Oregon Handout as of August 4, 2011 (2)

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11:16 AM

The Pure Virtual Braid Group is Quadratic[1]

Abstract Generalities

Dror Bar–Natan and Peter Lee in Oregon, August 2011
`http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/`
foots & refs on PDF version, page 3

Let K be a unital algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; so $K/I \xrightarrow[\epsilon]{\sim} \mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2: V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to \operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

✓ blue

The Overall Strategy. Consider the "singularity tower" of (K,I) (here "$:$" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\ I^{:p+1} \xrightarrow{\mu_{p+1}} I^{:p} \xrightarrow{\mu_p} I^{:p-1} \longrightarrow \cdots \longrightarrow K$$

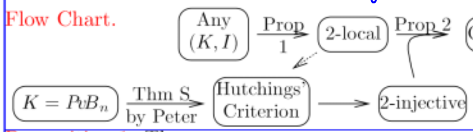
We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- How injective is this tower?
- What's $I^{:p}/\mu(I^{:p+1})$?

blue.

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq (I/I^2)^{\otimes p}=V^{\otimes p}$; denote $\pi: I^{:p}\to V^{\otimes p}$

Flow Chart.



Proposition 1. The sequence

$$\mathfrak{R}_p := \bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right) \xrightarrow{\partial} I^{:p} \xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2 := \ker\mu: I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, define $\psi: F\to F$ by $x_i\mapsto x_i+\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/\langle M\rangle$ (where M is a vector space of "moves") and $I\subset K$, then $I=J/\langle M\rangle$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:\langle M\rangle:J^{:p-j}$ and we have

$$\begin{tikzcd}
J^{:p} \arrow[r, "\mu_F", "\text{1-1}"'] \arrow[d, "\pi_p", "\text{onto}"'] & J^{:p-1} \arrow[d, "\text{onto}", "\pi_{p-1}"'] \\
I^{:p}=J^{:p}/\sum J^{:}:\langle M\rangle:J^{:} \arrow[r, "\mu"] & I^{:p-1}=J^{:p-1}/\sum J^{:}:\langle M\rangle:J^{:}
\end{tikzcd}$$

So $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}\left(J^{:}:\langle M\rangle:J^{:}\right)\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle:J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}=:\sum_{j=1}^{p-1}\mathfrak{R}_{p,j}$.

$\mathfrak{R}_2$ is simpler than may seem! It's an "augmentation bimodule" ($I\mathfrak{R}_2=0=\mathfrak{R}_2I$ thus $xr=\epsilon(x)r=r\epsilon(x)=rx$ for $x\in K$ and $r\in\mathfrak{R}_2$), and hence $\mathfrak{R}_2=\pi_2(\mu_F^{-1}M)$.

$$\begin{tikzcd}
J^{:2} \arrow[r, "\mu_F", "\text{1-1}"'] \arrow[d, "\pi_2"] & J\supset M \arrow[d, "\pi_1"] \\
I^{:2} \arrow[r, "\mu"] & I=J/\langle M\rangle
\end{tikzcd}$$

$\mathfrak{R}_p$ is simpler than may seem! In $\mathfrak{R}_{p,j}=I^{:j-1}:\mathfrak{R}_2:I^{p-j-1}$ the I factors may be replaced by $V=I/I^2$. Hence

$$\mathfrak{R}_p\simeq\bigoplus_{j=1}^{p-1}V^{\oplus j-1}\otimes\pi_2(\mu_F^{-1}M)\otimes V^{\otimes p-j-1}.$$

Why Care?

- In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly.
- In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z: K\to\hat{A}$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

still fishy.

The X Lemma (inspired by [Hut]).

$$\begin{tikzcd}
A_0 \arrow[dr, "\alpha_0"] & & C_0 \\
& B \arrow[ur, "\beta_0"] \arrow[dr, "\beta_1"] & \\
A_1 \arrow[ur, "\alpha_1"] & & C_1
\end{tikzcd}$$


Michael Hutchings
will compute homology for food

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$

$$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}.$$

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{tikzcd}
\mathfrak{R}_p \arrow[dr, "\partial"] & & I^{:p-1} \\
& I^{:p} \arrow[ur, "\mu_p"] \arrow[dr, "\pi"] & \\
I^{:p+1} \arrow[ur, "\mu_{p+1}"] & & V^{\otimes p}
\end{tikzcd}$$

Conclusion. We need to know that (K,I) is "syzygy complete" — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

Do I still need to put R_2 in?

$$\begin{tikzcd}
0 \arrow[r] & R_2 \arrow[r] \arrow[d, "\text{onto?}"] & I:I \arrow[r, "\mu_2"] \arrow[d, "\pi_2"] & I^2 \arrow[r] \arrow[d, "\pi_1"] & 0 \\
0 \arrow[r] & R_2 \arrow[r] & \frac{I}{I^2}\otimes\frac{I}{I^2} \arrow[r, "\bar\mu_2"] & I^2/I^3 \arrow[r] & 0
\end{tikzcd}$$

$\ker\bar\mu_2=\pi_2(\mu_2^{-1}(I^3))$

Seems to need that

$$0\to R_2 \xrightarrow{i} \frac{I}{I^2}\otimes\frac{I}{I^2} \xrightarrow{\mu_2} I^2/I^3\to 0$$

Seems to need that
$$I:I\to I^2\to I/I^3$$
is 2-injective; ~~That is, that~~ it clearly isn't.
recycled.

Looking at the long exact seq corresponding to the above in homology, to see that R_2 surjects on R_2 I need to know that

$$\mu_2(\ker\pi_2)=\ker\pi_1$$

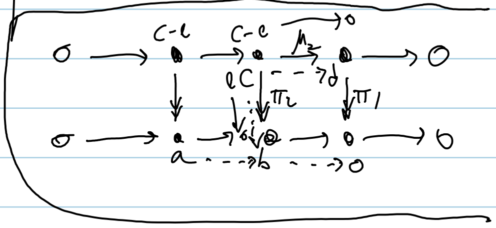
Clear as ✓

$$\mu_2(I^2:I)=I^3$$

to also know that $R_2\hookrightarrow R_2$, we need $\mu_2|_{\ker\pi_2}\hookrightarrow\ker\pi_1$

meaning $0=\ker\mu_2\cap\ker\pi_2$

$$=R_2\cap I\cdot I:I.$$



Claim. under $\pi: I^{:p}\to V^{\otimes p}$, $R_{p,i}\mapsto R_{p,i}$.

Namely,

$$\pi(I^{:m}:\underset{}{\overset{\ker(I:I\to I^2)\,\shortparallel}{R_2}}:I^{:n})=V^{\otimes m}\otimes \underset{\shortparallel\,\ker(V\otimes V\to I^2/I^3)}{R_2}\otimes V^{\otimes n}$$

✓ Put in!

The claim is the Peter-statement that $\operatorname{coker}\partial^{Ind}=U_m$; it does not follow merely from the hexagon.

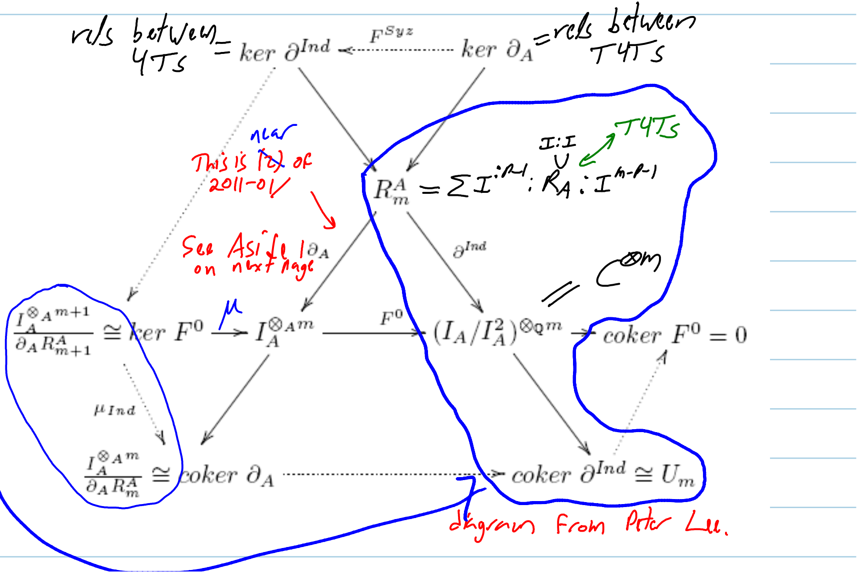


diagram from Peter Lee.

it does not follow merely from the hexagon.

$$\begin{tikzcd}
 & & & R^A_m \arrow[dl, "\partial_A"'] \arrow[dr, "\partial^{Ind}"] & & & \\
\frac{I_A^{\otimes_A m+1}}{\partial_A R^A_{m+1}} \cong \ker F^0 \arrow[rr, "\mu"] \arrow[dr, dotted, "\mu_{Ind}"'] & & I_A^{\otimes_A m} \arrow[rr, "F^0"] \arrow[dl] & & (I_A/I_A^2)^{\otimes_{\mathbb{Q}} m} \arrow[rr] \arrow[dr] & & \operatorname{coker} F^0 = 0 \\
 & \frac{I_A^{\otimes_A m}}{\partial_A R^A_m} \cong \operatorname{coker} \partial_A \arrow[rrrr, dotted] & & & & \operatorname{coker} \partial^{Ind} \cong U_m \arrow[ur, dotted] &
\end{tikzcd}$$

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diagram from Peter Lee.

The Pure Virtual Braid Group is Quadratic, II

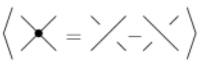
Examples and Interpretations

Dror Bar–Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

Example. (goes back to [Koh])


T. Kohno

$K=$  $I=$ 

$$(K/I^{p+1})^\star=(\text{invariants of type } p)=:\mathcal{V}_p$$

$$(I^p/I^{p+1})^\star=\mathcal{V}_p/\mathcal{V}_{p-1}\quad V=\langle t^{ij}|t^{ij}=t^{ji}\rangle=$$ 

$$\ker\bar\mu_2=\langle[t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle\text{4T relations}\rangle$$

$$A=q(K)=\begin{pmatrix}\text{horizontal chord dia-}\\\text{grams mod 4T}\end{pmatrix}=$$  4T

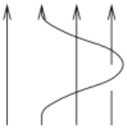
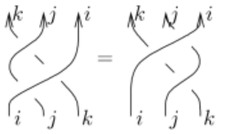
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group


L. Kauffman [Kau, KL]

$$\langle\sigma_{ij}:1\leq i\neq j\leq n\rangle\Big/\begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\\sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}$$

of "pure virtual braids" ("braids when you look", "blunder braids"):

$\sigma_{24}=$  R3: 

The Main Theorem [Lee]. PvB_n is quadratic.

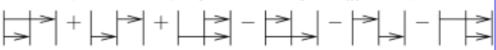
$A_n=q(PvB_n)$. [GPV]


Goussarov-Polyak-Viro

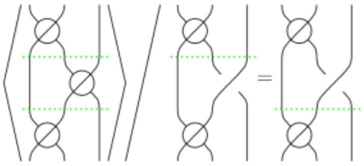
$I=$  with $\tilde\sigma_{ij}=\sigma_{ij}-1$, the "semi-virtual crossing".

$V=I/I^2=\left\langle\begin{matrix}\text{v-braids}\\\text{with one}\end{matrix}\right\rangle\Big/$ $=\langle a_{ij}\rangle_{1\leq i\neq j\leq n}$ $a_{24}=$ 

$$A_n=TV/\langle[a_{ij},a_{ik}]+[a_{ij},a_{jk}]+[a_{ik},a_{jk}],\ c^{ij}_{kl}=[a_{ij},a_{kl}]\rangle,$$

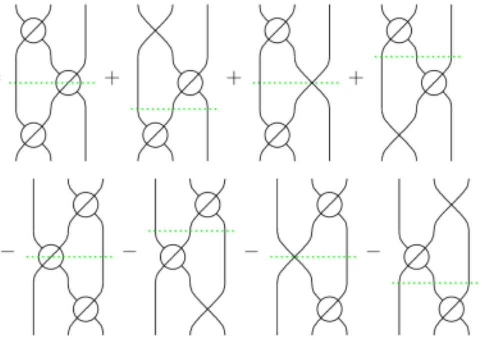
$y_{ijk}=$ 

$I^{:p}$.



James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.

$\mathfrak{R}_2(PvB_n)$ is generated as a vector space by C^{ij}_{kl} and

$Y_{ijk}:=$ 

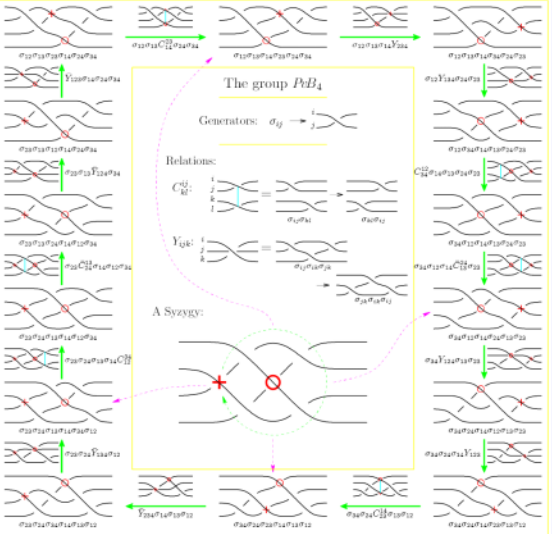
Syzygy Completeness, for PvB_n, means:

$$\mathfrak{R}_p=\bigoplus_{j=1}^{p-1}\mathfrak{R}_{p,j}\xrightarrow{\partial}I^{:p}\xrightarrow{\pi}V^{\otimes p}$$

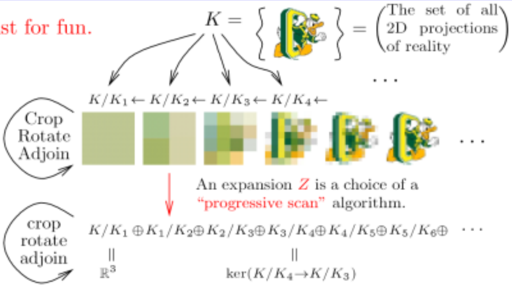
$$\{\tilde\sigma_{12}:\underline{Y_{345}}:[illegible]\tilde\sigma_{89}:\ldots\}\longrightarrow$$

$$\{\tilde\sigma_{12}:Y_{345}:[illegible]\tilde\sigma_{89}:\ldots\}\longrightarrow\{a_{12}y_{345}a_{89}\ldots\}$$

Is every relation between the y_{ijk}'s and the c^{ij}_{kl}'s also a relation between the Y_{ijk}'s and the C^{ij}_{kl}'s?



Just for fun.



$K=\{\ \}=$ (The set of all 2D projections of reality)

…

$K/K_1\leftarrow K/K_2\leftarrow K/K_3\leftarrow K/K_4\leftarrow$

Crop Rotate Adjoin

…

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$K/K_1\oplus K_1/K_2\oplus K_2/K_3\oplus K_3/K_4\oplus K_4/K_5\oplus K_5/K_6\oplus\ \cdots$

$\|$ $\mathbb{R}^3$ $\quad\|$ $\ker(K/K_4\to K/K_3)$