Oregon Handout as of July 26, 2011

July-26-11
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The Pure Virtual Braid Group is Quadratic[1]

Dror Bar–Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
Foots & refs on PDF version, page 3.

Presented to the great algebra masters of the Oregon School, in pursuit of their wisdom and advice, in acceptance that they know all and have seen all, and in dread that we will inflict boredom upon them.

Let K be an algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$.

Definition. Say that K is *quadratic* if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty}I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=Q(K)=\langle V=I/I^2\rangle/\langle\ker(\cancel{\mu}: V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (Q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

Why Care? • In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z: K\to\hat{A}$, becomes wonderful mathematics:

beautify

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torrosian [KV, AT]

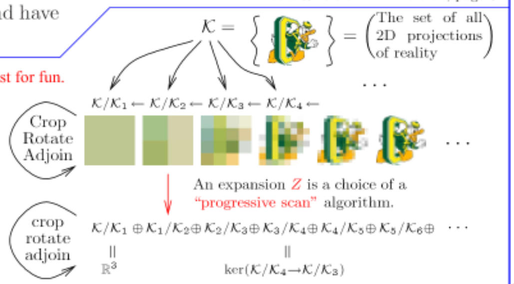

L. Kauffman

PvB_n is the group

$$\langle\sigma_{ij}:1\leq i\neq j\leq n\rangle\Big/\begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}$$

of "pure virtual braids" ("braids when you look", "blunder braids"):

Just for fun.



Example.


T. Kohno

$K=\langle\ \rangle\qquad I=\langle\ \rangle$

$m\to p$

$$(K/I^{m+1})^\star=(\text{invariants of type }m)=:\mathcal{V}_m$$

$$(I^m/I^{m+1})^\star=\mathcal{V}_m/\mathcal{V}_{m-1}\qquad C=\langle t^{ij}|t^{ij}=t^{ji}\rangle=\langle\ \rangle$$

$$\ker\mu_{11}=\langle[t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle 4T\text{ relations}\rangle$$

$$A=\begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix}=\langle\ \rangle\Big/4T$$

Z: universal finite type invariant, the Kontsevich integral.

The Main Theorem [Lee]. PvB_n is quadratic.

Flow Chart.

$$\boxed{\text{Any }(K,I)}\xrightarrow[2]{\text{Thm}}\boxed{\text{2-local}}\xrightarrow{\text{Thm 1}}\boxed{\text{Quadratic}}$$

$$\boxed{K=PvB_n}\xrightarrow[\text{by Peter}]{\text{Thm 4}}\boxed{\text{Syzygy complete}}\xrightarrow[3]{\text{Thm}}\boxed{\text{2-injective}}$$

2-Locality. The pair (K,I) is "2-local" if $\ker\mu_p: I^{:p}\to I^{:p-1}$ is $\sum_{j=1}^{p-1}\left(I^{:j-1}:R_2:I^{:p-j-1}\right)$, where $R_2:=\ker\mu: I^{:2}\to I$.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}K_{p-1}\longrightarrow\cdots\longrightarrow K_0=K$$

is "injective" if for all p, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots\longrightarrow K$$

is injective; i.e. if for all p, $\ker\delta_p\circ\delta_{p+1}=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if

$$\cdots\longrightarrow I^{:p+1}\xrightarrow{\mu_{p+1}}I^{:p}\xrightarrow{\mu_p}I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

is 2-injective, where ":" denotes $\otimes_K$ and μ is (always) multiplication. We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$.

Theorem 1. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get

$$\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu\left(I^{:p+1}/\ker\mu_{p+1}\right)}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}.$$

But trivially $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}\Big/\sum\left(I^{:j-1}:R_2:I^{:p-j-1}\right)$. But that's the degree p piece of $Q(K)$.

needs reread.