Free Group Rings vs. Free Algebras

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The General Case. If $K = F/M$ and $I \subset K$, then $I = J/M$ where $J \subset F$. Then $I^{:p} = J^{:p} / \sum J^{:j-1} : M : J^{:p-j}$ and we

$xy - 1 = (x-1)y + y - 1$

Here $K = \mathbb{Q}FG_n(x_n)$ & $F = FA_{2n}(x_i, y_i)$ so

$$M = \langle x_i y_i - 1 \rangle = \langle (x_i - 1)y_i + (y_i - 1) \rangle$$
$$= \langle (x_i - 1) + (y_i - 1) + (x_i - 1)(y_i - 1) \rangle$$

So $I = \langle \bar{x}_i, \bar{y}_i \rangle / \bar{x}_i + \bar{y}_i + \bar{x}_i \bar{y}_i = 0$

Hence $V = I/I^2 = \langle \bar{x}_i, \bar{y}_i \rangle_{v.s.} / \bar{y}_i = -\bar{x}_i$

$\mathfrak{R}_2$ is simpler than may seem! It's an "augmentation bimodule" ($I\mathfrak{R}_2 = 0 = \mathfrak{R}_2 I$ thus $xr = \epsilon(x)r = r\epsilon(x) = rx$ for $x \in K$ and $r \in \mathfrak{R}_2$), and hence $\mathfrak{R}_2 = \pi_2(\mu_F^{-1}M)$.

$$\begin{tikzcd} J^{:2} \arrow[r, "\mu_F"', "\text{1-1}"] \arrow[d, "\pi_2"] & J \supset M \arrow[d, "\pi_1"] \\ I^{:2} \arrow[r, "\mu"] & I = J/\langle M \rangle \end{tikzcd}$$

$\mathfrak{R}_p$ is simpler than may seem! In $\mathfrak{R}_{p,j} = I^{:j-1} : \mathfrak{R}_2 : I^{p-j-1}$ the I factors may be replaced by $V = I/I^2$. Hence

$$\mathfrak{R}_p \simeq \bigoplus_{j=1}^{p-1} V^{\oplus j-1} \otimes \pi_2(\mu_F^{-1}M) \otimes V^{\otimes p-j-1}.$$

$\mu_F^{-1}(M) = 0$

$R_2 = \pi_2(\mu_F^{-1} M)$
$= 0$

So $I^{:2} \xrightarrow{\mu} K$

is injective.

R_2 is $\ker\left(I/I^2 \otimes I/I^2 \to I^2/I^3\right)$