\\111\end{array}\right\}$$

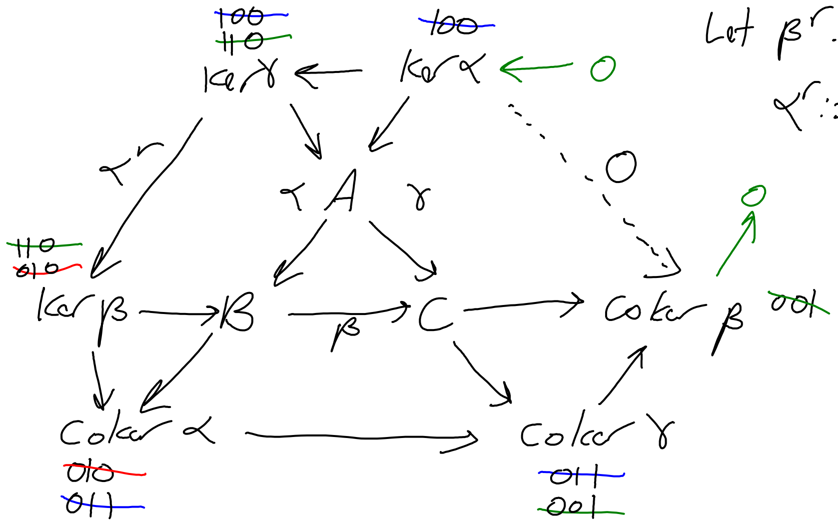
Q. Is there an invariant description for these factors?

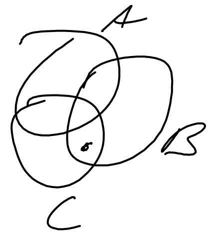
$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[rr, bend left, "\gamma"] & B \arrow[r, "\beta"'] & C \end{tikzcd}$$



Let $\beta^r := \beta|_{\operatorname{im}\alpha}$

$\alpha^r := \alpha|_{\ker\gamma}$





$100 = \ker\alpha \qquad 001 = \operatorname{coker}\beta \qquad 010 = \dfrac{\ker\beta}{\operatorname{Im}\alpha^r}$

$\dfrac{\ker\gamma}{\ker\alpha} = 110 = \ker\beta \cap \operatorname{Im}\alpha$

Question. For a general commutative diagram, are all incoming hom space describable using kers/cokers/the other ops of an Abelian category?

Probably not the right question.

Examples. $\operatorname{Hom}(1\to 0, A\xrightarrow{\alpha} B) = \ker\alpha$

$\operatorname{Hom}(0\to 1, A\xrightarrow{\alpha} B) = B$

$\operatorname{Hom}(1\to 1, A\xrightarrow{\alpha} B) = A$

$\operatorname{Hom}(A\xrightarrow{\alpha} B, 1\to 0) = A^*$

$\operatorname{Hom}(A\xrightarrow{\alpha} B, 0\to 1) = (\operatorname{coker}\alpha)^*$

$$\begin{tikzcd} \operatorname{Hom}(A \arrow[r] \arrow[d, dashed] & B, \arrow[d, dashed] & 1\to 1) = B^* \\ 1 \arrow[r, dashed] & 1 \end{tikzcd}$$

$\operatorname{Hom}(x\to y\to z, A\xrightarrow{\alpha} B\xrightarrow{\beta} C)$:

$0\to 0\to 1: C \qquad 0\to 1\to 0: \ker\beta \qquad 1\to 0\to 0: \ker\alpha$

$0\to 1\to 1: B \qquad 1\to 1\to 0: \ker\gamma \qquad 1\to 1\to 1: A$

The classical Hutchings case:

$$
\begin{tikzcd}
\{ \arrow[dd] & & \} \arrow[d] & \left\{\substack{\text{rels between}\\ T4T\text{'s}}\right\} \arrow[l, "\text{onto?}"'] \arrow[dl] & \\
& & K'_m \arrow[dl] \arrow[dr] & & \\
\dfrac{K^0_{m+1}}{\partial K'_{m+1}} \arrow[r, "\delta"] \arrow[dr, "1\text{-}1?"'] & K^0_m \arrow[d] \arrow[rr] & & \mathcal{D}^0_m \arrow[r] \arrow[d] & 0 \\
& K^0_m / \partial K'_m \arrow[rr] & & \mathcal{A}_m \arrow[ur] &
\end{tikzcd}
$$

The identification of this space remains a miracle

This seems like the right space only by the use of the diffability relation.

Peter's case: With

$$
\begin{tikzcd}
\sum_k I^{:k-1} : (\ker I:I \xrightarrow{\mu} I^2) : I^{:(n-k-1)} \arrow[d, "\Sigma"'] \arrow[dr] & \\
I^{:n} \arrow[r] & (I/I^2)^{\otimes n}
\end{tikzcd}
$$