Oregon Handout as of July 31, 2011

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global Q→q!

Improve typesetting throughout!

$R_2 =$

separate

The Pure Virtual Braid Group is Quadratic[1]

Dror Bar–Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

Foots & refs on PDF version, page 3.

Abstract Generalities Let K be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$. Definition. Say that K is quadratic if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty}I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=Q(K)=\langle V=I/I^2\rangle/\langle\ker(\bar\mu_2:V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (Q is a lovely functor). Then K is quadratic iff the obvious $\mu:A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

blue

$$\cdots\; I^{:p+1}\xrightarrow{\mu_{p+1}} I^{:p}\xrightarrow{\mu_p} I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$?
- How injective is this tower?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq(I/I^2)^{\otimes p}=V^{\otimes p}$.

\mathfrak

Flow Chart. Any (K,I) $\xrightarrow{\text{Prop 1}}$ 2-local $\xrightarrow{\text{Prop 2}}$ Quadratic; $K=PvB_n$ $\xrightarrow{\text{Thm S, by Peter}}$ Hutchings' Criterion $\longrightarrow$ 2-injective; 2-injective $\xrightarrow{\text{Prop 2}}$ Quadratic.

Proposition 1. The sequence

$$R_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:R_2:I^{:p-j-1}\right)\xrightarrow{\partial} I^{:p}\xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $R_2:=\ker\mu:I^{:2}\to I$; so (K,I) is "2-local".

blue?

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, denote $F\to F/J=\mathbb{F}$ by $x\mapsto[x]$ and define $\psi:F\to F$ by $x_i\mapsto x_i+[x_i]$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $R_2=R_p=0$, and hence the same is true for every J.

reread

The General Case. If $K=F/M$ and $I\subset K$, then $I=J/M$ where $J\subset F$. Then $I^{:p}=J^p/\sum J^{:j-1}:M:J^{:p-j}$ and we have

$$\begin{array}{ccc}
J^{:p} & \xrightarrow{\ \mu_F\ } & J^{:p-1}\\
\text{onto}\downarrow\pi_p & & \pi_{p-1}\downarrow\text{onto}\\
I^{:p}=J^{:p}/\sum J^{:}:M:J^{:} & \longrightarrow & I^{:p-1}=J^{:p-1}/\sum J^{:}:M:J^{:}
\end{array}$$

M_F Contre (1-1)

So $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}(J^{:}:M:J^{:})\right)=\pi_p\left(\sum J^{:}:\mu_F^{-1}(M):J^{:}\right)=\sum I^{:}:R_2:I^{:}$.

Take sum outside.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p (>0), $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Why Care? • In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z:K\to\hat{A}$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But trivially $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:R_2:I^{:p-j-1}\right)$. But that's the degree p piece of $Q(K)$.

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The X Lemma (inspired by [Hut]).

$$\begin{array}{ccccc}
A_0 & & & & C_0\\
 & \searrow^{\alpha_0} & & \nearrow^{\beta_0} & \\
 & & B & & \\
 & \nearrow_{\alpha_1} & & \searrow_{\beta_1} & \\
A_1 & & & & C_1
\end{array}$$

Michael Hutchings

will compute homology for food

If the above diagram is Conway (≍) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}$.

more space

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" lifts to a "topological syzygy".

$$\begin{array}{ccccc}
R_p & & & & I^{:p-1}\\
 & \searrow^{\partial} & & \nearrow^{\mu_p} & \\
 & & I^{:p} & & \\
 & \nearrow_{\mu_{p+1}} & & \searrow^{\pi} & \\
I^{:p+1} & & & & V^{\otimes p}
\end{array}$$

Conclusion. We need to know that (K,I) is **"syzygy complete"** — that every diagrammatic syzygy lifts to a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

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Example. (goes back to [Koh])

T. Kohno

$K = \langle \ \rangle \qquad I = \langle \ \rangle$

$(K/I^{p+1})^\star = (\text{invariants of type } p) =: \mathcal{V}_p$

$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji} \rangle$

$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4T \text{ relations} \rangle$

$A = \left(\text{horizontal chord diagrams mod 4T}\right) = \langle \ \rangle / 4T$

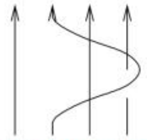
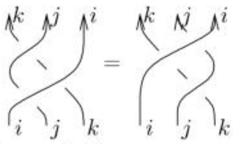
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$\langle \sigma_{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{matrix} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{matrix}$

L. Kauffman [Kau, KL]

of "pure virtual braids" ("braids when you look", "blunder braids"):

$\sigma_{24} =$  R3: 

The Main Theorem [Lee]. PvB_n is quadratic.

$A_n = Q(PvB_n)$. [GPV]

Goussarov-Polyak-Viro

$I = \langle \ \rangle$ with $\bar\sigma_{ij} = \sigma_{ij} - 1$, the "semi-virtual crossing".

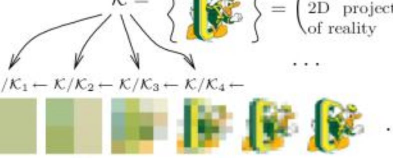
$V = I/I^2 : \langle$ v-Braids w/ one $\rangle / X = X$

$= \langle a_{ij} \rangle_{i \neq j}$

Derive loc, 6T. & draw

A box for $I^{\cdot p}$, like Peter's.

Just for fun. $\mathcal{K} = \{ \ \} =$ (The set of all 2D projections of reality)



$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow$

Crop Rotate Adjoin

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$

$\mathbb{R}^3 \qquad \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

A box for R_2:

Add one sample syzygy.

Figuring out R_2:

$$
\begin{tikzcd}
J^{\otimes 2} \arrow[r, "\mu^F"', "1-1"] \arrow[d, "\pi_2"'] & J \supset M \arrow[d, "\pi_1"] \\
I^{\otimes 2} \arrow[r, "\mu"'] & I = J/M
\end{tikzcd}
$$

$\operatorname{Ker} \mu = \pi_2(\mu_F^{-1}(M))$

in principle computable.