CHECKING THE HUTCHINGS CRITERION

"Topological syzygies" include the "Zamolodchikov tetrahedron"

$F^{syz}(Zam.) = $ Zam. with $a_{ij} \mapsto (\sigma_{ij}+1)$ keeping only lowest degree in σ_{ij}

Query: $F^{syz}(Zam) \overset{?}{=} \ker \partial$?

What is $\ker \partial$ (syzygies in pvb_n)

FACT: $A = pvb_n$ is Koszul (BEER, Lee)

HENCE: $\bigoplus_{i<j}(R_{p,i} \cap R_{p,j}) \to \bigoplus_i R_{p,i} \to V^{\otimes p} \to A^p \to 0$

is exact, where

$$\begin{tikzcd} & R_{p,i} \arrow[dr, "+"] & \\ R_{p,i}\cap R_{p,j} \arrow[ur] \arrow[dr] \arrow[rr, dashed] & & V^{\otimes p} \\ & R_{p,j} \arrow[ur, "-"'] & \end{tikzcd}$$

"Trivial syzygies" $R_{p,i} \cap R_{p,j}$ if $|i-j|>1$

Non-trivial: $R_{p,i} \cap R_{p,i+1} \cong V^{\otimes i} \otimes A^{!3} \otimes V^{\otimes p-i-3}$

via: $(\Delta^!_{1,1} \otimes \mathrm{Id}) \circ \Delta^!_{2,1} : A^{!3*} \xrightarrow{\sim} R \otimes V \cap V \otimes R$ (*)

where $\Delta^!_{\bullet,1}$ is dual to $m^! : A^{!\bullet} \otimes A^{!1} \to A^{!\bullet+1}$

BASIS $A_n^{!k}$ has basis indexed by unordered partitions of $[n] = \{1,\dots,n\}$ into $(n-k)$ ordered subsets "Lah numbers $L(n,n-k)$"

"Chain gangs" via $\left.\begin{matrix} i_1 \to i_2 \to \dots \to i_k \\ j_1 \to \dots \to j_\ell \end{matrix}\right\} \leftrightarrow a_{i_1 i_2} \wedge \dots \wedge a_{i_{k-1} i_k} \wedge a_{j_1 j_2} \wedge \dots$

Ind degree 3 primarily get all chains

$i_1 \to i_2 \to i_3 \to i_4$

These are easily calculated by (*) and $= F^{syz}(Zam)$

Also get $\left.\begin{matrix} i_1 \to i_2 \to i_3 \\ i_4 \to i_5 \end{matrix}\right\}$ ie $i_1\ i_2\ i_3\ i_4\ i_5$

and $\left.\begin{matrix} i_1 \to i_2 \\ i_3 \to i_4 \\ i_5 \to i_6 \end{matrix}\right\}$ ie $i_1\ i_2\ i_3\ i_4\ i_5\ i_6$

Proof of Basis

A relations: $[a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}] = 0$

$[a_{ij}, a_{ke}] = 0$

$A^!$ relations:

$a_{ij} \wedge a_{ik} = a_{ij} \wedge a_{jk} - a_{ik} \wedge a_{kj}$ (V)

$a_{ik} \wedge a_{jk} = a_{ij} \wedge a_{jk} - a_{ji} \wedge a_{ik}$ (A)

$a_{ij} \wedge a_{ji} = 0$ (No loop)

Idea: V: V-join, Λ: A-join }: relations replace A/V-joins by directed segments.

- Define a "Defect" function on (AS) monomials st the maximal term in a relation is preserved by multiplication that produces no loops ("Multiplicative")
- Verify that Defect 0 monomials = Chain Gangs
- Defect 0 monomials generate $A^!$:
  - If a monomial's graph has a loop, it is 0.
  - Multiplicative ⇒ forests can be reduced to Defect 0 monomials.
- Prove that Defect 0 monomials are Indep.

Defect
- A pair of vertices is "unordered" if there is no oriented sequence of edges between them.
- Defect of a tree is # unordered pairs of vertices in it.
- Defect of a forest is sum of defects of its trees.
- Clear that Defect 0 monomials ⟷ Chain Gangs.
- MAX: In (V) and (A) the join terms have maximal defect.

Multiplicativity Induction on # of vertices in relation

Show: if vertices a, b are ordered in join term, they are ordered in other terms.

Hence: adding edges either increases Defect or preserves it. Now apply "MAX"

3 Cases: Case I: $a \to b$ new edge with a, b new vertices: clear. (connected components are same in all terms)

Case II: new edge $a - b$ with (a) new, (b) old vertex, (c) any other old vertex.

Suppose $a \to \dots \to c$ in join term, then in fact $a \to b \to \dots \to c$, hence $b \to \dots \to c$ in old join term, hence $b \to \dots \to c$ in all other old terms, hence $a \to b \to \dots \to c$ in all new terms.

Case III new edge $a - b$ with (a)(b) old vertices, (c) any other old vertex

Let (c), (d) old vertices in components of (a), (b) resp.

If $c \to \dots \to d$ in join term, then in fact $c \to \dots \to a \to b \to \dots \to d$, hence $c \to \dots \to a$ and $b \to \dots \to d$ in old join, hence $c \to \dots \to a$ and $b \to \dots \to d$ in other old terms, so $c \to \dots \to d$ also.

Loops are 0 Oriented loops:

Get shorter oriented loop, use induction and

Unoriented loops:

Get shorter loops, use induction and ditto.

Chain Gangs are Linearly Independent

- Ie, all reductions of forests to chain gangs give same result: use induction on size of defect.
- True for forests of defect 1 as ∃! reduction.
- Suppose true for defect ≤ p.

  Let F be forest w/ Defect p+1, ≥ reductions.

$$\begin{tikzcd} & F \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[dr, "b"'] & & B \arrow[dl, "a"] \\ & C & \end{tikzcd}$$

- If a, b do not overlap, have common reduction C.
- Defect A, B ≤ p so apply induction.

$$\begin{tikzcd} & F \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[d, "a'"'] & & B \arrow[d, "b'"] \\ A' \arrow[dr, "a''"'] & & B' \arrow[dl, "b''"] \\ & C & \end{tikzcd}$$

- If a, b overlap, find a', a'', b', b'' leading C
- Apply induction.

Overlaps 3 types (by inspection)

One checks "by hand" that all ways of reducing give same result.

KOSZULNESS "A quotient of an exterior algebra with quadratic Grobner basis (as exterior algebra) is Koszul (Yuzvinsky).

Order on AS monomials:
- order generators numerically: $a_{12} < a_{13} < \dots < a_{21} < a_{23} \dots$
- Order factors in monomial in increasing order.
- Order monomials length-lexigographically. (ignore signs, $0 < \alpha \neq 0$)
- Multiplicative: if $\alpha < \beta$ then $\alpha\gamma < \beta\gamma$ if $\beta\gamma \neq 0$.
- Relations are equivalent to relations with maximal terms: all $a_{ij} \wedge a_{jk}$, $i<k<j$

Constancy of Dimension

are all length 2 legal monomials

Down Up

Let $S_n^{(k)}$ be legal monomials of length k.

- $S_n^{(k)}$ generates $A_n^{!k}$ by multiplicativity.
- $\dim S_n^{(k)} = L(n, n-k) = \dim A_n^{!k}$ by BASIS Chain Gangs.

Up Graphs: $\dim S_n^{Up(k)} = s(n, n-k)$

Down Graphs: $\dim S_n^{Down(k)} = 1$.

Up-Down Graphs: Given collection of Up Graphs, only allow Down arrows between roots of the Up Graphs.

- Counting: (a) Divide [n] into ℓ cycles and place Up Graphs on the cycles. Let m_i be root of cycle C_i.
- (b) Partition $M = \{m_1, \dots, m_\ell\}$ as $M_1 \sqcup \dots \sqcup M_k$ and place Down Graphs on the M_i.

- $\dim S_n^{(n-k)} = \sum_{\ell=k}^{n} s(n,\ell) S(\ell,k) = L(n,k) = \dim A_n^{!(n-k)}$