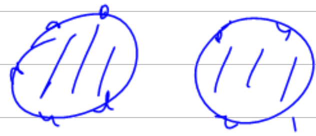
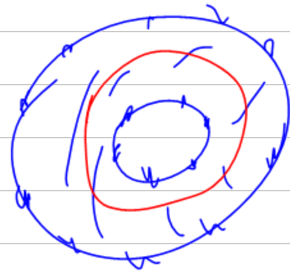
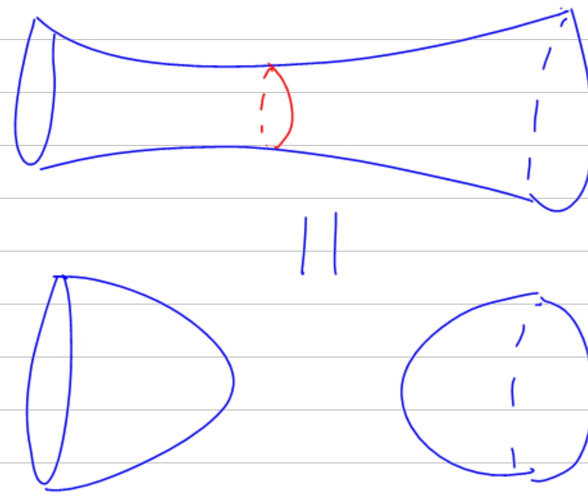


## Neck cutting

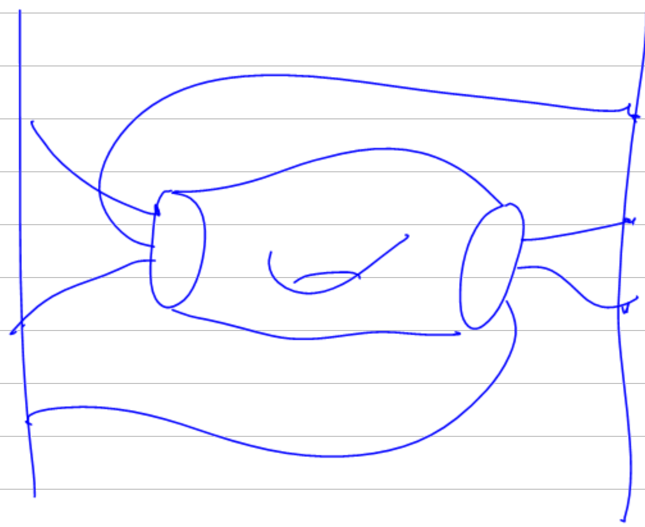
$$[\,,\,] : \overbrace{A \otimes A} \longrightarrow A$$

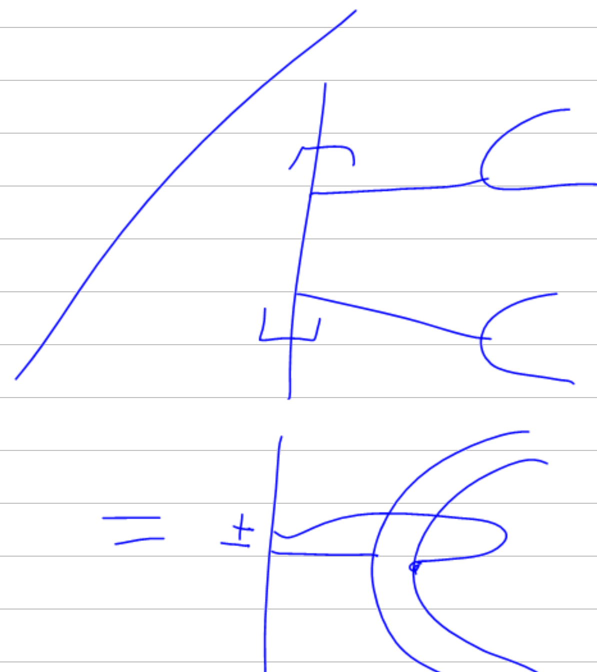
$$\delta : A \longrightarrow \underbrace{A \otimes A}$$





Not respect grading.

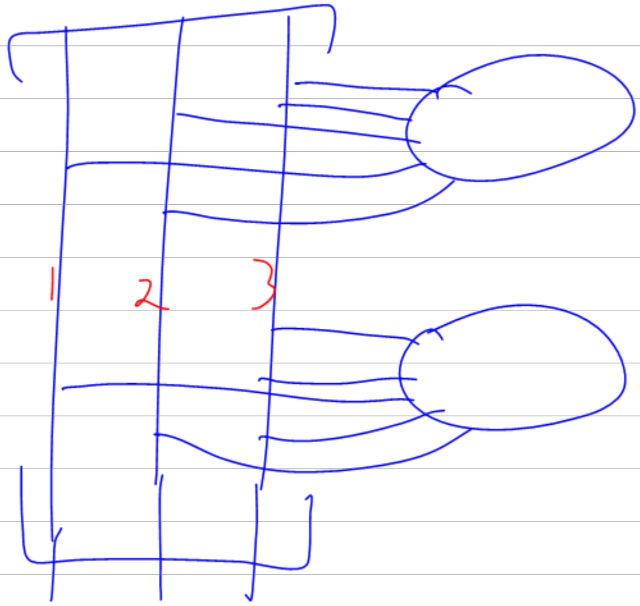





**Bracket**

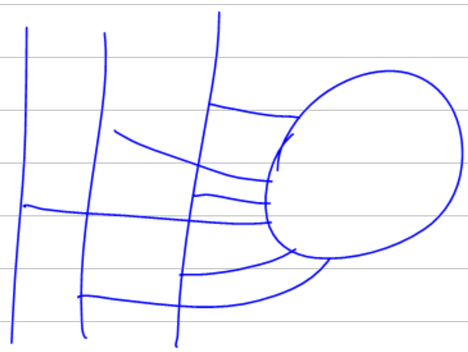
/ $g > 0$

$b > 1$




$=$

Co-bracket: $\mathcal{S} / _{bc>2}$



$\mathcal{S}\mathcal{G}F$ $\qquad$ $\mathrm{gr}\,\mathcal{S} : \mathrm{gr}\,F$

$\mathcal{G} = \mathrm{gr}\,F$

If $\mathrm{gr}\,\mathcal{S} = 0$ $\qquad$ $\mathrm{gr}'\mathcal{S} : \mathcal{G}_{\cancel{\infty}} \to \mathcal{G}_{\cancel{\infty}+1}$

$$\left(\mathcal{S}^c / _{g>1}\right)\Big|_{bc \geq k} = F_k$$

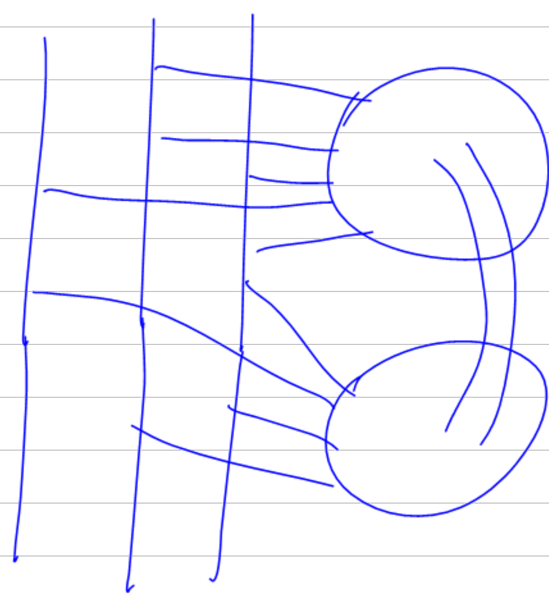
$G = \operatorname{gr} F = \prod F_k / F_{k+1}$

$G_1 = F_1 / F_2 = CW$

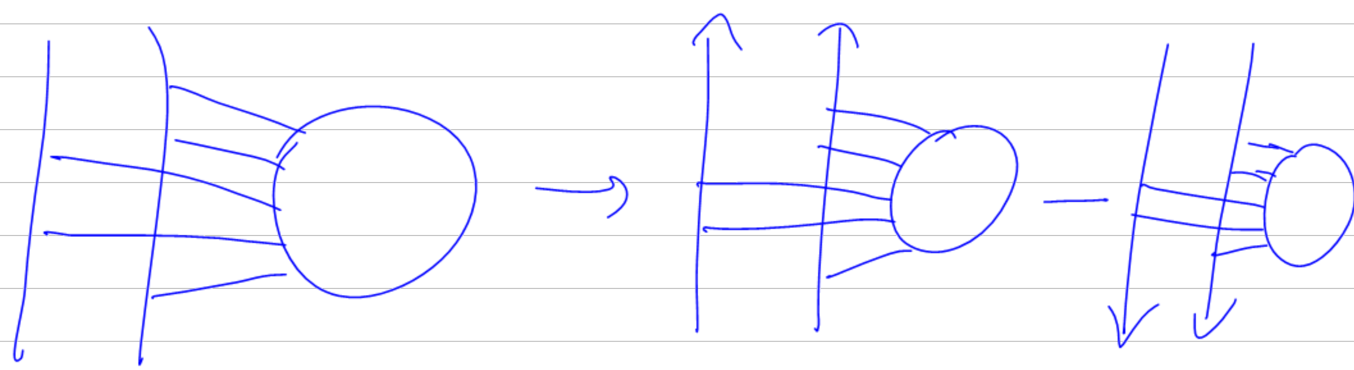
$S \in G_2 F$ flip dancer over.

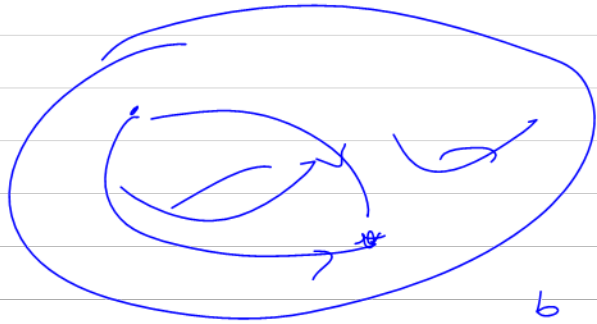
$\operatorname{gr} S = 0$

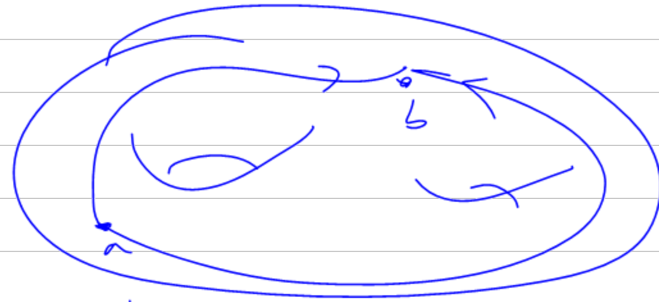
$\operatorname{gr} S \big|_{G_1} = \delta$

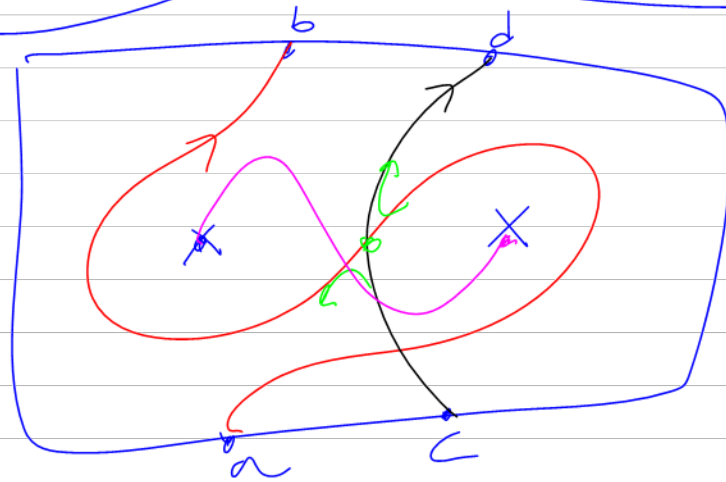


$S$:

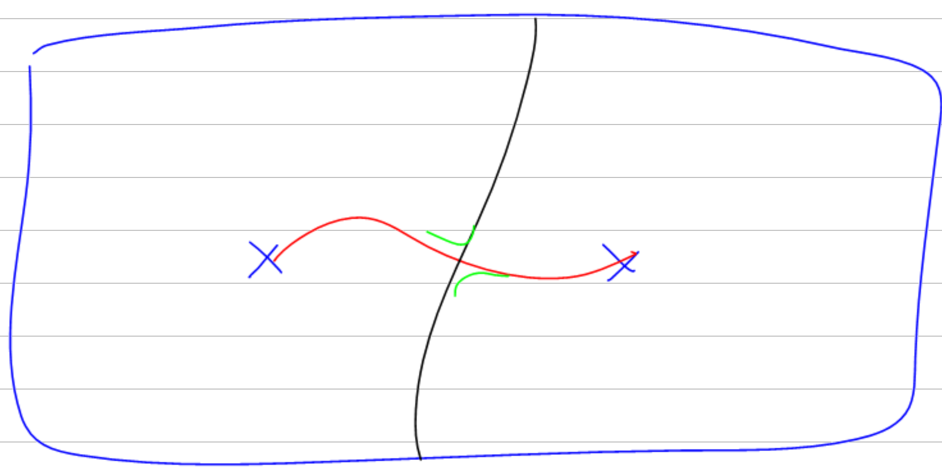


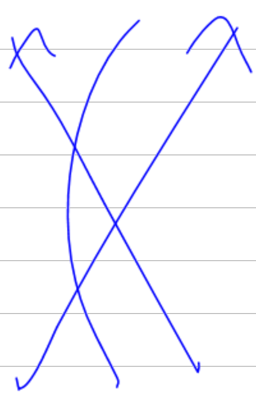


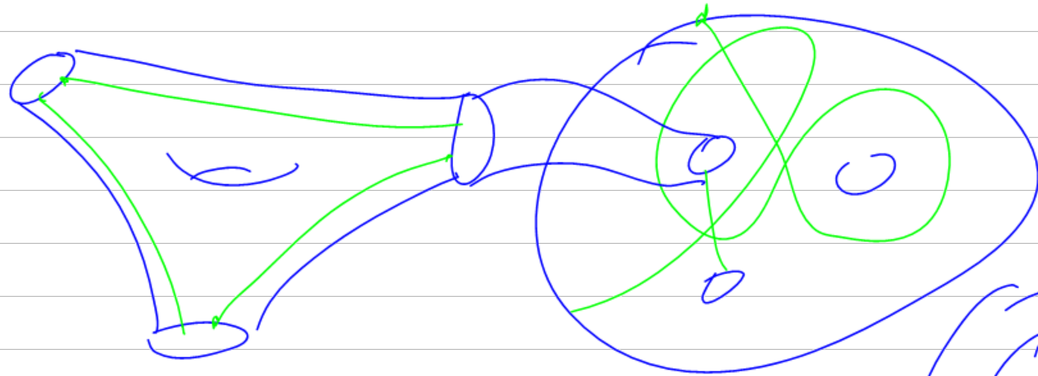


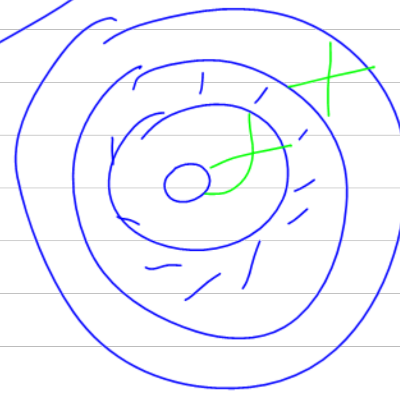

$$\pi_1(a,b)\otimes\pi_1(c,d)\to\pi_1(a,d)\otimes\pi_1(c,b)$$





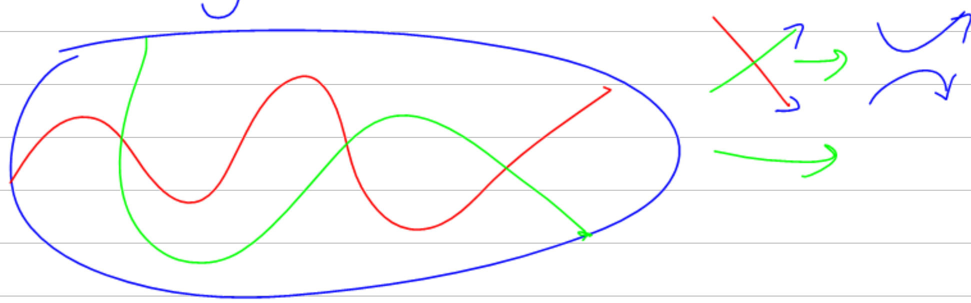


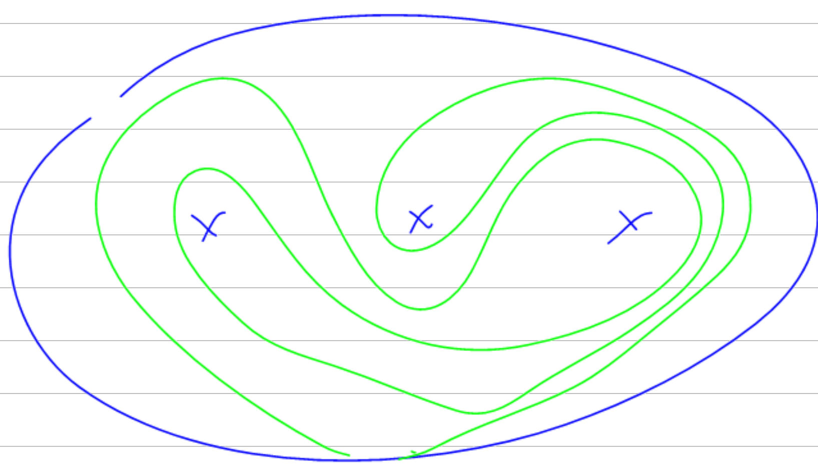
1. Disjoint union
2. Unary "add tunnel"



3. stacking "Folding"

4.





$QFG(x_1 \ldots x_n) \supset I = VS\langle w-1 \rangle$

$= I\langle \underbrace{x_i - 1}_{t_i} \rangle$

$x_1 x_2 x_3 - 1 = (x_1 - 1) + x_1(x_2 - 1)$
$+ x_1 x_2 (x_3 - 1)$

$x^{-1} - 1 = -x^{-1}(x-1)$

<u>Claim</u> $gr\,QG = A\langle t_i \rangle$ $\quad I^k / I^{k+1}$

$\phi$

$t_{\alpha_1} \cdots t_{\alpha_k} \longmapsto [(x_{\alpha_1} - 1) \cdots (x_{\alpha_k} - 1)]$

$I^k = $ l.c. of

$w_1(x_{\alpha_1} - 1) w_2 (x_{\alpha_2} - 1) \cdots w_k (x_{\alpha_k} - 1)$

<u>Aside</u> $(x_i - 1) w = \sum w_j (x_j - 1)$

<u>Pf</u> $(x_i w - 1) - (w - 1) = \nearrow$

$I^k = VS\langle w(x_{\alpha_1} - 1) \cdots (x_{\alpha_k} - 1) \rangle$

$w \prod (x_{\alpha_i} - 1) = \prod (x_{\alpha_i} - 1) + \underbrace{(w-1) \prod (x_{\alpha_i} - 1)}_{I^{k+1}}$

$$\mathcal{I}^k/\mathcal{I}^{k+1} = \left\langle \prod (x_{\alpha_i}-1) \right\rangle$$

$\boxed{Z \text{ is } 1\text{-}1}$

$$Z : FG_n \longrightarrow \widehat{FA}_n$$

$$x_i^{\pm 1} \longrightarrow e^{\pm t_i} \qquad x_i \to 1+t_i$$

$$x_i^{-1} \to 1 - t_i + t_i^2 - t_i^3 + \dots$$

$$(x_i - 1) \longmapsto e^{t_i} - 1 = t_i(\dots)$$

$$\operatorname{gr} Z : \mathcal{I}^k/\mathcal{I}^{k+1} \longrightarrow \text{deg } k \text{ in } FA.$$

---

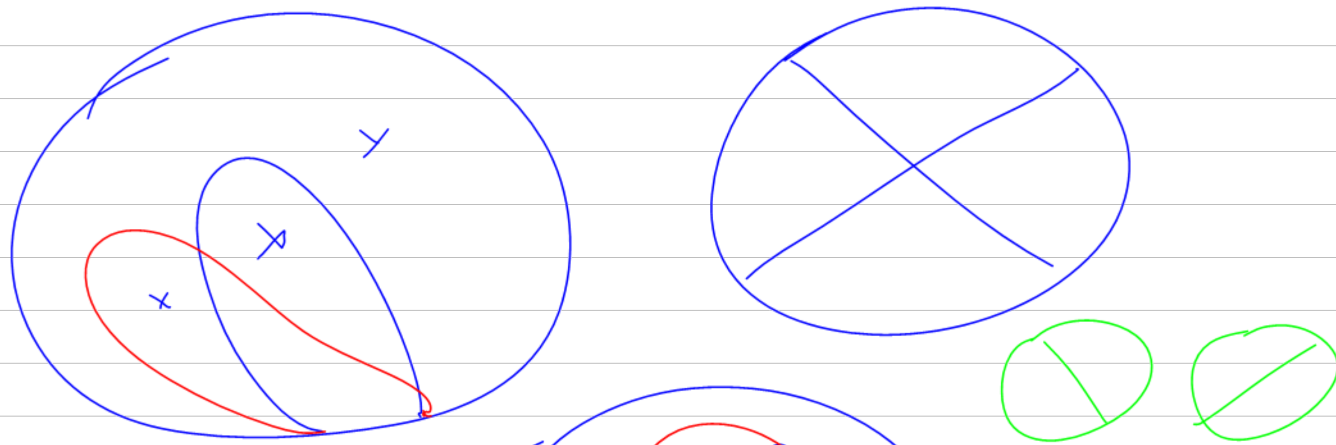
$$Z(\alpha) = Z(\beta) \Rightarrow Z(\underbrace{\alpha\beta^{-1}}_{\gamma}) = 1$$

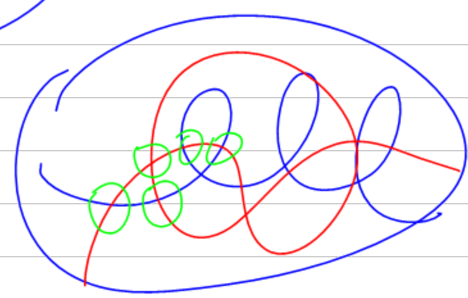
$$Z(\gamma) = (1 + p_1 t_{i_1} + \dots)(1 + p_2 t_{i_2} + \dots) \dots (1 + p_k t_{i_k} + \dots)$$

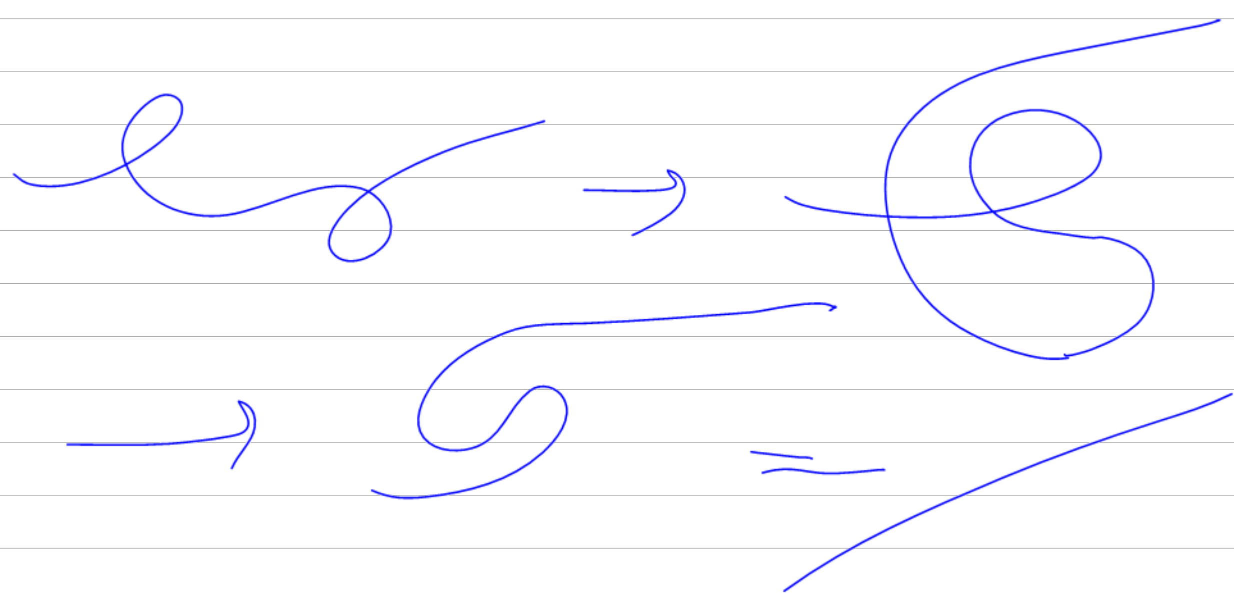
$$\gamma = x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k} \qquad i_\alpha \neq i_{\alpha+1} \quad p_i \neq 0$$

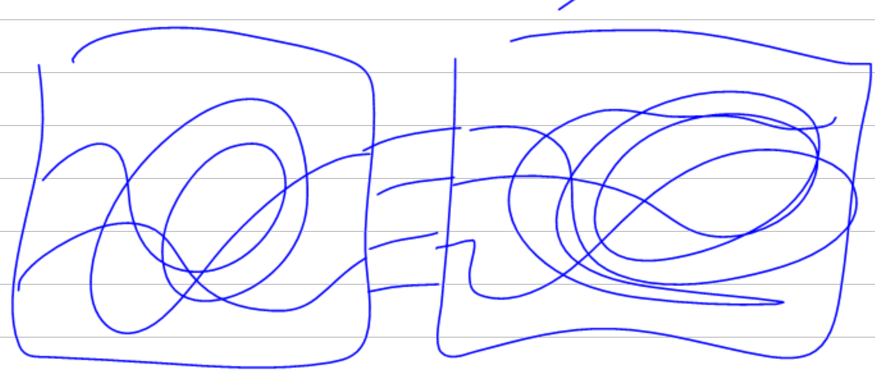
$$\prod_{\alpha=1}^{k} t_{i_\alpha}^{p_\alpha} \qquad i_\alpha \neq i_{\alpha+1} \quad p_\alpha \neq 0 \quad \text{"}k\text{-runs"}$$

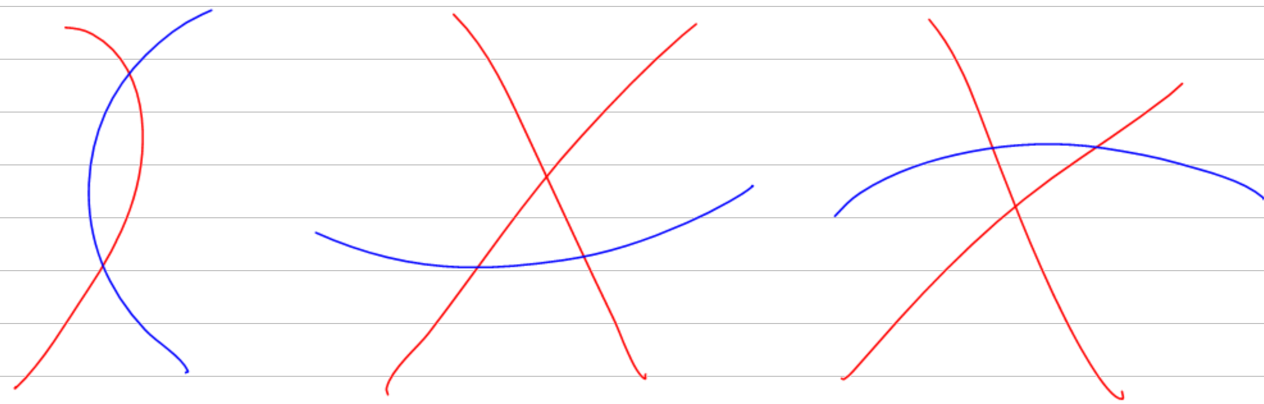
max runs min deg $\quad \underbrace{\prod p_i}_{\neq 0} \ \underbrace{\prod t_{i_\alpha}}_{\neq 0}$

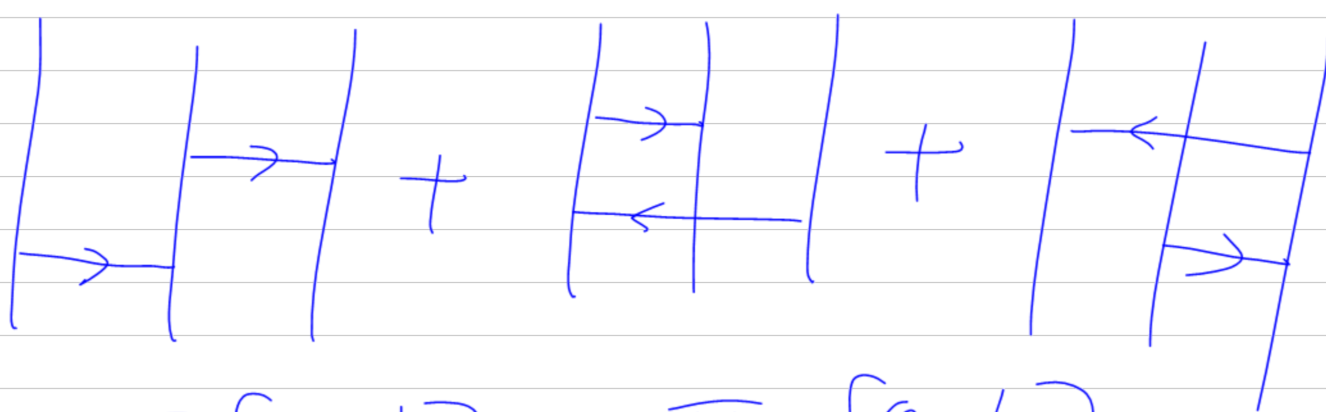




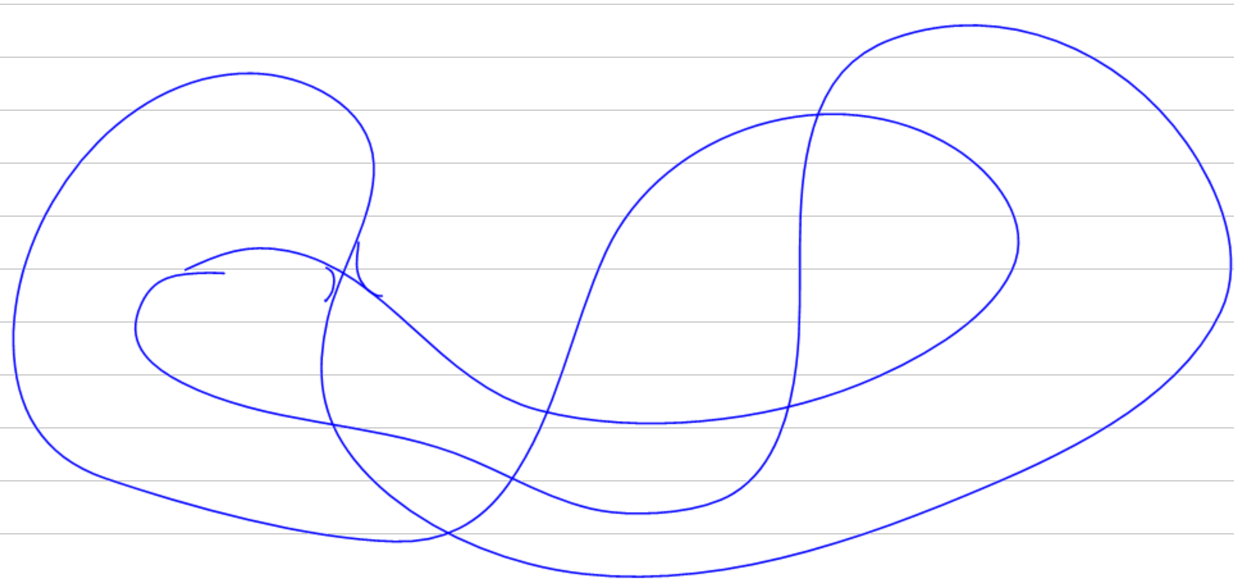


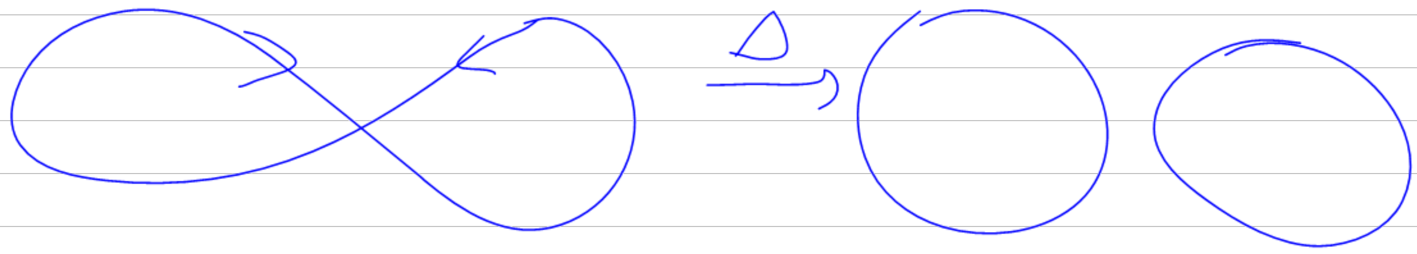


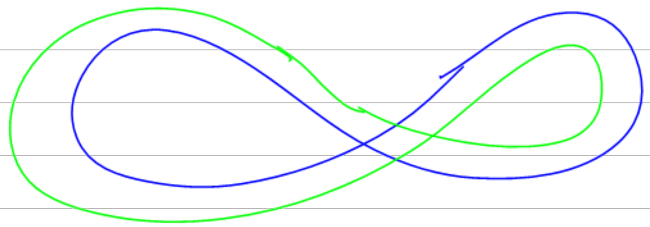




$\pi_1[a,b]$ $\pi_2[a,b]$







$\varphi : F_n \to F_n$ $\quad \varphi(F_n^{(k)}) \subset F_n^{(k)}$ $\quad F_n^{(k)} \subset \mathbb{Q}F_n$ $\quad F_n^{(k)} = I^k$

$\varphi : F_n \to \mathbb{Q}F_n$ $\quad I = \{\sum a_i g_i : \sum a_i = 0\}$

$x, y \in F_n$ $\quad (1-x)(1-y) = 1 - x - y + xy \in I^2$

$$\varphi(1) - \varphi(x) - \varphi(y) + \varphi(xy) \in I^2 = \sum (a_i - b_i)(c_i - d_i)$$

W.L.O.G $\quad \varphi(1) = 1$

$$1 - \varphi(x) - \varphi(y) + \varphi(xy) = (a-b)(c-d) = (a-b)(\underbrace{a^{-1}}_{a^{-1}a} - d)$$

$$= (1 - ba^{-1})(1 - ad)$$

So $\varphi(x) = u \quad \varphi(y) = v$

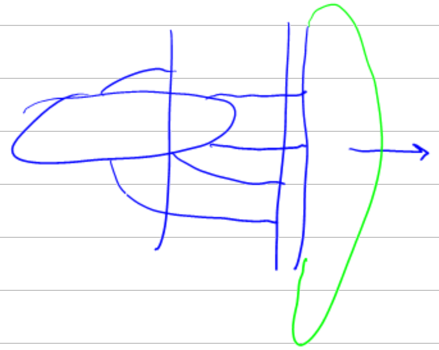
$\varphi(xy) = uv$

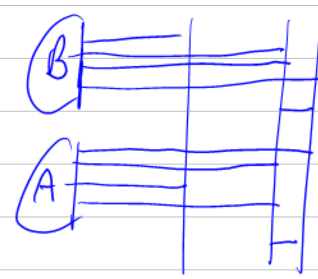
$$= (1-u)(1-v) = 1 - u - v + uv$$

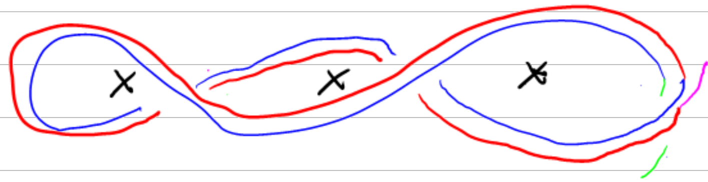
Talk idea: "Links in a Pole Dancing Studio: A Reading of Massuyeau, Alekseev, Kawazumi, Kuno, and Naef".

Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau, Alekseev, Kawazumi, Kuno, and Naef.

We study the pole-strand and strand-strand double filtration on the space of links in a pole dancing studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and perhaps simplify substantial parts of the work of the aforementioned authors on expansions for the Goldman-Turaev Lie bi-algebra.

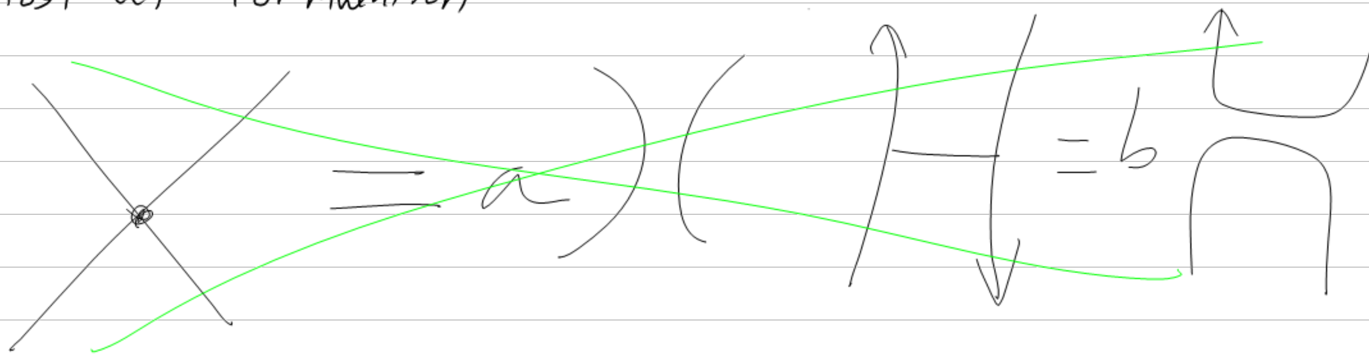


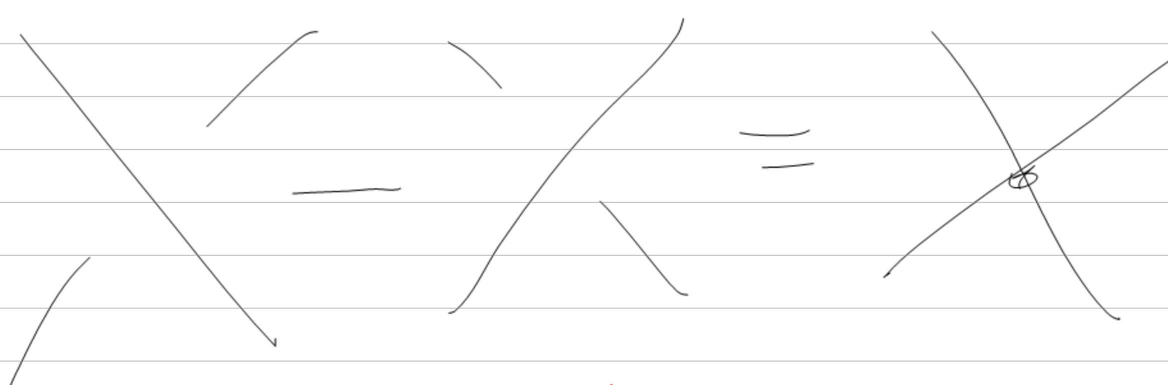


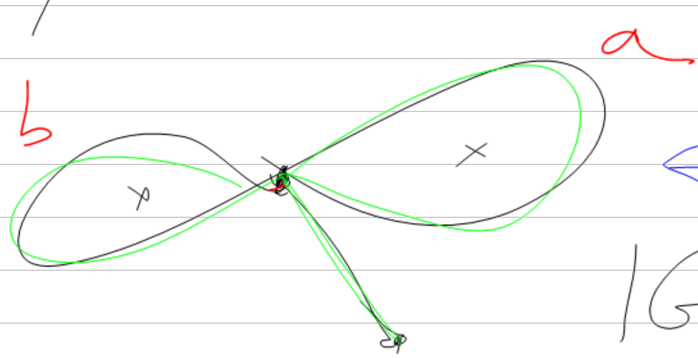



July 29 Agenda:
1. The abstract formulation.
2. The relation(s) coming from $\Delta$.
3. $\mathbb{RP}^2$ ?

## The Abstract Formulation



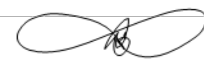

$\longleftrightarrow \; G \times G$

$|GG| = \dfrac{G \times G}{\mathrm{Ad}\, G}$

$(a,b) \sim (g^{-1}ag,\, g^{-1}bg)$

$|GG| \longrightarrow |G| \times |G|$

$|GG| \longrightarrow |G| \qquad (a,b) \to ab$



$b \quad \mathbb{Q}|G| \otimes \mathbb{Q}|G| \longrightarrow \mathbb{Q}|GG| \longrightarrow |G|$

$L \otimes L \to L_2$

$|GGG| \to |GG| \times |G|$

$\delta \quad |G| \longrightarrow \mathbb{Q}|GG| \longrightarrow |G| \times |G|$

* Really?

* What replaces Lie-bialg axioms

* Expansions?

$$\begin{array}{ccc} K & \xrightarrow{Z} & A \\ \downarrow{\scriptstyle b,\delta} & & \downarrow \\ K & \xrightarrow{Z} & A \end{array}$$

$|GG| \rightsquigarrow |AA|$

$A$ : Free alg : words

$|A|$ : cyclic words $/ xw = wx$

$$|AA| = \underset{A\otimes A}{(w_1, w_2)} \Big/ (xw_1, w_2) + (w_1, xw_2) \sim (w_1 x, w_2) + (w_1, w_2 x)$$

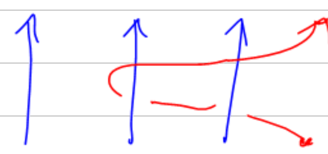
$$|AA| \to |A| \otimes |A| \qquad (w_1, w_2)$$

$$(w_1, w_2) \quad |AA| \to |A| \qquad w_1 w_2$$

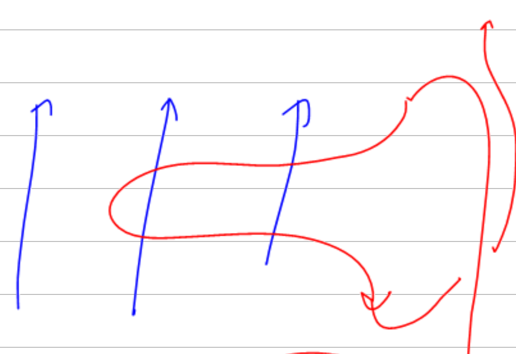
$* \; KV2_0$

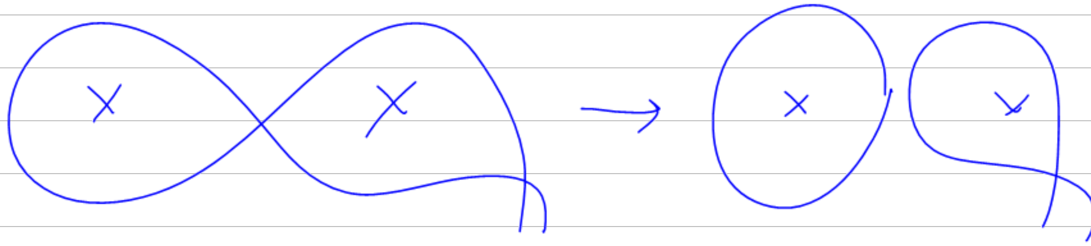
$$G \overset{\ell}{\rightrightarrows} K \xrightarrow{\pi} G \qquad \ell /\!/ \pi = r /\!/ \pi$$

$$G \xrightarrow{\ell} \text{ascend. rep.}$$



$$G \longrightarrow \text{desc rep.}$$





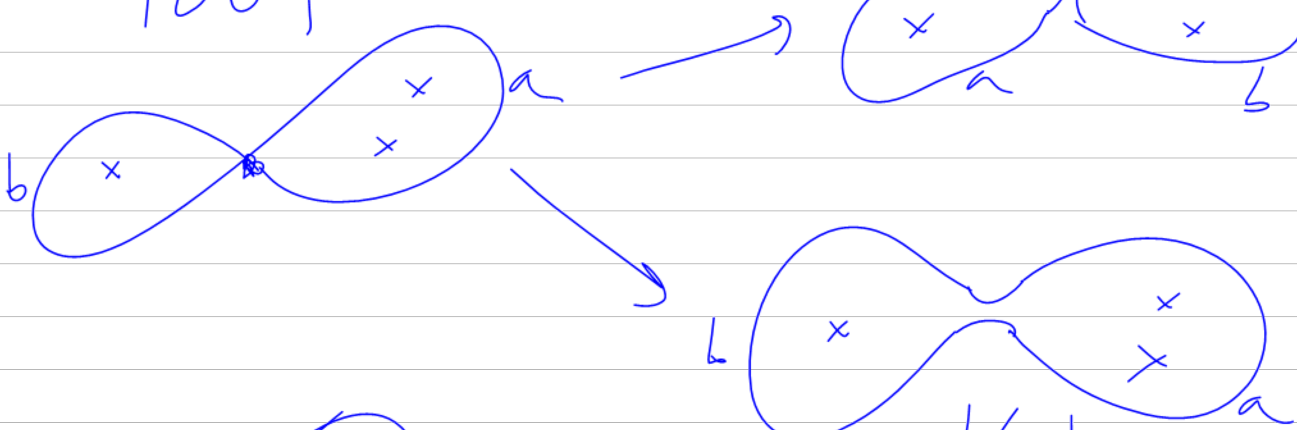
$$\begin{tikzcd}
G \arrow[r, mapsto] \arrow[d] & {|G| \times G} \arrow[d, "\pi // \mathrm{alt}"] \\
{|G|} \arrow[r, "\delta"] & {|G| \otimes |G|}
\end{tikzcd}$$

Really?

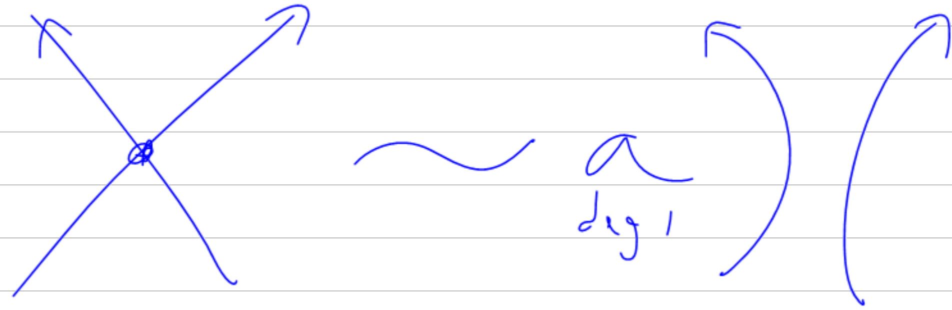
$Z(lw)$

$Z(rw) = op(Z(lw))$

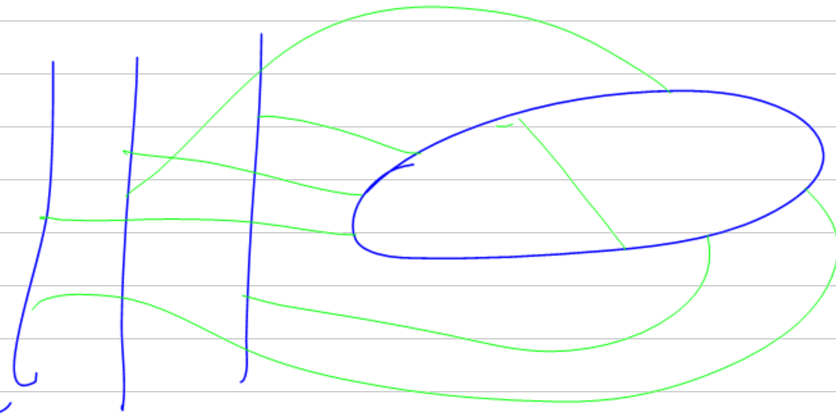
$|GG|$ $\quad$ $|G| \times |G|$



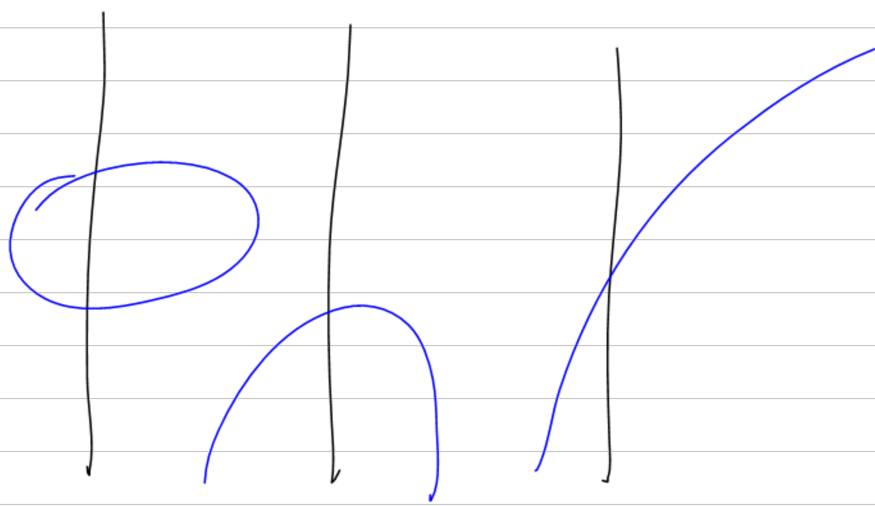

$|G|$

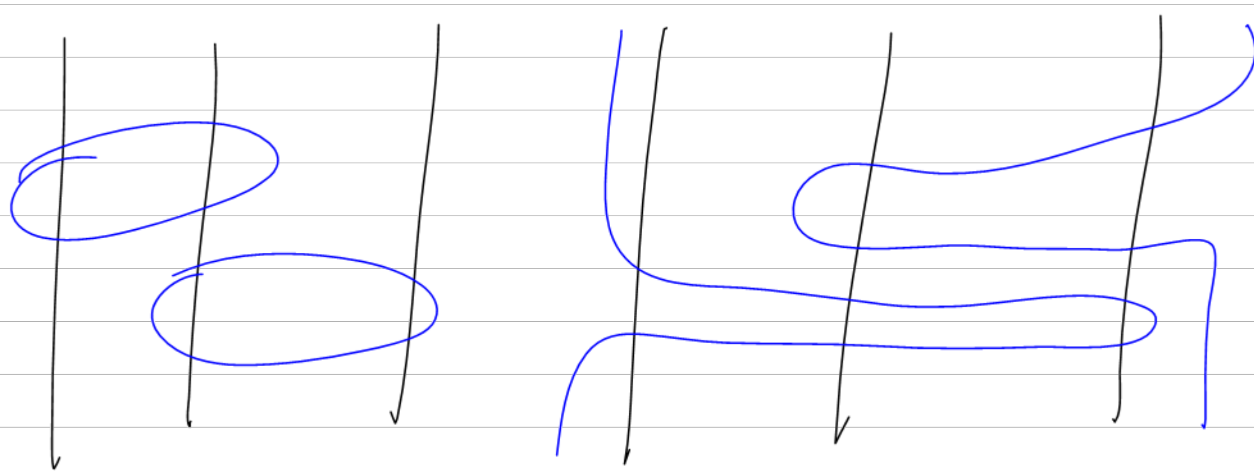


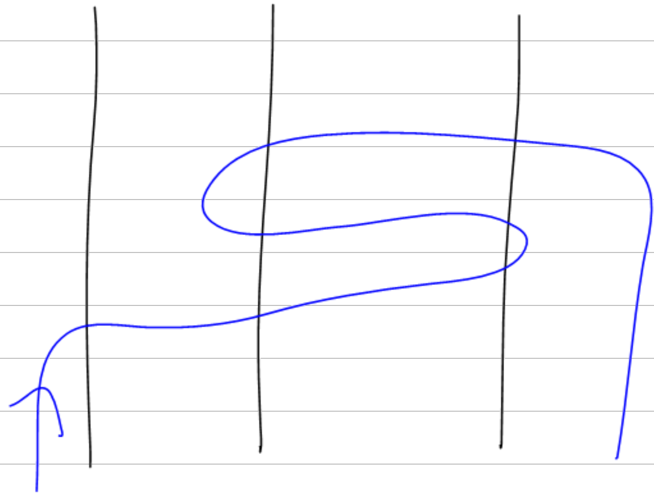

$$|AA| = A_{*,1}/A_{*,2}$$



$$|A| \longrightarrow |AA|$$



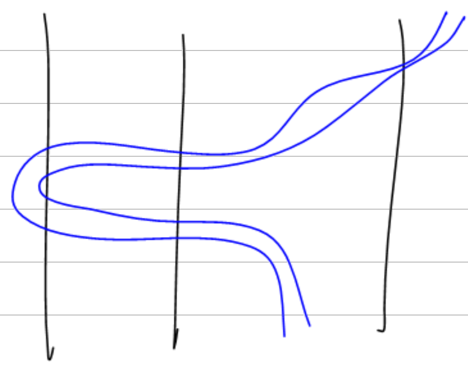




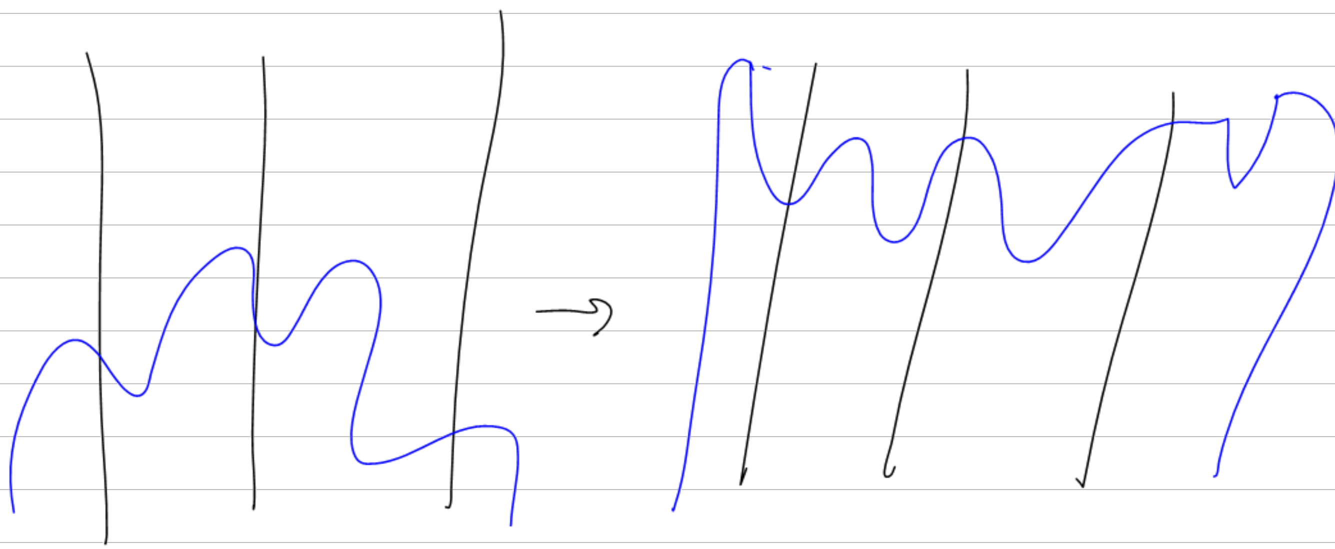
$$\pi \longrightarrow \pi \times |\pi|$$

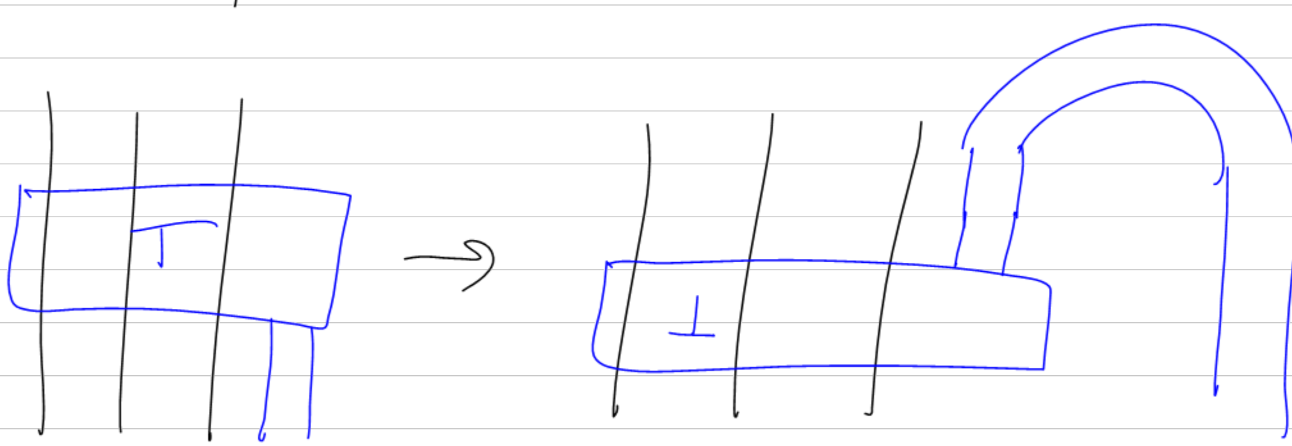
$$\Downarrow$$

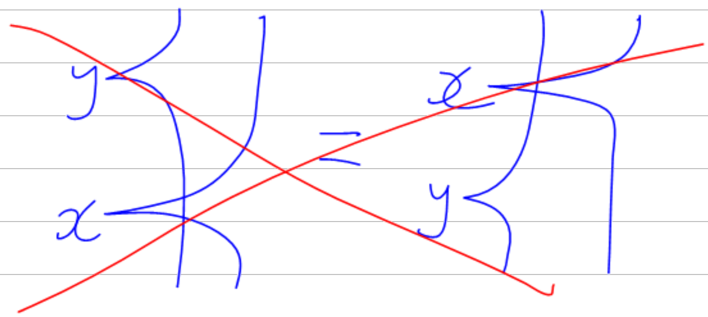
$$|\pi| \longrightarrow |\pi \times |\pi||$$



Gwnnilim

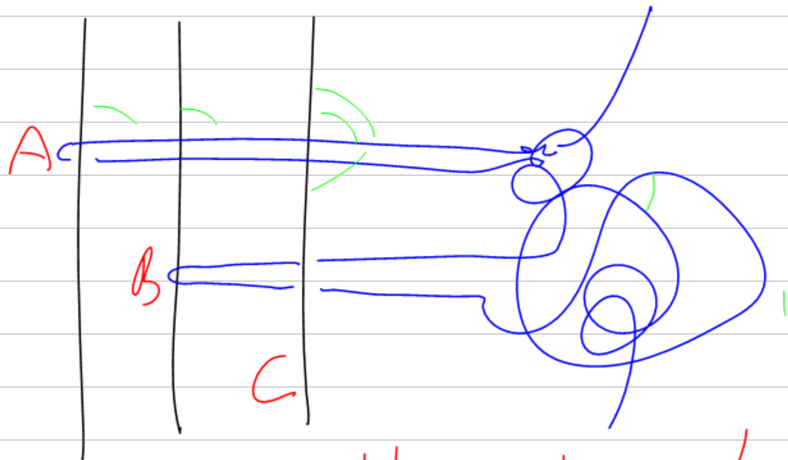




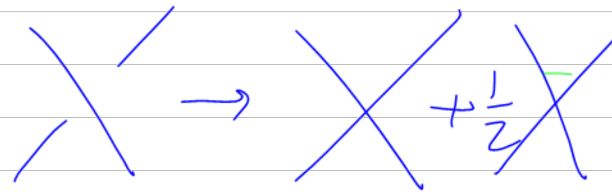



$|G|$

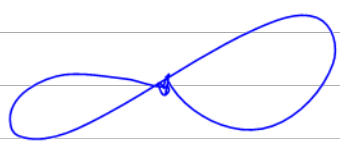
$|GG|$

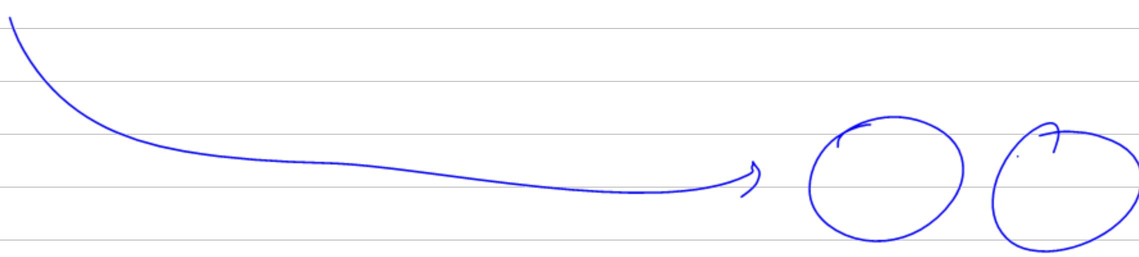


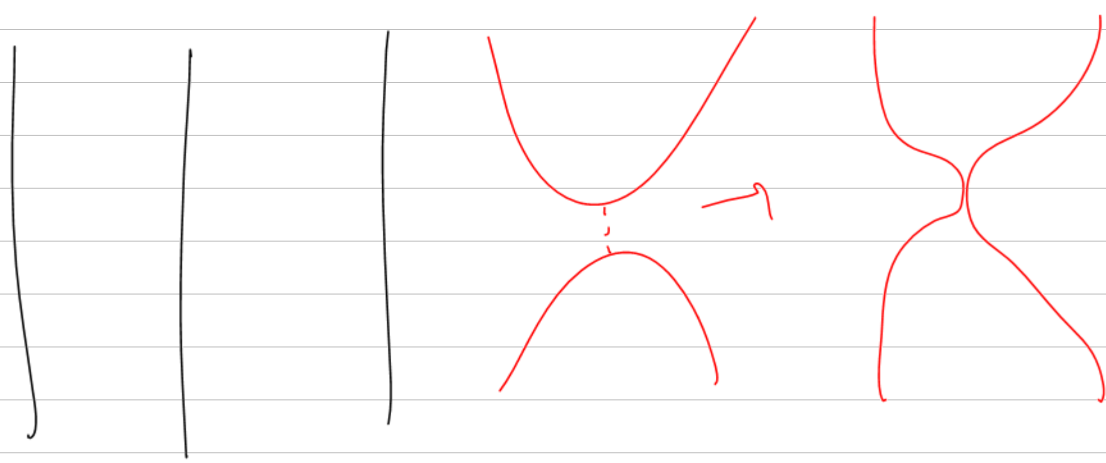

Humbert

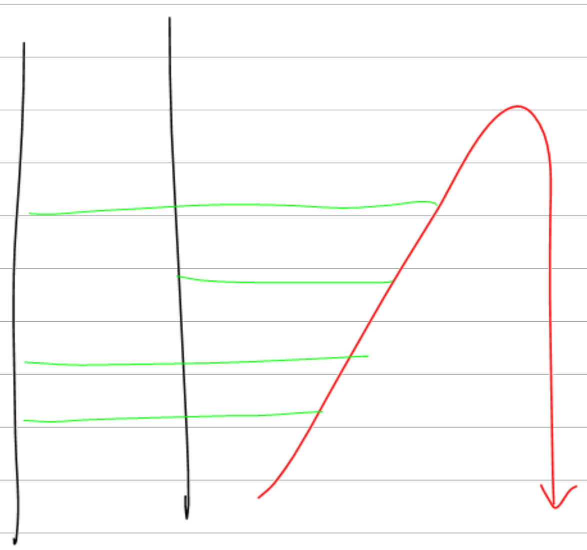


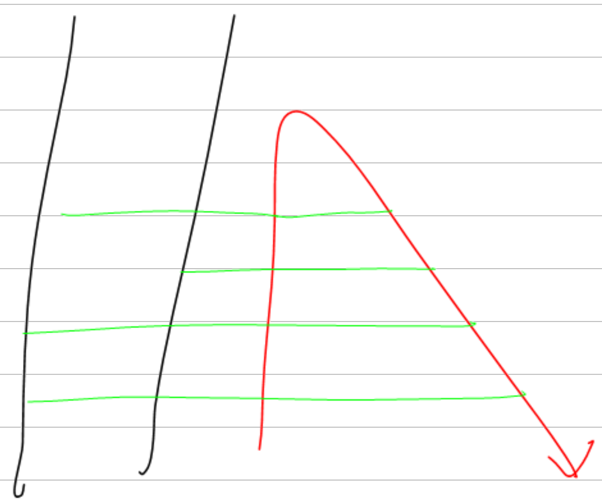


 $\longleftrightarrow |GG| \longrightarrow |G| \otimes |G|$









Consider $\lambda_{0,1} : \pi \to \mathcal{K}(\cap)$ as on the right.

Let $\mu : \pi \to \pi \otimes |\pi|$ be $\mu(\gamma) = \hbar^{-1}(\lambda_0(\gamma) - \lambda_1(\gamma))$

$\lambda_1$ is obtained from $\lambda_0$ by flipping at all self-intersections from ascending to descending, so $\mu$ is is the AKKN $\mu$ and

$$\delta \sim \mu // \underbrace{tr \otimes 1}_{tr_1} // Alt$$

[I am not careful about framings!!]

we can see that there are $\lambda_{0,1}^a : A \to \underbrace{\mathcal{A}^{/2}(\cap)}_{\mathcal{A}^{/1}(\cap)}$ such that

$$\begin{tikzcd} \pi \arrow[r, "\lambda_i"] \arrow[d, "W"'] & \mathcal{K}^{/2}(\cap) \arrow[d, "Z^{/2}"] \arrow[dl, "Z^{/1}"'] \\ A \arrow[r, "\lambda_i^a"'] & \mathcal{A}^{/2}(\cap) \end{tikzcd}$$

Furthermore, $\lambda_1 = \lambda_0 // F // C_1$

so $\lambda_1^a = \lambda_0^a // \underset{\text{flip}}{F} // \underset{\text{conjugation}}{C_1}$

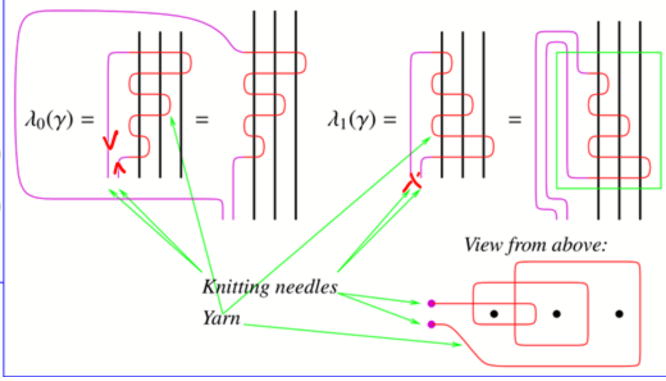
and then, in $\mathcal{A}^{/2}(\bigcirc\bigcirc)$,

$$\hbar(\gamma // \delta // W) = \hbar(\gamma // \mu // tr_1 // Alt // W) =$$
$$= \hbar(\gamma // \mu // W // tr_1 // Alt)$$
$$= \hbar(\gamma // \mu // Z // tr_1 // Alt)$$
$$= \gamma // (\lambda_0 - \lambda_1) // Z // tr_1 // Alt)$$
$$= \gamma // Z^{/1} // (\lambda_0^a - \lambda_1^a) // tr_1 // Alt)$$
$$= \gamma // Z^{/1} // \lambda_0^a // (I - F // C_1) // tr // Alt) = \gamma // Z^{/1} // \lambda_0^a // (I - F) // tr // Alt)$$
$$= \gamma // Z^{/1} // \hbar \cdot \eta^a // tr_1 // Alt = \hbar(\gamma // W // \delta^a)$$

And so $\delta // W = W // \delta^a$

**Unignoring the Complications.** We need $\lambda_0$ and $\lambda_1$ such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping ~~all~~ at self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.




If Kontsevich likes $\lambda_0$ and $\lambda_1$ (namely if there are $\lambda_i^a$ with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then $\eta$ will have a compatible algebraic companion $\eta^a$:

$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

In AKKN1:

**Proposition 5.9.** *The map $\mu : \mathbb{K}\pi \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$ satisfies the product formula*

$$\mu(xy) = \mu(x)(1 \otimes y) + (1 \otimes x)\mu(y) + (|\ | \otimes 1)\kappa(x, y)$$

*for any $x, y \in \mathbb{K}\pi$, and we have $\mu(\gamma_i) = 0$ for any $i$. Moreover, these two properites characterise the map $\mu$.*

*Proof.* The product formula follows from Proposition 5.3. Notice that a suitable smoothing of $\nu\gamma_i$ has no self-intersections and that its rotation number is $-1/2$. Thus $i(\gamma_i) = \nu\gamma_i$ and $\mu(\gamma_i) = 0$. The last statement follows from the fact that $\pi$ is generated by $\{\gamma_i\}_i$. ☐

**Proposition 5.10.** *The composition map*

$$\pi \xrightarrow{i} \pi^+ \xrightarrow{|\ |^+} \hat{\pi}^+ \xrightarrow{\delta^+} |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|.$$

*descends to a map $\delta^+ : \hat{\pi} \to |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|$. For any $\gamma \in \pi$, we have*

$$\delta^+(|\gamma|) = -\mathrm{Alt}(1 \otimes |\ |)\mu(\gamma) + |\gamma| \wedge \mathbf{1}. \tag{39}$$

*Its $\mathbb{K}$-linear extension $\delta^+ : |\mathbb{K}\pi| \cong \mathbb{K}\hat{\pi} \to |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|$ is a lift of the Turaev cobracket in the sense that $\varpi^{\otimes 2} \circ \delta^+ = \delta \circ \varpi$, where $\varpi : |\mathbb{K}\pi| \to |\mathbb{K}\pi|/\mathbb{K}\mathbf{1}$ is the natural projection.*