An Ideal-Theoretic Hutchings Theory

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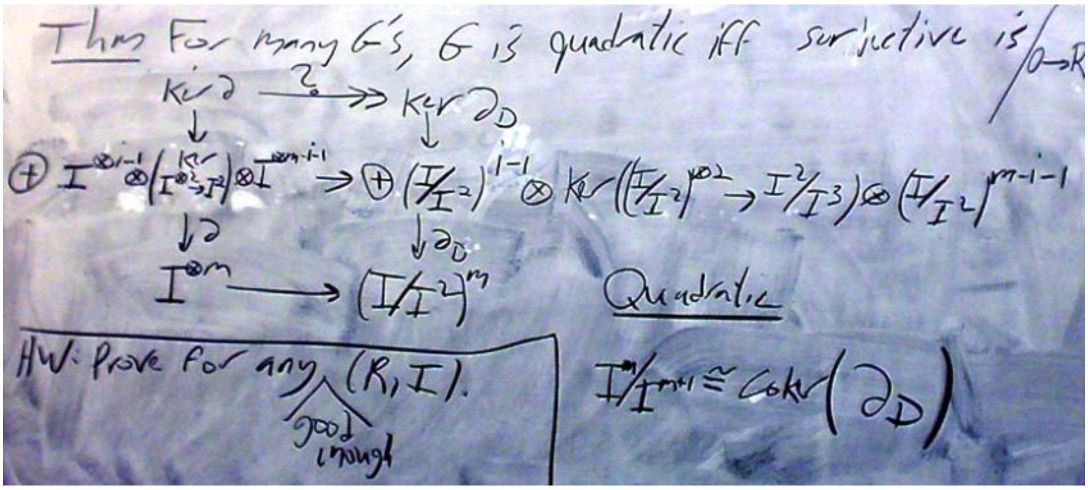
7:09 AM



From http://katlas.math.toronto.edu/drorbn/bbs/show?shot=LeeP-110107-160837.jpg

Definition. Given an ideal I in a (non-commutative) ring K (w/ product μ) set:

$$\mathcal{D}_1 = I/I^2 \qquad \mathcal{D}_m = \mathcal{D}_1^{\otimes m} \qquad \mathcal{G}_m = I^m/I^{m+1}$$

change to $\otimes_{\otimes}$

$$R_2^D = \ker\left(\mathcal{D}_2 \xrightarrow{\mu} \mathcal{G}_2\right) \subset \mathcal{D}_2$$

$$R_m^D = \bigoplus \mathcal{D}_{i-1} \otimes R_2^D \otimes \mathcal{D}_{m-i-1} \subset \bigoplus_m \mathcal{D}_m$$

$\partial_D : R_m^D \longrightarrow \mathcal{D}_m$ the sum of the obvious inclusions.

$A_m = \operatorname{coker}(\partial_D)$.

(note that there is $\overbrace{\text{a surjection}}^{\text{well-defined}}$ $\mu : A_m \to \mathcal{G}_m$)

$$K_m = I^{\otimes m} \qquad R_2^T = \ker\left(K_2 \xrightarrow{\mu} I^2\right)$$

$$R_m^T = \bigoplus K_{i-1} \otimes R_2^T \otimes K_{m-i-1} \subset \bigoplus_m K_m$$

$\partial_T : R_m^T \longrightarrow K_m$ the sum of the obvious inclusions.

Definition. (K,I) is "quadratic" if $\mu : A_m \longrightarrow \mathcal{G}_m$ is an isomorphism.

"if" might be sufficient.

Theorem. (K,I) is quadratic iff the map $\ker\partial_T \longrightarrow \ker\partial_D$ coming from the diagram below is a surjection:

$$\ker\partial_T \longrightarrow R_m^T \xrightarrow{\partial_T} K_m$$

$$\begin{array}{ccccc} \downarrow & & \downarrow \pi & & \downarrow \pi \\ \ker \partial_D & \longrightarrow & R_m^D & \xrightarrow{\partial_D} & \mathcal{D}_m \end{array}$$

The way that this may be different than Hutchings is that Hutchings also assumes that

$$\operatorname{im} \partial_T = \ker(\mu : K_m \longrightarrow K_{m-1})$$

What means $\ker \delta \supset \ker \delta^2$?

$$K_{n+1} \xrightarrow{\delta} K_n \xrightarrow{\delta} K_{n-1} \xrightarrow{\delta} K_{n-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} K_0 = k$$

Claim If $\ker \delta = \ker \delta^2$, then $\dfrac{\delta^n K_n}{\delta^{n+1} K_{n+1}} \cong \dfrac{K_n}{\delta K_{n+1} + \ker \delta}$

Proof $\dfrac{\delta^n K_n}{\delta^{n+1} K_{n+1}} \xleftarrow[\delta]{\sim} \dfrac{\delta^{n-1} K_n}{\delta^n K_{n+1}} \xleftarrow[\delta]{\sim} \cdots \xleftarrow[\delta]{\sim} \dfrac{\delta K_n}{\delta^2 K_{n+1}} \xleftarrow[\delta]{\sim} \dfrac{K_n}{\delta K_{n+1} + \ker \delta}$

$0 = [\delta \alpha] \longleftarrow [\alpha]$

means $\delta\beta = \alpha \in \delta^{n-1} K_n$, $\delta\alpha \in \delta^{n+1} K_{n+1} = \delta^2(\delta^{n-1} K_{n+1})$

so $\underset{\delta^2\beta}{\delta\alpha} = \delta^2 \gamma$, $\gamma \in \delta^{n-1} K_{n+1}$

So $\delta^2(\beta - \gamma) = 0$ so

$\delta(\beta - \gamma) = 0$ so

$\alpha = \delta\gamma \in \delta^n K_{n+1}$

$0 = [\delta\alpha] \longleftarrow [\alpha]$

means $\delta\alpha \in \delta^2 K_{n+1}$ so $\exists \beta \in K_{n+1}$

$\delta\alpha = \delta^2\beta \Rightarrow \delta(\alpha - \delta\beta) = 0$

$\Rightarrow \alpha = \underset{\delta K_{n+1}}{\underset{\cap}{\delta\beta}} + \underset{\ker \delta}{\underset{\cap}{(\alpha - \delta\beta)}}$

In our case, $\dfrac{\delta^n K_n}{\delta^{n+1} K_{n+1}} \cong \dfrac{K_n}{\delta K_{n+1} + \ker \delta}$ becomes $I^n / I^{n+1} \cong$ not the right thing?

Question. What is $K_n / \delta K_{n+1}$, i.e., $I^{\otimes n} / \delta I^{\otimes(n+1)}$?

$$(I/I^2) \otimes \cdots \otimes (I/I^2) \longleftrightarrow I^{\otimes n} / \delta I^{\otimes(n+1)} \quad ✓$$

Question. What is $R_m^T / \delta R_{m+1}^T$? First, what's δ?

$$R_m^T = \bigoplus I^{\otimes(i-1)} \otimes (\ker I^{\otimes 2} \to I^2) \otimes I^{\otimes(m-i-1)}$$

? The literal analog of K_m' seems more to be like

Lo $\qquad \bigoplus_i I^{\otimes(n-2)} \otimes_{\mathbb{Q}} \underbrace{\left(\ker I^{\otimes 2} \to I^2\right)}_{\text{at position } i}$ $\Big\}$ but there's no ∂_T map here.

$$\begin{array}{ccccccccc|ccccccc}
\cdots \xrightarrow{\delta} & \mathcal{K}^1_{n+1} & \xrightarrow{\delta} & \mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\delta} \cdots & & & \mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}^1_{n-1} & \longrightarrow 0 & \\
& \downarrow b & & \downarrow b & & \downarrow b & & & & \downarrow b & & \downarrow b & & \downarrow \bar{b} & & \text{(exact rows).} \\
\cdots \xrightarrow{\delta} & \mathcal{K}_{n+1} & \xrightarrow{\delta} & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\delta} \cdots, & & & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}_{n-1} & \longrightarrow 0 &
\end{array}$$

Pasted from <http://www.math.toronto.edu/~drorbn/papers/Species/1_4Hutchings_theory_integra.html>

$$\begin{array}{ccccccccccccc}
\mathcal{K}_n & & \mathcal{K}_{n-1} & & \mathcal{K}_{n-1} & & \mathcal{K}^1_{n-1} & & \mathcal{K}^1_{n-1} & & \mathcal{K}^1_{n-1} & & \bar{\mathcal{K}}^1_{n-1} \\
\uplus & & \cup & & \cup & & \uplus & & \uplus & & \uplus & & \uplus \\
\dfrac{\ker \delta^2}{\ker \delta} & \xrightarrow{\delta} & \ker \delta \cap \operatorname{im} \delta & \overset{?}{=} & \operatorname{im} b \cap \ker \pi & \xleftarrow{b} & \dfrac{\ker \pi \circ b}{\ker b} & = & \dfrac{\ker \bar{b} \circ \pi}{\ker b} & = & \dfrac{\pi^{-1}(\ker \bar{b})}{\ker b} & \xrightarrow{\pi} & \dfrac{\ker \bar{b}}{\pi(\ker b)}.
\end{array}$$

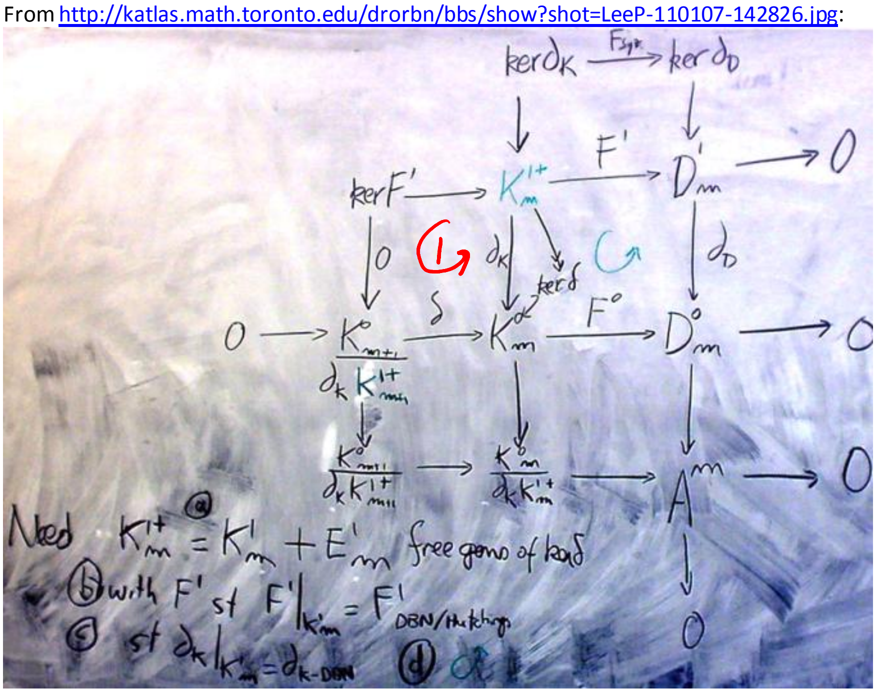
perspective | switch

Question. Do we need $\operatorname{im} b = \ker \delta$ above? Perhaps an inclusion $\operatorname{im} b \subset \ker \delta$ is enough? No, that seems to be the wrong direction.

Question. How much is δ'_n, as in $K'_n \xrightarrow{\delta'_n} K'_{n-1} \xrightarrow{\pi} \bar{K}'_{n-1} \to 0$, used/is necessary in the above?

Ans. We need to know that δ'_n is a surjection. Otherwise we seem not to care about K'_n; I don't see that we ever use the exactness of $K'_n \xrightarrow{b} K^0_n \xrightarrow{\delta} K^0_{n-1}$.

From http://katlas.math.toronto.edu/drorbn/bbs/show?shot=LeeP-110107-142826.jpg:



Big issue (?): Is (1) commutative?

(b) might be hard to meet.

From papers/Fundamental:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{K}^1_{m+1} \arrow[r, "\delta"] \arrow[d] & \ker \partial|_{\mathcal{K}^1_m} \arrow[r, "F"] \arrow[d] & \ker \partial|_{\mathcal{D}^1_m} \arrow[d] \arrow[ddll, dashed, rounded corners] & \\
 & \mathcal{K}^1_{m+1} \arrow[r, "\delta"] \arrow[d, "0"'] \arrow[dr, phantom, "1"] & \mathcal{K}^1_m \arrow[r, "F"] \arrow[d, "\partial"'] & \mathcal{D}^1_m \arrow[r] \arrow[d, "\partial"'] \arrow[dl, "2\ \exists ?"] & 0 \\
0 \arrow[r] & \mathcal{K}^0_{m+1}/\partial\mathcal{K}^1_{m+1} \arrow[r, "\delta"] \arrow[d] & \mathcal{K}^0_m \arrow[r, "F"] \arrow[d] & \mathcal{D}^0_m \arrow[d] & (\longrightarrow\ 0) \\
 & \mathcal{K}^0_{m+1}/\partial\mathcal{K}^1_{m+1} \arrow[r, "\delta"] \arrow[d] & \mathcal{K}^0_m/\partial\mathcal{K}^1_m \arrow[r] \arrow[d] & \mathcal{A}^r_m \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The commutativity of 1 allows the existence of 2.