POLY-TIME KNOT POLYNOMIALS VIA SOLVABLE APPROXIMATIONS

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ABSTRACT. Following Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], we construct the first poly-time-computable knot polynomials since Alexander's [Al, 1928]. We use some new commutator-calculus techniques and a family of Lie algebra $sl_2^{\leqslant k}$ which are solvable yet at the same time they make progressively better approximations of the simple Lie algebra sl_2. The resulting invariants are the strongest genuinely-computable knot invariants presently available and they seem to contain information about some classical topologically-defined knot invariants.

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1. Introduction

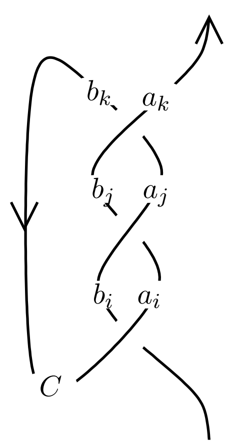
(The impatient reader may skip straight to Section 1.2, "Statement of the Main Theorem")

1.1. **From Algebras to Invariants.** There is a standard "quantum algebra" methodology that associates a framed knot invariant to certain triples (U, R, C), where U is a unital algebra and $R \in U \otimes U$ and $C \in U$ are invertible (see e.g. [Oh, Section 4.2]). For convenience, we recall this methodology in Aside 1.1.

The best algebras with which to apply this methodology, at least as of 2017, are certain completions $\hat{\mathcal{U}}(\mathfrak{g})$ of the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of semi-simple Lie algebras $\mathfrak{g}$ (or their quantizations). But these algebras are infinite dimensional, and the sum in Aside 1.1 is infinite and not immediately computable.

The dogma solution is to pick a finite dimensional representation of $\mathfrak{g}$ and use it to represent all the elements appearing in Aside 1.1, effectively replacing the algebra by the algebra of endomorphisms of some finite dimensional vector space. This turns the sum finite; yet if the knot K has n crossings, our sum becomes a sum over n indices $i_1, \dots, i_n$. Thus there

Draw K as a long knot in the plane so that at each crossing the two crossing strands are pointing up, and so that the two ends of K are pointing up.
Put a copy of $R = \sum a_i \otimes b_i$ on every positive crossing of K with the "a" side on the over-strand and the "b" side on the under-strand, labeling these a's and b's with distinct indices $i, j, k, \dots$ (similarly put copies of $R^{-1} = \sum a'_i \otimes b'_i$ on the negative crossings; these are absent in our example). Put a copy of $C^{\pm 1}$ on every cuap where the tangent to the knot is pointing to the right (meaning, a C on every such cup and a C^{-1} on every such cap).
Form an expression $z(K)$ in U by multiplying all the a, b, C letters as they are seen when traveling along K and then summing over all the indices, as shown.



$$z(K) = \sum_{i,j,k} b_i a_j b_k C a_i b_j a_k$$

If R and C satisfy some conditions dictated by the standard Reidemeister moves of knot theory, the resulting $z(K)$ is a knot invariant.
Abstractly, $z(K)$ is obtained by tensoring together several copies of $R^{\pm 1} \in U^{\otimes 2}$ and $C^{\pm 1} \in U$ to get an intermediate result $z_0 \in U^{\otimes S}$, where S is a finite set with two elements for each crossing of K and one element for each right-pointing cuap. We then multiply the different tensor factors in z_0 in an order dictated by K to get an output in a single copy of U.

Aside 1.1. The standard methodology on an example knot.

are exponentially-many summands to consider and it takes an exponential amount of time to compute $z(K)$, limiting its computation only to relatively small knots.[1,2] In addition, by choosing a specific representation of $\mathfrak{g}$, one looses the good behaviour of z under strand-doubling. In Section 9 we explain why such good behaviour is a desirable property.

Alternatively, one may extract finite-type [BN1, CDM] information out of z by reducing modulo appropriate filtrations of U and its tensor powers. Invariants of type d are computable in time less than or equal to $O(n^d)$ [BN2], and thus for small d, they are effectively computable. But there are only a few invariants of sufficiently small type d, they are not very powerful, and there are some no-go theorems that limit the power of any finite number of finite-type invariants to resolve certain topological questions [Ng, St].

Our approach to the computation of $z(K)$ is different. Instead of working directly in $U^{\otimes S}$ (see Aside 1.1), we work in relatively small[3] spaces $\mathcal{F}(S)$ of "closed-form formulas for elements of $U^{\otimes S}$". For this to work, we need to ensure that the fundamentals R and C would be described by "closed-form formula", and that the most basic operations necessary for the computation of z, namely multiplication of factors in $U^{\otimes S}$, can be implemented "in closed form".

In practice, the kind of terms that appear within formulas for R and C are exponentials of the form $\mathbb{e}^{\xi x}$, where x is a generator of U and ξ is a formal scalar variable, their iterated derivatives $(\partial_\xi)^k \mathbb{e}^{\xi x} = x^k \mathbb{e}^{\xi x}$, and exponentials of quadratics like $\mathbb{e}^{\lambda xy}$ or $\mathbb{e}^{\lambda x \otimes y}$, with scalar λ and $x, y \in U$. We then need to multiply several such exponentials and differentiated exponentials, and we need to learn how to bring such products into some pre-chosen "canonical order". In the standard $U \sim \hat{\mathcal{U}}(\mathfrak{g})$ case, where $\mathfrak{g}$ is semi-simple, this is complicated. Yet if $\mathfrak{g}$ is solvable, this is often easy (see Aside 1.2). Wouldn't it be nice if it was possible to approximate semi-simple Lie algebras with solvable ones?

In this paper we exploit the little-known fact that this is (nearly) possible. Precisely, given a semisimple $\mathfrak{g}$, there exists a Lie algebra $\mathfrak{g}^\epsilon$ defined over the ring $\mathbb{Q}[\epsilon]$ of polynomials in a formal variable ϵ (in other words, $\mathfrak{g}^\epsilon$ is a "one-parameter family of Lie algebras"), so that

(1) If ϵ is fixed to be some constant not equal to zero, then $\mathfrak{g}^\epsilon$ is isomorphic to $\mathfrak{g}^+ := \mathfrak{g} \oplus \mathfrak{h}$, which is the original $\mathfrak{g}$ with an additional copy of its own (Abelian) Cartan subalgebra $\mathfrak{h}$ added.
(2) At $\epsilon = 0$, $\mathfrak{g}^0$ is solvable. Furthermore, $\mathfrak{g}^\epsilon$ is solvable in a formal neighborhood of $\epsilon = 0$: for any natural number $k \geqslant 0$ the reduction $\mathfrak{g}^{\leqslant k}$ of $\mathfrak{g}^\epsilon$ to the ring $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$ is solvable as a Lie algebra over $\mathbb{Q}$ (whose dimension is $(k+1)\dim\mathfrak{g}$).

As k gets larger, the solvable $\mathfrak{g}^{\leqslant k}$ is closer and closer to $\mathfrak{g}^\epsilon$, as the reduction modulo $\epsilon^{k+1} = 0$ means less and less, and so at least informally, $\mathfrak{g}^{\leqslant k} \xrightarrow[k\to\infty]{} \mathfrak{g}^+ \sim \mathfrak{g}$. See also Aside 1.3.

It remains to sketch why $\mathfrak{g}^\epsilon$ exists. The short, precise, but jargon-heavy answer is in the next paragraph. A jargon-free example, in the case of $\mathfrak{g} = gl_n$, is in Aside 1.4.

[1] "Divide and conquer" methods often improve the computation time to $O(e^{c\sqrt{n}})$ for some constant c. Utilizing this, the simplest of these "quantum invariants", the Jones, HOMFLY-PT and Kauffman polynomials, corresponding to sl_2, sl_N and so_N in their defining representations, can be computed for surprisingly large knots even though ultimately $e^{c\sqrt{n}}$ grows more rapidly than any polynomial.

[2] Note that almost any time the phrases "braided monoidal category" or "TQFT" are used within low dimensional topology, some tensor powers of some vector spaces need to be considered at some point, and dimensions grow exponentially. Thus our criticism applies in these cases too [BN5].

[3] Ranks grow polynomially in $|S|$.

Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{b}^+$ and $\mathfrak{b}^-$ be its upper and lower Borel subalgebras, respectively, Then $(\mathfrak{b}^+)^*$ is $\mathfrak{b}^-$, and as the latter has a Lie bracket, it follows that $\mathfrak{b}^+$ has a co-bracket δ. In fact, $\mathfrak{b}^+$ along with its bracket $[\cdot,\cdot]$ and co-bracket δ is a "Lie bialgebra", and one may recover $\mathfrak{g}^+ = \mathfrak{g}\oplus\mathfrak{h} = \mathfrak{b}^-\oplus\mathfrak{b}^+$ as the "Drinfel'd double" $\mathcal{D}(\mathfrak{b}^+,[\cdot,\cdot],\delta)$ of $\mathfrak{b}^+$ (see e.g. [ES, Chapter 4]). By a quick inspection, the axioms of a Lie bialgebra are homogeneous in δ: meaning that $(\mathfrak{b}^+,[\cdot,\cdot],\epsilon\delta)$ is again a Lie bialgebra for any scalar ϵ, and one may set $\mathfrak{g}^\epsilon := \mathcal{D}(\mathfrak{b}^+,[\cdot,\cdot],\epsilon\delta)$. The required properties are all easy to check. Perhaps the most interesting is the solvability of $\mathfrak{g}^0$: indeed $\mathfrak{g}^0 = I\mathfrak{b}^+ := (\mathfrak{b}^+)^*\rtimes\mathfrak{b}^+$ with $(\mathfrak{b}^+)^*$ regarded as an Abelian Lie algebra and $\mathfrak{b}^+$ acts on $(\mathfrak{b}^+)^*$ using the co-adjoint action, and then the solvability of $I\mathfrak{b}^+$ easily follows from the solvability of $\mathfrak{b}^+$. It is worth noting that the knot-theoretic significance of $\mathfrak{b}^*\rtimes\mathfrak{b}$ for a general Lie algebra $\mathfrak{b}$ was studied extensively in the context of "w-knots" in [BND1, BND2, BND3, BN4, BN3], and that these studies along with the observations in this paragraph were in some sense the starting points for our current study.

We would have loved our own story a lot better if it had ended here. Namely if for any semi-simple $\mathfrak{g}$ and any $k\geqslant 0$ we knew how to construct R and C in "small" spaces $\mathcal{F}(S)$ of formulas for elements of $\hat{\mathcal{U}}(\mathfrak{g}^{\leqslant k})^{\otimes S}$ and if we knew how to efficiently "multiply" in $\mathcal{F}(S)$. This in fact is almost true: the only thing we miss are explicit formulas for R and C. In order to obtain such formulas we first have to replace $\mathfrak{g}^\epsilon$ by its "quantized" version $\mathfrak{g}^\epsilon_q$, which is obtained from $(\mathfrak{b}^+,[\cdot,\cdot],\epsilon\delta)$ using Drinfel'd's "quantum double" construction. Very little of substance changes in the formulas associated with $\mathfrak{g}^\epsilon_q$ as opposed to $\mathfrak{g}^\epsilon$; they are only just a bit uglier.

Indeed, here's a reoredering exercise that we will care about deeply later in this paper. The semi-simple Lie algebra sl_2 is generated by elements y, a, and x with relations $[a,x]=2x$, $[a,y]=-2y$, and $[x,y]=a$. If η_i, α_i, and ξ_i are scalars we have can reorder $\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x}\mathbb{e}^{\eta_2 y}\mathbb{e}^{\alpha_2 a}\mathbb{e}^{\xi_2 x}$ to become $\mathbb{e}^{\eta_0 y}\mathbb{e}^{\alpha_0 a}\mathbb{e}^{\xi_0 x}$ where

$$(\eta_0,\alpha_0,\xi_0)=\left(\frac{e^{-2\alpha_1}\eta_2}{\eta_2\xi_1+1}+\eta_1,\alpha_1+\alpha_2+\log\left(\eta_2\xi_1+1\right),\frac{\xi_1\left(e^{-2\alpha_2}+\eta_2\xi_2\right)+\xi_2}{\eta_2\xi_1+1}\right).$$

In the solvable Lie algebra sl_2^0 obtained from sl_2 by adding a central generator c and replacing the last sl_2 relation with $[x,y]=c$ while keeping $[a,x]=2x$ and $[a,y]=-2y$ (thus separating the roles of a as a "number operator" and as a "Heisenberg-like commutator"), we have $\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x}\mathbb{e}^{\eta_2 y}\mathbb{e}^{\alpha_2 a}\mathbb{e}^{\xi_2 x}=\mathbb{e}^{\eta_0 y}\mathbb{e}^{\alpha_0 a}\mathbb{e}^{\xi_0 x}\mathbb{e}^{\gamma_0 c}$, where

$$(\eta_0,\alpha_0,\xi_0,\gamma_0)=\left(e^{-2\alpha_1}\eta_2+\eta_1,\alpha_1+\alpha_2,e^{-2\alpha_2}\xi_1+\xi_2,\eta_2\xi_1\right).$$

The sl_2^0 formulas are visibly simpler than the sl_2 formulas. What is even more important is that the iterated derivatives of the sl_2^0 formulas stay within a finite dimensional space of expressions. This is not the case for the sl_2 formulas.
Notes. • *The formulas within this Aside are proven in Section 12.1.* • *Over $\mathbb{C}$, sl_2^0 is isomorphic to the "diamond Lie algebra" of* [Ki, Chapter 4.3], *which is sometimes called "the Nappi-Witten algebra"* [NW].

Aside 1.2. Reordering differentiated exponentials is easier in the solvable case then in the semi-simple case.

In addition, we have so far worked out in detail only the case of $=sl_2$. Almost everything seems to generalize to arbitrary semi-simple $\mathfrak{g}$, and we hope to return to the more general case in a later publication.

1.2. Statement of the Main Theorem. MORE.

1.3. Section Summaries and Dependencies. MORE.

1.4. Acknowledgement. MORE.

2. Rotational Virtual Tangles

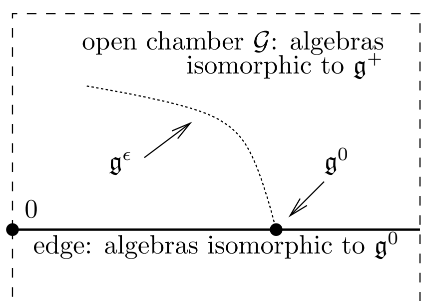
(This section can be read independently of the rest of this paper).
More.

3. The Zeroth Example in Detail

MORE.

4. The Lie Algebra $sl_2^{\alpha,\beta}$ and its Quantization

Semisimple Lie algebras are famously "rigid" and allow no deformations [Ha] *(within the universe of Lie algebras; outside of it, there's "quantum groups", of course). How is this consistent with the existence of the family $\mathfrak{g}^\epsilon$?*
The short answer is that $\mathfrak{g}^\epsilon$ is a deformation of the solvable $\mathfrak{g}_0$, not of $\mathfrak{g}$. So $\mathfrak{g}$ is a deformation of $\mathfrak{g}_0$, but $\mathfrak{g}_0$ is not a deformation but a contraction [IW, Gi] *of $\mathfrak{g}$.*



Some cells in $\mathcal{B}(V)$

It is perhaps a bit clearer to think in terms of the "space of Lie brackets". Given a vector space V, a Lie bracket on V is an element $b \in V^* \otimes V^* \otimes V$ which satisfies a linear equation (being anti-symmetric) and a quadratic equation (Jacobi; it is quadratic as a function of b). So we can consider the variety $\mathcal{B}(V)$ of all Lie brackets on V. A very schematic depiction is on the right. Within $\mathcal{B}(V)$, Lie algebras isomorphic to some specific semisimple $\mathfrak{g}^+$ make an open chamber $\mathcal{G}$. Indeed that's the meaning of rigidity — when you move a tiny bit away from $\mathfrak{g}^+$ what you see is isomorphic to $\mathfrak{g}^+$. Yet the closure $\bar{\mathcal{G}}$ of $\mathcal{G}$ contains other Lie algebras, including $\mathfrak{g}^0$.
Note that every cell in $\mathcal{B}$ contains in its closure the 0 bracket, belonging to the Abelian Lie algebra $\mathfrak{a}$ on *V. If a path $\mathfrak{g}^\epsilon$ is chosen as above but with $\mathfrak{g}^0 = \mathfrak{a}$, then in the same sense as above, $\mathfrak{g}^\epsilon$ is nilpotent in a neighborhood of $\epsilon = 0$. So in the same* sense as above, *every Lie algebra can be approximated by nilpotent Lie algebras. Why are we not exploiting this fact in this paper? Because the knot invariants that arise from nilpotent approximation are finite type invariants and with solvable approximation we do better.*

Aside 1.3. How is this possible? The moduli of Lie algebras.

MORE. (This section can be read independently of the rest of this paper).

5. Computations in (Quantized) Enveloping Algebras: Trading Non-Commutativity for Non-Linearity

(This section can be read independently of the rest of this paper).
More.

6. Pushforwards of Pure Distributions

(This section can be read independently of the rest of this paper).
Content: Pushforwards in general, interpretation using pairings, the difficulty in the general case, central-nilpotent-diagonal and the main theorem.
More.

7. The General $sl_2^{\leqslant k}$ Invariant

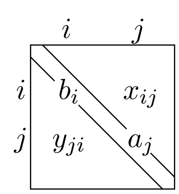
MORE.

8. Bulk Stitching

MORE.

9. Behaviour under strand reversal and strand doubling

The Lie algebra gl_n^+, namely gl_n plus an additional n-dimensional Abelian factor of "diagonal matrices", is the direct sum (as a vector space) of two subalgebras of gl_n: the upper triangular matrices ◹ and the lower triangular matrices ◺. With some ambiguity regarding diagonal matrices, gl_n^ϵ is obtained from $gl_n^+ =$ ◹ $\oplus$ ◺ by selectively multiplying some of the structure constants of the latter by ϵ. In summary form, this is [◹, ◹] = ◹, [◺, ◺] = ϵ◺, and [◹, ◺] = ◺ + ϵ◹, which stands for "brackets within ◹ are unchanged, brackets within ◺ are multiplied by ϵ, and in a bracket of something in ◹ with something in ◺, the part of the output $z(K)$ in ◺ is unchanged and the part in ◹ is multiplied by ϵ".



Even more concretely, in terms of generators and relations, we have that gl_n^+ is generated by $\{x_{ij}, y_{ji} \colon 1 \leqslant i < j \leqslant n\} \cup \{a_i, b_i \colon 1 \leqslant i \leqslant n\}$, with relations

$$[x_{ij}, x_{kl}] = \delta_{j=k}x_{il} - \delta_{l=i}x_{kj}, \qquad [y_{ij}, y_{kl}] = \epsilon\delta_{j=k}y_{il} - \epsilon\delta_{l=i}y_{kj},$$

$$[x_{ij}, y_{kl}] = \delta_{j=k}(\epsilon\delta_{j<k}x_{il} + \delta_{i=l}(b_i + \epsilon a_i)/2 + \delta_{i>l}y_{il}) - \delta_{l=i}(\epsilon\delta_{k<j}x_{kj} + \delta_{k=j}(b_j + \epsilon a_j)/2 + \delta_{k>j}y_{kj}),$$

$$[a_i, x_{jk}] = (\delta_{i=j} - \delta_{i=k})x_{jk}, \qquad [b_i, x_{jk}] = \epsilon(\delta_{i=j} - \delta_{i=k})x_{jk},$$

$$[a_i, y_{jk}] = (\delta_{i=j} - \delta_{i=k})y_{jk}, \qquad [b_i, y_{jk}] = \epsilon(\delta_{i=j} - \delta_{i=k})y_{jk},$$

where δ_{cond} is 1 if cond is true and is 0 otherwise. As matrices, x_{ij} is the upper triangular matrix with 1 in position ij and 0 elsewhere, y_{ji} is the lower triangular matrix with 1 at ji and 0 elsewhere, and a_i and b_i are both diagonal with 1 at ii and 0 elsewhere, except a_i is regarded as an upper triangular matrix and b_i as a lower triangular matrix.

Aside 1.4. A solvable approximation of gl_n.

MORE.

10. Complexity

MORE.

11. Experimental Results

MORE.

12. Odds and Ends

12.1. **Proofs of the Formulas in Aside 1.2.** We can carry out the required computations in a faithful matrix representation of sl_2 / sl_2^0; such representations automatically extend to faithful representations of exponentials and products of exponentials in $\hat{\mathcal{U}}(sl_2)$ / $\hat{\mathcal{U}}(sl_2^0)$. And as we have computers, we may as well use them (sources in ωεβ/). First, we enter the sl_2 matrices and verify their commutation relations:

```
y = {{0, 0}, {1, 0}}; a = {{1, 0}, {0, -1}}; x = {{0, 1}, {0, 0}};
{a.x - x.a == 2 x, a.y - y.a == -2 y, x.y - y.x == a}
```

```
{True, True, True}
```

We then declare that `𝔼[M]` means $\mathbb{e}^M$, when M is a matrix, and verify the sl_2 formulas in Aside 1.2:

```
𝔼[M_?MatrixQ] := MatrixExp[M];
Simplify[𝔼[η1 y].𝔼[α1 a].𝔼[ξ1 x].𝔼[η2 y].𝔼[α2 a].𝔼[ξ2 x] == 𝔼[η0 y].𝔼[α0 a].𝔼[ξ0 x] /.
  {η0 → η1 + (𝕖^(-2 α1) η2)/(1+η2 ξ1), α0 → α1 + α2 + Log[1 + η2 ξ1], ξ0 → (ξ2+ξ1 (𝕖^(-2 α2)+η2 ξ2))/(1+η2 ξ1)}]
```

```
True
```

(The truth, of course, is that we originally used a computer to `Solve` for η_0, α_0, and ξ_0, but once the formulas are found, we only need to check them).

In the case of sl_2^0, the matrix representation is a bit more complicated, the formulas are a bit simpler, and the end result is the same `true`:

```
y = {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}}; a = {{0, 0, 0}, {0, 2, 0}, {0, 0, 0}}; x = {{0, 0, 0}, {0, 0, 1}, {0, 0, 0}}; c = {{0, 0, -1}, {0, 0, 0}, {0, 0, 0}};
{a.x - x.a == 2 x, a.y - y.a == -2 y, x.y - y.x == c, x.c == c.x, y.c == c.y, a.c == c.a}
```

```
{True, True, True, True, True, True}
```

```
Simplify[
 𝔼[η1 y].𝔼[α1 a].𝔼[ξ1 x].𝔼[η2 y].𝔼[α2 a].𝔼[ξ2 x] == 𝔼[η0 y].𝔼[α0 a].𝔼[ξ0 x].𝔼[γ0 c] /.
  {η0 → η1 + 𝕖^(-2 α1) η2, α0 → α1 + α2, ξ0 → 𝕖^(-2 α2) ξ1 + ξ2, γ0 → η2 ξ1}]
```

True

$\square$

13. Tables

MORE.

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14. Recycling

This section does not exist in a respectable math paper.

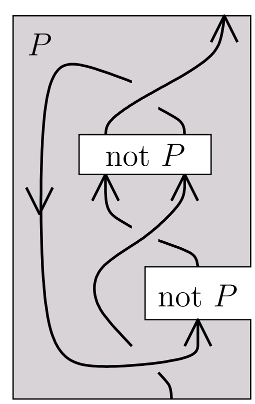
14.1. **Recycled 170704.** There is a standard "quantum algebra" methodology that associates a framed knot / tangle invariant to certain triples (U, R, C), where U is a unital algebra and $R \in U \otimes U$ and $C \in U$ are invertible (see e.g. [Oh, Section 4.2]). In Aside 14.1 we provisionally explain what we mean by "tangle", and the "quantum algebra" methodology is recalled in Aside 1.1.

14.1.1. *Formulas and Meta-Algebras.* Our approach to the computation of $z(K)$ is different. Instead of working directly in $U^{\otimes S}$, our invariant $Z(K)$ takes values in spaces $\mathcal{F}(S)$ of "formulas for elements of $U^{\otimes S}$" that have an "value map" $\mathbb{V}\colon \mathcal{F}(S) \to U^{\otimes S}$, taking a formula in $\mathcal{F}(S)$ to its value in $U^{\otimes S}$, for which $z = Z /\!\!/ \mathbb{V}$.[4] We make sure that the following five properties hold:

(1) There are simple and easy to compute (constant time) formulas for the invariants of a crossing and of a cuap.
(2) There are operations on $\mathcal{F}(S)$ that mirror standard operations on the space $U^{\otimes S}$ and on the space $\mathcal{K}(S)$ of S-component tangles, so that a diagram of the following nature commutes:

[4]We use properly ordered compositions! $f /\!\!/ g$ means "do f then g", often obfuscated using "$g \circ f$".

Like elsewhere, for us a "tangle" K is a part of a (multi-component, oriented, framed) knot in a part P of a plane in which an "up" direction is declared. Unlike elsewhere, we do not insist that P would be a disk; it may be a union of disks with a few sub-disks removed. We do insist, however, that the ends of K would lie within the boundary ∂P of P and would be up-going there. We also insist that the components ("strands") of K would be intervals (i.e., not circles), and that they would be placed in a bijection with some finite set S of "strand labels".
In Section 2 we replace this provisional definition with "rotational virtual tangles" in the spirit of [Ka].



Aside 14.1. Provisionally, what we mean by a "tangle".

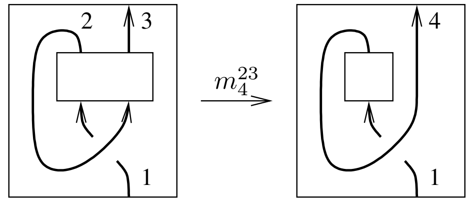
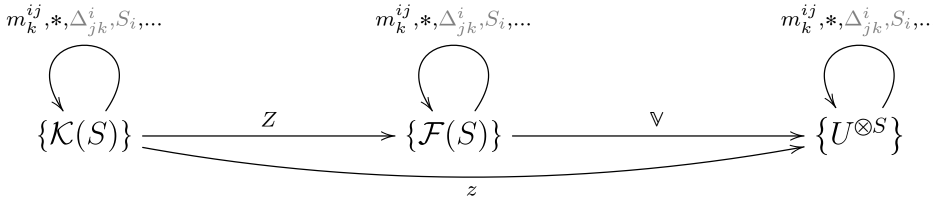


FIGURE 14.1. Stitching.



m_k^{ij}: "stitching" m_k^{ij}: "meta-multiplication" m_k^{ij}: "multiplication"

The most important of these operations is the operation m_k^{ij}, defined whenever $i \neq j \in S$ and $k \notin S \backslash \{i, j\}$. On tangles, it is "stitching": the operation $\mathcal{K}(S) \to \mathcal{K}((S\backslash\{i,j\}) \cup \{k\})$ that takes the head of component i in a tangle K and stitches it to the tail of component j, renaming the resulting single component k, as in Figure 14.1[5]. Clearly from the construction in Aside 1.1, the corresponding operation on $\{U^{\otimes S}\}$ is "multiply tensor factor i with tensor factor j, storing the result in tensor factor k. We have a "meta-multiplication" operation $m_k^{ij}\colon \mathcal{F}(S) \to \mathcal{F}((S\backslash\{i,j\}) \cup \{k\})$ which takes "the formula for an element ζ in $U^{\otimes S}$" to "the formula for $m_k^{ij}(\zeta)$", and which likewise intertwines Z. Namely, we have $\mathbb{V}/\!\!/m_k^{ij} = m^{ij}/\!\!/\mathbb{V}$ and $m_k^{ij}/\!\!/Z = Z/\!\!/m_k^{ij}$.

(3) Similarly, if $S_1 \cap S_2 = \emptyset$, there is a "disjoint union" operation $*\colon \mathcal{K}(S_1) \times \mathcal{K}(S_2) \to \mathcal{K}(S_1 \cup S_2)$.[6] The corresponding operation on $\{\mathcal{U}^S\}$ is the tensor product operation $* = \otimes\colon U^{\otimes S_1} \times U^{\otimes S_2} \to U^{\otimes(S_1 \cup S_2)}$. We ensure that there is a compatible $*\colon \mathcal{F}(S_1) \times \mathcal{F}(S_2) \to \mathcal{F}(S_1 \cup S_2)$.

(4) $\mathbb{V}$ is injective. A formula is determined its value.

(5) The rank of $\mathcal{F}(S)$ (over some ring $\mathcal{R}$ of Laurent polynomials which we will specify later) grows polynomially in the size $|S|$ of S, and all the operations on $\mathcal{F}(S)$ are computable using a polynomial number of ring operations.

These five properties taken together are almost enough for what we want. If K is an n-crossing tangle, it be presented as some stitching of a disjoint union of n individual crossings floating indepedently. Hence by using (1)–(3), a formula $Z(K)$ for the invariant $z(K)$ can be computed using $O(n)$ stitchings and unions. By (4), that formula is in itself an invariant. Finally, by (5), $Z(K)$ can be computed using a polynomial number of ring operations (and some combinatorial overhead which amounts to much less).

To show that the computation of Z is poly-time it remains to bound the complexity of the ring elements that we encounter, and hence the complexity of ring operations among them. This is done in Section ??.

[5]The careful reader will notice that stitching is only partially defined, for the head of i must lie next to the tail of j for m_k^{ij} to make sense, and that it is sometimes ill-defined, for there may be more than one path connecting the head of i with the tail of j. Please accept our assurances that these issues do not lead to any difficulties, and that they are fully resolved in Section 2.

[6]As in footnote 5, there is a minor placement issue here. It is resolved in Section 2.

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