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The Pure Virtual Braid Group is Quadratic[1]

Dror Bar-Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
Foots & refs on PDF version, page 3.

remove space.

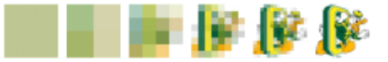
Let K be an algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$. Definition. Say that K is quadratic if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty}I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=Q(K)=\langle V=I/I^2\rangle/\langle\ker(\bar\mu_2:V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (Q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

Just for fun.

$\mathcal{K}=\Big\{$  $\Big\}=\Big($The set of all 2D projections of reality$\Big)$

$\mathcal{K}/\mathcal{K}_1\leftarrow\mathcal{K}/\mathcal{K}_2\leftarrow\mathcal{K}/\mathcal{K}_3\leftarrow\mathcal{K}/\mathcal{K}_4\leftarrow$ ...

Crop Rotate Adjoin



An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$\mathcal{K}/\mathcal{K}_1\oplus\mathcal{K}_1/\mathcal{K}_2\oplus\mathcal{K}_2/\mathcal{K}_3\oplus\mathcal{K}_3/\mathcal{K}_4\oplus\mathcal{K}_4/\mathcal{K}_5\oplus\mathcal{K}_5/\mathcal{K}_6\oplus\cdots$

$\| \;\mathbb{R}^3$ $\quad\| \;\ker(\mathcal{K}/\mathcal{K}_4\to\mathcal{K}/\mathcal{K}_3)$

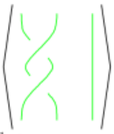
Why Care? • In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z:K\to\hat A$, becomes wonderful mathematics:

beautify

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torrosian [KV, AT]

Example.

T. Kohno

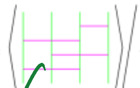
$K=\Big\langle$  $\Big\rangle$ $\quad I=\Big\langle$  $\Big\rangle$

ref?

$(K/I^{p+1})^\star=$ (invariants of type p) $=:\mathcal{V}_p$

$(I^p/I^{p+1})^\star=\mathcal{V}_p/\mathcal{V}_{p-1}\quad C=\langle t^{ij}|t^{ij}=t^{ji}\rangle=\Big\langle$  $\Big\rangle$

$\ker\bar\mu_2=\langle[t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle 4T\text{ relations}\rangle$

$A=\Big($horizontal chord diagrams mod 4T$\Big)=\Big\langle$  $\Big\rangle\Big/4T$

Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle\sigma_{ij}:1\le i\ne j\le n\rangle\Big/\begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}$$

L. Kauffman

of "pure virtual braids" ("braids when you look", "blunder braids"):

A

The Main Theorem [Lee]. PvB_n is quadratic.

Flow Chart. special | general

Any (K,I) $\xrightarrow{\text{Thm 2}}$ 2-local $\xrightarrow{\text{Thm 1}}$ Quadratic

$K=PvB_n$ $\xrightarrow[\text{by Peter}]{\text{Thm 4}}$ Syzygy complete $\xrightarrow{\text{Thm 3}}$ 2-injective

2-Locality. The pair (K,I) is "2-local" if the sequence

$$R_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:R_2:I^{:p-j-1}\right)\xrightarrow{\partial}I^{:p}\xrightarrow{\mu_p}I^{:p-1}$$

is exact, where $R_2:=\ker(\mu_2:I^{:2}\to I)$

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}K_{p-1}\longrightarrow\cdots\longrightarrow K_0=K$$

is "injective" if for all p, $\ker(\delta_p)=0$. It is "2-injective" if its "1-reduction"

reduce to std. with

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots\longrightarrow K$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if

$$\cdots\longrightarrow I^{:p+1}\xrightarrow{\mu_{p+1}}I^{:p}\xrightarrow{\mu_p}I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

it's sing tower is 2-injective.

is 2-injective, where ":" denotes $\otimes_K$ and μ is (always) multiplication. We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$.

Theorem 1. If (K,I) is 2-local and 2-injective, it is quadratic.

reread.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get

$$\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}.$$

But trivially $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}\Big/\sum\left(I^{:j-1}:R_2:I^{:p-j-1}\right)$. But that's the degree p piece of $Q(K)$.

A: overall strategy

Consider the "singularity tower" of (K, I)

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("$:$" = "$\otimes_K$", μ is (always) multiplication)

$$\to I^{:p+1} \to I^{:p} \to I^{:p-1} \to \dots \to K$$

1. What's $I^{:p}/\mu(I^{:p+1})$?
2. How injective is it?

<u>Lemma</u> $I^{:p}/\mu(I^{:p+1}) \cong (I/I^2)^{\otimes p} = V^{\otimes p}$.