Pensieve header: The “Speedy” engine.

```
In[∘]:= Once[<< KnotTheory`];
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

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ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[∘]:= PP_ = Identity; $k = 2; γ = 1; ℏ = 1;
```

The “Speedy” Engine

Internal Utilities

Canonical Form:

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
          Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := Module[
          {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
          Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] × (Times @@ vs^ps)]
         ];
        CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; 𝔼[L, Q, P] stands for $e^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] × 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

In[ ]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

In[ ]:=
```
U2l = {B_i_^p_. :> e^(-p ħ γ b_i), B^p_. :> e^(-p ħ γ b), T_i_^p_. :> e^(-p ħ t_i), T^p_. :> e^(-p ħ t), 𝒜_i_^p_. :> e^(p γ α_i), 𝒜^p_. :> e^(p γ α)};
l2U = {e^(c_. b_i_+d_.) :> B_i^(-c/(ħ γ)) e^d, e^(c_. b+d_.) :> B^(-c/(ħ γ)) e^d,
   e^(c_. t_i_+d_.) :> T_i^(-c/ħ) e^d, e^(c_. t+d_.) :> T^(-c/ħ) e^d,
   e^(c_. α_i_+d_.) :> 𝒜_i^(c/γ) e^d, e^(c_. α+d_.) :> 𝒜^(c/γ) e^d,
   e^ℰ_ :> e^(Expand@ℰ)};
```

Derivatives in the presence of exponentiated variables:

In[ ]:=
```
D_b[f_] := ∂_b f - ħ γ B ∂_B f; D_(b_i_)[f_] := ∂_(b_i) f - ħ γ B_i ∂_(B_i) f;
D_t[f_] := ∂_t f - ħ T ∂_T f; D_(t_i_)[f_] := ∂_(t_i) f - ħ T_i ∂_(T_i) f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_(α_i_)[f_] := ∂_(α_i) f + γ 𝒜_i ∂_(𝒜_i) f;
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

In[ ]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P;
Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
Zip_{ζ_,ζs___}[P_] :=
 (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> (D_{ζ*,d}[f])) /. ζ* → 0 /.
  ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|e^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

```
In[◦]:= QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

```
In[◦]:= LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    Zs = zs /. {b → B, t → T, α → 𝒜};
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    Zrule = Join[zrule,
      zrule /. r_Rule :> ((U = r[[1]] /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. (Zrule ⋃ ζrule);
    EEQ[ps___] := EEQ[ps] =
      (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
    CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
      Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
          Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ] ];
```

```
In[◦]:= B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
    Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
     ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

𝔼 morphisms with domain and range.

In[∘]:=
```
B_is_List[𝔼_d1_→r1_[L1_, Q1_, P1_], 𝔼_d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] // 𝔼_d2_→r2_[L2_, Q2_, P2_] :=
  B_r1⋂d2[𝔼_d1→r1[L1, Q1, P1], 𝔼_d2→r2[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_d1_→r1_[L1_, Q1_, P1_] 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼_(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] × 𝔼[L2, Q2, P2]);
𝔼_dr_[L_, Q_, P_]_$k_ := 𝔼_dr @@ 𝔼[L, Q, P]_$k;
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

𝔼[Λ]

In[∘]:=
```
𝔼_dr_[Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼_dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, ⅇ^(Λ-Λ0)]_$k /. l2U]
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
    ReleaseHold[Hold[
        SD[op_nisp,$k_Integer, Block[{i, j, k}, op_isp,$k = ℰ; op_nis,$k]];
        SD[op_isp, op_{is},$k]; SD[op_sis__, op_{sis}];
      ] /. {SD → SetDelayed,
        isp → {is} /. {i → i_, j → j_, k → k_},
        nis → {is} /. {i → ii, j → jj, k → kk},
        nisp → {is} /. {i → ii_, j → jj_, k → kk_}
      }] ]]
```

The Objects

Symmetric Algebra Objects

In[∘]:=
```
sm_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} := 𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_{i_} := 𝔼_{{}→{i}}[0];
sη_{i_} := 𝔼_{{i}→{}}[0];
```

In[∘]:=
```
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

Booting Up QU

In[∘]:=
```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

In[∘]:=
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^{-ϵ β_i} η_j) y_k]]
```

Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$. As a map $\mathrm{P}:\mathbb{A}\otimes\mathbb{B}\to Q$.

$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

In[∘]:=
```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum_{k=1}^{$k+1} ((1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1 - e^{k γ ϵ ℏ}))],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
        (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_j α_i / ℏ, η_j ξ_i / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
        (R_{1,2} // ((P_{{i,1},0})_{$k} (P_{{2,j},$k-1})_{$k}))[3]]]]
```

In[∘]:=
```
R_{1,2} // P_{2,3}
```

Out[∘]=
```
𝔼_{{3}→{1}}[b_1 β_3, y_1 η_3, 1 + O[ϵ]^3]
```

In[∘]:=
```
(R1,2 // ((P{i,1},0)2 (P{2,j},1)2)) [3]
```

Out[∘]= $1 + \left(-\frac{1}{8}\eta_j^2 \xi_i^2 - \frac{\eta_j^3 \xi_i^3}{4\hbar} - \frac{\eta_j^4 \xi_i^4}{16\hbar^2}\right)\epsilon^2 + O[\epsilon]^3$

In[∘]:=
```
Define[aSi = (aσi→2 R̄1,i) // P2,1,
  aS̄i = 𝔼{i}→{i}[-ai αi, -xi 𝒜i ξi, 1 + If[$k == 0, 0, (aS̄{i},$k-1)$k[3] -
      ((aS̄{i},0)$k // aSi // (aS̄{i},$k-1)$k) [3]]]]
```

In[∘]:=
```
Define[bSi = bσi→1 Ri,2 // aS2 // P2,1,
  bS̄i = bσi→1 Ri,2 // aS̄2 // P2,1,
  aΔi→j,k = (R1,j R2,k) // bm1,2→3 // Pi,3,
  bΔi→j,k = (Rj,1 Rk,2) // am1,2→3 // P3,i]
```

In[∘]:=
```
Define[
  dmi,j→k = ((sϒi→4,4,1,1 // aΔ1→1,2 // aΔ2→2,3 // aS̄3) (sϒj→-1,-1,-4,-4 // bΔ-1→-1,-2 // bΔ-2→-2,-3)) //
    (P1,-3 P3,-1 am2,-4→k bm4,-2→k) ]
```

NB. We use the co-algebra structure B tensor A^{cop}. This has the benefit of making our algebra quasi-triangular in the traditional sense of the word.
Watch out: $\Delta_{i\to j,k}$ means *j* is to the RIGHT of strand k and dS looks like an S.

In[∘]:=
```
Define[dσi→j = aσi→j bσi→j,
  dϵi = sϵi, dηi = sηi,
  dSi = sϒi→1,1,2,2 // (bS1 aS̄2) // dm2,1→i,
  dS̄i = sϒi→1,1,2,2 // (bS̄1 aS2) // dm2,1→i,
  dΔi→j,k = (bΔi→1,3 aΔi→4,2) // (dm3,4→k dm1,2→j) ]
```

In[∘]:=
```
Define[Ci = 𝔼{}→{i}[0, 0, Bi^(1/2) ℯ^(-ℏ ϵ ai/2)]$k,
  C̄i = 𝔼{}→{i}[0, 0, Bi^(-1/2) ℯ^(ℏ ϵ ai/2)]$k,
  ci = 𝔼{}→{i}[0, 0, Bi^(1/4) ℯ^(-ℏ ϵ ai/4)]$k,
  c̄i = 𝔼{}→{i}[0, 0, Bi^(-1/4) ℯ^(ℏ ϵ ai/4)]$k,
  Kinki = (R1,3 C̄2) // dm1,2→1 // dm1,3→i,
  K̄inki = (R̄1,3 C2) // dm1,2→1 // dm1,3→i,
  ρi = (c1 c̄3 dSi) // dm1,i→i // dmi,3→i] (*ρ reverses a strand*)
```

Note. $t = -\epsilon a + \gamma b$ and $b = t/\gamma + \epsilon a/\gamma$

In[∘]:=
```
Define[b2ti = 𝔼{i}→{i}[αi ai + βi (ϵ ai + ti) / γ + ξi xi + ηi yi],
  t2bi = 𝔼{i}→{i}[αi ai + τi (-ϵ ai + γ bi) + ξi xi + ηi yi]]
```

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $d\Delta_{1\to1,2}$
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // dS_1
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, y_1]$ // dS_1
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $\overline{dS}_1$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0, 0, (x_1 + x_2) - \hbar a_2 x_1 \epsilon + \frac{1}{2}\hbar^2 a_2^2 x_1 \epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -x_1 + (\hbar x_1 - \hbar a_1 x_1)\epsilon + \left(-\frac{1}{2}\hbar^2 x_1 + \hbar^2 a_1 x_1 - \frac{1}{2}\hbar^2 a_1^2 x_1\right)\epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -\frac{y_1}{B_1} + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -x_1 - \hbar a_1 x_1 \epsilon - \frac{1}{2}\left(\hbar^2 a_1^2 x_1\right)\epsilon^2 + O[\epsilon]^3\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, (1 + \epsilon a_1 \hbar) x_1]$ // dS_1

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -x_1 + \left(\frac{\hbar^2 x_1}{2} - \hbar^2 a_1 x_1 + \frac{1}{2}\hbar^2 a_1^2 x_1\right)\epsilon^2 + O[\epsilon]^3\right]$

In[∘]:= $((-1 + \hbar) x_1 + (1 - \hbar) a_1 x_1)$ // Expand

Out[∘]= $-x_1 + \hbar x_1 + a_1 x_1 - \hbar a_1 x_1$

In[∘]:= $t2b_1\, t2b_2$ // $P_{2,1}$

Out[∘]= $\mathbb{E}_{\{1,2\}\to\{\}}\left[\frac{\alpha_2 \tau_1}{\hbar}, \frac{\eta_1 \xi_2}{\hbar}, 1 + \left(\frac{\eta_1^2 \xi_2^2}{4\hbar} - \frac{\tau_1 \tau_2}{\hbar}\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, y_1]$ // $b\Delta_{1\to1,2}$
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, y_1]$ // $d\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0, 0, (B_2 y_1 + y_2) + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0, 0, (B_2 y_1 + y_2) + O[\epsilon]^2\right]$

In[∘]:= $(R_{1,2}$ // $bS_1) \equiv \overline{R}_{1,2}$
$(R_{1,2}$ // $aS_2) \equiv \overline{R}_{1,2}$

Out[∘]= True

Out[∘]= True

$\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $d\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[0, 0, (x_1 + x_2) - \hbar a_2 x_1 \epsilon + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $a\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0, 0, (x_1 + x_2) - \hbar a_1 x_2 \epsilon + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $(\overline{aS})_1$
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // aS_1

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -x_1 + (\hbar x_1 - \hbar a_1 x_1) \in + O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -x_1 - \hbar a_1 x_1 \in + O[\in]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, b_1 y_2]$ // $bm_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, b_1 y_1 - y_1 \in + O[\in]^2\right]$

In[∘]:= $a\Delta_{i\to1,2}$ // aS_1 // $am_{1,2\to1}$
$a\Delta_{i\to1,2}$ // aS_2 // $am_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0, 0, 1 + O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0, 0, 1 + O[\in]^2\right]$

In[∘]:= $a\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[a_1 \alpha_1 + a_2 \alpha_1, x_1 \xi_1 + x_2 \xi_1, 1 + \left(-\hbar a_1 x_2 \xi_1 + \frac{1}{2} \hbar x_1 x_2 \xi_1^2\right) \in + O[\in]^2\right]$

Testing

co-associativity

In[∘]:= $(d\Delta_{1\to1,2}$ // $d\Delta_{2\to2,3}) \equiv (d\Delta_{1\to2,3}$ // $d\Delta_{2\to1,2})$

Out[∘]= True

algebra morphism

In[∘]:= $(d\Delta_{i\to1,2}\, d\Delta_{j\to3,4}$ // $dm_{1,3\to i}$ // $dm_{2,4\to j}) \equiv (dm_{i,j\to k}$ // $d\Delta_{k\to i,j})$

Out[∘]= True

associativity

In[∘]:= $(dm_{1,2\to k}$ // $dm_{k,3\to k}) \equiv (dm_{2,3\to k}$ // $dm_{1,k\to k})$

Out[∘]= True

antipode

In[∘]:= $d\Delta_{i\to1,2}$ // dS_1 // $dm_{1,2\to1}$
$d\Delta_{i\to1,2}$ // dS_2 // $dm_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0, 0, 1 + O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0, 0, 1 + O[\in]^2\right]$

quasi-triangular axioms

In[∘]:= $(R_{1,3} \,//\, d\Delta_{1\to 1,2}) \equiv (R_{1,3}\, R_{2,4} \,//\, dm_{3,4\to 3})$
$(R_{1,3} \,//\, d\Delta_{3\to 2,3}) \equiv (R_{1,3}\, R_{0,2} \,//\, dm_{1,0\to 1})$
$(d\Delta_{i\to k,j}\, R_{1,2} \,//\, dm_{j,1\to 1} \,//\, dm_{k,2\to 2}) \equiv (R_{1,2}\, d\Delta_{i\to j,k} \,//\, dm_{1,j\to 1} \,//\, dm_{2,k\to 2})$

Out[∘]= True

Out[∘]= True

Out[∘]= True

In[∘]:= $(R_{1,2} \,//\, aS_2) \equiv (\overline{R}_{1,2})$

Out[∘]= True

In[∘]:= $(R_{1,2} \,//\, dS_1) \equiv (\overline{R}_{1,2})$
$(R_{1,2} \,//\, \overline{dS}_2) \equiv (\overline{R}_{1,2})$

Out[∘]= True

Out[∘]= True

In[∘]:= $QQ_{s\_,r\_} := R_{11,22}\, R_{33,44} \,//\, dm_{11,44\to s} \,//\, dm_{22,33\to r}$
$\overline{QQ}_{s\_,r\_} := \overline{R}_{22,11}\, \overline{R}_{44,33} \,//\, dm_{11,44\to s} \,//\, dm_{22,33\to r}$

In[∘]:= $QQ_{1,2}\, \overline{QQ}_{3,4} \,//\, dm_{1,3\to 1} \,//\, dm_{2,4\to 2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\, 0,\, 1 + O[\epsilon]^2\right]$

Drinfeld element u

In[∘]:= $u_{i\_} := R_{11,22} \,//\, dS_{22} \,//\, dm_{22,11\to i}$
$\overline{u}_{i\_} := \overline{R}_{11,22} \,//\, \overline{dS}_{22} \,//\, dm_{22,11\to i}$
$\overline{uu}_{i\_} := \overline{R}_{11,22} \,//\, \overline{dS}_{22} \,//\, dm_{11,22\to i}$
$\overline{u2}_{i\_} := \overline{R}_{11,22} \,//\, dS_{11} \,//\, dm_{11,22\to i}$
$\overline{u3}_{i\_} := R_{11,22} \,//\, dS_{11} \,//\, dS_{11} \,//\, dm_{22,11\to i}$

In[∘]:= $u_i\, \overline{u}_j \,//\, dm_{i,j\to i}$
$u_i\, \overline{uu}_j \,//\, dm_{i,j\to i}$
$u_i\, \overline{u2}_j \,//\, dm_{i,j\to i}$
$u_i\, \overline{u3}_j \,//\, dm_{i,j\to i}$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\, 0,\, 1 + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\, 0,\, B_i - \hbar\, a_i\, B_i\, \epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\, 0,\, B_i - \hbar\, a_i\, B_i\, \epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\, 0,\, 1 + O[\epsilon]^2\right]$

In[∘]:= $(\mathbf{u_1}\ //\ \mathbf{dS_1})$

$\mathbf{R_{11,22}}\ //\ \mathbf{dS_{22}}\ //\ \mathbf{dm_{11,22\to i}}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-\hbar\, a_1 b_1,\ -\frac{\hbar\, x_1 y_1}{B_1},\ 1+\left(\frac{\hbar^2 x_1 y_1}{B_1}-\frac{\hbar^2 a_1 x_1 y_1}{B_1}-\frac{3\hbar^3 x_1^2 y_1^2}{4 B_1^2}\right)\in+\left(-\frac{\hbar^3 x_1 y_1}{2 B_1}+\frac{\hbar^3 a_1 x_1 y_1}{B_1}-\frac{\hbar^3 a_1^2 x_1 y_1}{2 B_1}+\frac{5\hbar^4 x_1^2 y_1^2}{2 B_1^2}-\frac{5\hbar^4 a_1 x_1^2 y_1^2}{2 B_1^2}+\frac{\hbar^4 a_1^2 x_1^2 y_1^2}{2 B_1^2}-\frac{67\hbar^5 x_1^3 y_1^3}{36 B_1^3}+\frac{3\hbar^5 a_1 x_1^3 y_1^3}{4 B_1^3}+\frac{9\hbar^6 x_1^4 y_1^4}{32 B_1^4}\right)\in^2+O[\in]^3\Big]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\Big[-\hbar\, a_i b_i,\ -\frac{\hbar\, x_i y_i}{B_i},\ 1+\left(\frac{\hbar^2 x_i y_i}{B_i}-\frac{\hbar^2 a_i x_i y_i}{B_i}-\frac{3\hbar^3 x_i^2 y_i^2}{4 B_i^2}\right)\in+\left(-\frac{\hbar^3 x_i y_i}{2 B_i}+\frac{\hbar^3 a_i x_i y_i}{B_i}-\frac{\hbar^3 a_i^2 x_i y_i}{2 B_i}+\frac{5\hbar^4 x_i^2 y_i^2}{2 B_i^2}-\frac{5\hbar^4 a_i x_i^2 y_i^2}{2 B_i^2}+\frac{\hbar^4 a_i^2 x_i^2 y_i^2}{2 B_i^2}-\frac{67\hbar^5 x_i^3 y_i^3}{36 B_i^3}+\frac{3\hbar^5 a_i x_i^3 y_i^3}{4 B_i^3}+\frac{9\hbar^6 x_i^4 y_i^4}{32 B_i^4}\right)\in^2+O[\in]^3\Big]$$

In[∘]:= $$\left(\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ B_1{}^{-1}\left(1+\in a_1\hbar+\frac{\in^2}{2}a_1{}^2\hbar^2\right)\right]\mathbf{u_2}\ //\ \mathbf{dm_{1,2\to1}}\right)\equiv(\mathbf{u_1}\ //\ \mathbf{dS_1})$$

Out[∘]= True

In[∘]:= $\mathbf{u_1}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-\hbar\, a_1 b_1,\ -\frac{\hbar\, x_1 y_1}{B_1},\ B_1+\left(-\hbar\, a_1 B_1-\hbar^2 x_1 y_1-\hbar^2 a_1 x_1 y_1-\frac{3\hbar^3 x_1^2 y_1^2}{4 B_1}\right)\in+\left(\frac{1}{2}\hbar^2 a_1^2 B_1-\frac{1}{2}\hbar^3 x_1 y_1+\frac{1}{2}\hbar^3 a_1^2 x_1 y_1-\frac{\hbar^4 x_1^2 y_1^2}{2 B_1}+\frac{\hbar^4 a_1 x_1^2 y_1^2}{4 B_1}+\frac{\hbar^4 a_1^2 x_1^2 y_1^2}{2 B_1}-\frac{13\hbar^5 x_1^3 y_1^3}{36 B_1^2}+\frac{3\hbar^5 a_1 x_1^3 y_1^3}{4 B_1^2}+\frac{9\hbar^6 x_1^4 y_1^4}{32 B_1^3}\right)\in^2+O[\in]^3\Big]$$

q

In[∘]:= $(\mathbf{u_1}\ //\ \mathbf{dS_1})\ \overline{\mathbf{u3}}_2\ //\ \mathbf{dm_{1,2\to1}}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{1}{B_1}+\frac{\hbar\, a_1\in}{B_1}+O[\in]^2\right]$$

In[∘]:= $(\mathbf{u_1}\ //\ \mathbf{d\Delta_{1\to2,1}})\equiv\left(\overline{\mathbf{QQ}}_{1,2}\ \mathbf{u_3}\ \mathbf{u_4}\ //\ \mathbf{dm_{1,3\to1}}\ //\ \mathbf{dm_{2,4\to2}}\right)$

Out[∘]= True

In[∘]:= $\mathbb{E}_{\{\}\to\{i\}}[0,\ 0,\ x_i]\ //\ \mathbf{dS_i}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ -x_i+(\hbar\, x_i-\hbar\, a_i x_i)\in+O[\in]^2\right]$$

In[∘]:= **Kink₁**

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1 b_1,\ \hbar\, x_1 y_1,\ \frac{1}{\sqrt{B_1}}+\left(\frac{\hbar\, a_1}{2\sqrt{B_1}}-\frac{\hbar^3 x_1^2 y_1^2}{4\sqrt{B_1}}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= **(u₁ // dS₁) u₂ // dm₁,₂→₁**
(u₁ // dS₁) u₂ // dm₂,₁→₁

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[-2\hbar\, a_1 b_1,\ \frac{(-\hbar-\hbar B_1)\, x_1 y_1}{B_1^2},\ B_1+\left(-\hbar\, a_1 B_1+\frac{a_1(-2\hbar^2-\hbar^2 B_1)\, x_1 y_1}{B_1}+\frac{(-7\hbar^3-8\hbar^3 B_1-3\hbar^3 B_1^2)\, x_1^2 y_1^2}{4 B_1^3}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[-2\hbar\, a_1 b_1,\ \frac{(-\hbar-\hbar B_1)\, x_1 y_1}{B_1^2},\ B_1+\left(-\hbar\, a_1 B_1+\frac{a_1(-2\hbar^2-\hbar^2 B_1)\, x_1 y_1}{B_1}+\frac{(-7\hbar^3-8\hbar^3 B_1-3\hbar^3 B_1^2)\, x_1^2 y_1^2}{4 B_1^3}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= **(u₁ // dS₁)**
u₂

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\, a_1 b_1,\ -\frac{\hbar\, x_1 y_1}{B_1},\ 1+\left(\frac{\hbar^2 x_1 y_1}{B_1}-\frac{\hbar^2 a_1 x_1 y_1}{B_1}-\frac{3\hbar^3 x_1^2 y_1^2}{4 B_1^2}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[-\hbar\, a_2 b_2,\ -\frac{\hbar\, x_2 y_2}{B_2},\ B_2+\left(-\hbar\, a_2 B_2-\hbar^2 x_2 y_2-\hbar^2 a_2 x_2 y_2-\frac{3\hbar^3 x_2^2 y_2^2}{4 B_2}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= **R₁,₂**
$\overline{\mathbf{R}}_{1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 b_1,\ \hbar\, x_2 y_1,\ 1-\frac{1}{4}\left(\hbar^3 x_2^2 y_1^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[-\hbar\, a_2 b_1,\ -\frac{\hbar\, x_2 y_1}{B_1},\ 1+\left(-\frac{\hbar^2 a_2 x_2 y_1}{B_1}-\frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= **C₁**

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \sqrt{B_1}-\frac{1}{2}\left(\hbar\, a_1\sqrt{B_1}\right)\epsilon+O[\epsilon]^2\right]$

The Knot Tensors

In[∘]:=
```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
  kR̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
  km_{i,j→k} = ((t2b_i t2b_j) // dm_{i,j→k} // b2t_k ) /. {t_k → t, T_k → T, τ_{i|j} → 0},
  kC_i = (C_i // b2t_i ) /. T_i → T,
  kC̄_i = (C̄_i // b2t_i ) /. T_i → T,
  kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
  kKink̄_i = Kink̄_i // b2t_i /. {t_i → t, T_i → T}]
```

In[∘]:=
```
Define[tm_{i,j→k} = t2b_i // t2b_j // dm_{i,j→k} // b2t_k]
Define[tΔ_{i→j,k} = t2b_i // dΔ_{i→j,k} // b2t_j // b2t_k]
Define[tS_i = t2b_i // dS_i // b2t_i]
Define[tS̄_i = t2b_i // dS̄_i // b2t_i]
Define[tR_{i,j} = R_{i,j} // b2t_i // b2t_j, tR̄_{i,j} = R̄_{i,j} // b2t_i // b2t_j]
Define[tC_i = C_i // b2t_i , tC̄_i = C̄_i // b2t_i ]
Define[tKink_i = Kink_i // b2t_i , tKink̄_i = Kink̄_i // b2t_i]
Define[tBS_{i,j→k} =
   tC_3 tC_4 tΔ_{i→r1,l1} tΔ_{j→r2,l2} // tS̄_{r1} // tS_{r2} // tm_{l1,r2→k} // tm_{k,3→k} // tm_{k,4→k} // tm_{k,r1→k} // tm_{k,l2→k}]
```

In[∘]:=
```
R_{1,3} R_{2,6} // dm_{3,6→3}
R_{1,3} // dΔ_{1→2,1}
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\ \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\ 1 + \left(-\frac{1}{4}\,\hbar^3\, B_2^2\, x_3^2\, y_1^2 - \frac{1}{4}\,\hbar^3\, x_3^2\, y_2^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\ \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\ 1 + \left(-\frac{1}{4}\,\hbar^3\, B_2^2\, x_3^2\, y_1^2 - \frac{1}{4}\,\hbar^3\, x_3^2\, y_2^2\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:=
```
tR_{1,2} // tS̄_1 // tS̄_1 // tm_{1,2→1}
(tR_{1,2} // tS̄_1 // tS̄_1 // tm_{1,2→1} // tS_1) 𝔼_{{}→{2}}[0, 0, T_2 (1 - 2 ϵ ℏ a_1)] // tm_{1,2→1}
(tR_{1,2} // tS̄_1 // tS̄_1 // tm_{2,1→1}) 𝔼_{{}→{2}}[0, 0, T_2 (1 - 2 ϵ ℏ a_1)] // tm_{1,2→1}
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ 1 + \left(\hbar\, a_1^2 + \hbar^2\, x_1\, y_1 - \frac{1}{4}\,\hbar^3\, x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ 1 + \left(\hbar\, a_1^2 - \hbar^2\, x_1\, y_1 - \frac{1}{4}\,\hbar^3\, x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ 1 + \left(\hbar\, a_1^2 - \hbar^2\, x_1\, y_1 - \frac{1}{4}\,\hbar^3\, x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:=
```
𝔼_{{}→{2}}[0, 0, x_2] // dS_2
```

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0,\ 0,\ -x_2 - \hbar\, a_2\, x_2\, \epsilon + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2] \mathbin{//} \overline{dS}_2$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, -\frac{y_2}{B_2} + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2] \mathbin{//} \overline{dS}_2 \mathbin{//} \overline{dS}_2$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, y_2 + \hbar\, y_2\, \epsilon + O[\epsilon]^2\right]$

In[∘]:= $tm_{i,j\to k}$
$tR_{i,j}$
$\overline{tR}_{i,j}$
tC_i
$\overline{tC}_i$
$tKink_i$
$\overline{tKink}_i$
$t\Delta_{i\to j,k}$
tS_i

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,\alpha_i + a_k\,\alpha_j + t_k\,\tau_i + t_k\,\tau_j,\ y_k\,\eta_i + \frac{y_k\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,\xi_i}{\mathcal{A}_j} + \frac{(1-T_k)\,\eta_j\,\xi_i}{\hbar} + x_k\,\xi_j,$
$1 + \left(2\,a_k\,T_k\,\eta_j\,\xi_i + \frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j} + \frac{(1-3\,T_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i} + \frac{(1-3\,T_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j} + \frac{\left(1-4\,T_k+3\,T_k^2\right)\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon +$
$O[\epsilon]^2\Big]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,t_i,\ \hbar\,x_j\,y_i,\ 1 + \left(\hbar\,a_i\,a_j - \frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,t_i,\ -\frac{\hbar\,x_j\,y_i}{T_i},\ 1 + \left(-\hbar\,a_i\,a_j - \frac{\hbar^2\,a_i\,x_j\,y_i}{T_i} - \frac{\hbar^2\,a_j\,x_j\,y_i}{T_i} - \frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,T_i^2}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \sqrt{T_i} - \hbar\,a_i\,\sqrt{T_i}\,\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \frac{1}{\sqrt{T_i}} + \frac{\hbar\,a_i\,\epsilon}{\sqrt{T_i}} + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,t_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{T_i}} + \left(\frac{\hbar\,a_i}{\sqrt{T_i}} + \frac{\hbar\,a_i^2}{\sqrt{T_i}} - \frac{\hbar^3\,x_i^2\,y_i^2}{4\,\sqrt{T_i}}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,t_i,\ -\frac{\hbar\,x_i\,y_i}{T_i},\ \sqrt{T_i} + \left(-\hbar\,a_i\,\sqrt{T_i} - \hbar\,a_i^2\,\sqrt{T_i} - \frac{2\,\hbar^2\,a_i\,x_i\,y_i}{\sqrt{T_i}} - \frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,T_i^{3/2}}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i + a_k\,\alpha_i + t_j\,\tau_i + t_k\,\tau_i,\ y_j\,\eta_i + T_j\,y_k\,\eta_i + x_j\,\xi_i + x_k\,\xi_i,$
$1 + \left(-\hbar\,a_j\,T_j\,y_k\,\eta_i + \frac{1}{2}\,\hbar\,T_j\,y_j\,y_k\,\eta_i^2 - \hbar\,a_j\,x_k\,\xi_i + \frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\Big]$

Out[∘]= $$\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i - t_i\,\tau_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{T_i} - x_i\,\mathcal{A}_i\,\xi_i + \frac{(\mathcal{A}_i - T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,T_i},\right.$$
$$1 + \left(\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{T_i} - \frac{\hbar\,a_i\,y_i\,\mathcal{A}_i\,\eta_i}{T_i} - \frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,T_i^2} - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i + \frac{2\,a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{T_i} - \frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{T_i} + \frac{(-\mathcal{A}_i + T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{T_i} + \frac{y_i\left(3\,\mathcal{A}_i^2 - T_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2\,T_i^2} - \frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 + \frac{x_i\left(3\,\mathcal{A}_i^2 - T_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2\,T_i} + \frac{\left(-3\,\mathcal{A}_i^2 + 4\,T_i\,\mathcal{A}_i^2 - T_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\,\hbar\,T_i^2}\right)\epsilon + O[\epsilon]^2\Bigg]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[a_i\,t_i,\ x_i\,y_i,\ \frac{1}{\sqrt{T_i}} + \left(\frac{a_i}{\sqrt{T_i}} + \frac{a_i^2}{\sqrt{T_i}} - \frac{x_i^2\,y_i^2}{4\,\sqrt{T_i}}\right)\epsilon + O[\epsilon]^2\right]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[-a_i\,t_i,\ -\frac{x_i\,y_i}{T_i},\ \sqrt{T_i} + \left(-a_i\,\sqrt{T_i} - a_i^2\,\sqrt{T_i} - \frac{2\,a_i\,x_i\,y_i}{\sqrt{T_i}} - \frac{3\,x_i^2\,y_i^2}{4\,T_i^{3/2}}\right)\epsilon + O[\epsilon]^2\right]$$

In[∘]:=
```
Z@Knot[3, 1]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]= $$\mathbb{E}_{\{\}\to\{0\}}\left[0,\ 0,\ \frac{T}{1 - T + T^2} + \left(\frac{a\left(-2\,T\,\hbar + 2\,T^3\,\hbar\right)}{1 - 2\,T + 3\,T^2 - 2\,T^3 + T^4} + \frac{-2\,T\,\hbar + 3\,T^2\,\hbar - 2\,T^3\,\hbar + T^4\,\hbar}{1 - 3\,T + 6\,T^2 - 7\,T^3 + 6\,T^4 - 3\,T^5 + T^6} + \frac{x\,y\left(-2\,T\,\hbar^2 - 2\,T^2\,\hbar^2\right)}{1 - 2\,T + 3\,T^2 - 2\,T^3 + T^4}\right)\epsilon + O[\epsilon]^2\right]$$

In[∘]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
    xs and a length 2n list of rotation numbers rots. Crossing
    sites are indexed 1 through 2n, and rots⟦k⟧ is the rotation
    between site k-1 and site k. RVK is also a casting operator
    converting to the RVK presentation from other knot presentations.";
```

In[ ]:=
```
RVK[pd_PD] := PP_RVK@Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]  PositiveQ@x
                           { Xm[x[[2]], x[[1]]]    True      ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten@Replace[front, k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]];
              {1 - l, k + 1, l}),
            _Xp | _Xm :> {}
           }), {1}],
      Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
     ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[ ]:=
```
rot[i_, 0] := 𝔼_{}→{i}[0, 0, 1];
rot[i_, n_] := rot[i, n, $k];
rot[i_, n_, k_] := Module[{j},
   rot[i, n, k] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] \overline{kC}_j] // km_{i,j→i}];
```

In[ ]:=
```
Width[pd_PD] :=
 Max[Length /@ FoldList[Complement[#1 ⋃ #2, #1 ⋂ #2] &, {}, List @@ List @@@ pd]]
```

In[ ]:=
```
ThinPosition[K_] := Module[{todo, done, pd, c},
   todo = List @@ PD@K; done = {}; pd = PD[];
   While[todo =!= {},
    AppendTo[pd, c = RandomChoice@MaximalBy[todo, Length[done ⋂ List @@ #] &]];
    todo = DeleteCases[todo, c];
    done = done ⋃ List @@ c];
   pd ];
ThinPosition[K_, n_] := First@MinimalBy[Table[ThinPosition[K], n], Width];
```

In[∘]:=
```
Z[K_] := Z[RVK@EchoFunction[Width]@ThinPosition[K, 100]];
Z[rvk_RVK] := Monitor[ PP"Z"@Module[{ξ, done, st, c, χ, i, j, k},
    ξ = 1; done = {}; st = Range[2 Length[rvk[[1]]]]; $M = {};
    Do[AppendTo[$M, c];
     {i, j} = List @@ c;
     χ = (c /. {_Xp :> kR_{i,j} OverBar[kKink_0], _Xm :> OverBar[kR_{i,j}] kKink_0}) // km_{j,0→j};
     Do[χ = (rot[0, rvk[[2, k]]] χ) // km_{0,k→k}, {k, {i, j}}];
     ξ *= χ;
     Do[
      If[MemberQ[done, k + 1], ξ = ξ // km_{k,k+1→k}; st = st /. k + 1 → k];
      If[MemberQ[done, k - 1], ξ = ξ // km_{st[[k-1]],k→st[[k-1]]}; st = st /. k → st[[k - 1]]],
      {k, {i, j}}];
     done = done ⋃ {i, j},
     {c, rvk[[1]]}
    ];
    CF /@ (ξ /. {x_1 → x, y_1 → y, a_1 → a})
   ], {Length@$M, $M}]
```

In[∘]:=
```
Z31 = Z@Knot[3, 1]
```

Get: ParentDirectory[File] in $Path is not a string.

KnotTheory: Loading precomputed data in PD4Knots`.

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Out[∘]=

$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{T}{1-T+T^2}+\left(\frac{a\left(-2T+2T^3\right)}{1-2T+3T^2-2T^3+T^4}+\frac{-2T+3T^2-2T^3+T^4}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T-2T^2\right)x\,y}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+$$

$$\left(\frac{a^2\left(2T+2T^2-12T^3+2T^4+2T^5\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{a\left(4T-4T^2-14T^3+16T^4-10T^5+4T^6\right)}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\right.$$

$$\frac{4T-11T^2+6T^3-2T^5+4T^6-2T^7+T^8}{2-10T+30T^2-60T^3+90T^4-102T^5+90T^6-60T^7+30T^8-10T^9+2T^{10}}+$$

$$\frac{a\left(4T+12T^2-12T^3-8T^4\right)x\,y}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(2T-2T^2-4T^3-4T^4-2T^5+2T^6\right)x\,y}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+$$

$$\left.\frac{\left(3T+9T^2+3T^3\right)x^2y^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}\right)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:= $cw = \left(\left(\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{(-2T-2T^2)}{1-2T+3T^2-2T^3+T^4} \text{ /. } \{T\to T_1\}\right]\right) w_i \text{ // } tm_{1,i\to i}\right)[[3]]$

Out[∘]= $\left(\frac{-T_i+T_i^3}{1-2T_i+3T_i^2-2T_i^3+T_i^4}+\frac{a_i(-2T_i+2T_i^3)}{1-2T_i+3T_i^2-2T_i^3+T_i^4}+\frac{(-2T_i-2T_i^2)x_i y_i}{1-2T_i+3T_i^2-2T_i^3+T_i^4}\right)+ \left(\left(-2a_iT_i-2a_i^2T_i+4a_iT_i^2+4a_i^2T_i^2-4a_iT_i^4-4a_i^2T_i^4+2a_iT_i^5+2a_i^2T_i^5- 4a_iT_ix_iy_i+4a_iT_i^2x_iy_i+4a_it_iT_i^2x_iy_i+4a_iT_i^3x_iy_i-4a_iT_i^4x_iy_i- 4a_it_iT_i^4x_iy_i-2T_ix_i^2y_i^2+t_iT_ix_i^2y_i^2+2t_iT_i^2x_i^2y_i^2+2T_i^3x_i^2y_i^2+t_iT_i^3x_i^2y_i^2\right)\epsilon\right)/ \left(t_i-4t_iT_i+8t_iT_i^2-10t_iT_i^3+8t_iT_i^4-4t_iT_i^5+t_iT_i^6\right)+ \left(\left(-6a_i^2T_i^2x_iy_i+3a_i^2t_iT_i^2x_iy_i+6a_i^2T_i^3x_iy_i+3a_i^2t_iT_i^3x_iy_i+6a_i^2T_i^4x_iy_i- 3a_i^2t_iT_i^4x_iy_i-6a_i^2T_i^5x_iy_i-3a_i^2t_iT_i^5x_iy_i-a_iT_ix_i^2y_i^2-5a_iT_i^2x_i^2y_i^2+ 4a_it_iT_i^2x_i^2y_i^2+a_iT_i^3x_i^2y_i^2+6a_it_iT_i^3x_i^2y_i^2+5a_iT_i^4x_i^2y_i^2+2a_it_iT_i^4x_i^2y_i^2\right)\epsilon^2\right)/ \left(-t_i+5t_iT_i-12t_iT_i^2+18t_iT_i^3-18t_iT_i^4+12t_iT_i^5-5t_iT_i^6+t_iT_i^7\right)+O[\epsilon]^3$

In[∘]:= `Coefficient[(Z31[[3]] /. {T → T_i, x → x_i, y → y_i, a → a_i}) - ϵ cw, ϵ] // Together // Factor`

Out[∘]= $-\frac{(-1+T_i)^2T_i(1+T_i^2)}{(1-T_i+T_i^2)^3}$

In[∘]:= `Coefficient[(Z31[[3]] /. {T → T_i, x → x_i, y → y_i, a → a_i}) - ϵ cw, ϵ^2 y_i^2 x_i^2] // Together // Factor`

Out[∘]= $\frac{2T_i(1+t_i-T_i+t_iT_i-6t_iT_i^2+T_i^3+t_iT_i^3-T_i^4+t_iT_i^4)}{t_i(-1+T_i)^2(1-T_i+T_i^2)^3}$

In[∘]:= $w_1 w_2 \text{ // } tm_{1,2\to 1}$

In[∘]:= $cw2 = \left(\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \frac{2T_i(1+t_i-T_i+t_iT_i-6t_iT_i^2+T_i^3+t_iT_i^3-T_i^4+t_iT_i^4)}{t_i(-1+T_i)^2(1-T_i+T_i^2)^3}\epsilon^2+O[\epsilon]^3\right] w_1 w_2 \text{ // } tm_{1,2\to 1} \text{ // } tm_{1,i\to i}\right)[[3]]$

Out[∘]= $\left(\left(T_i+4a_iT_i+4a_i^2T_i+t_iT_i+4a_it_iT_i+4a_i^2t_iT_i-3T_i^2-12a_iT_i^2-12a_i^2T_i^2-t_iT_i^2-4a_it_iT_i^2- 4a_i^2t_iT_i^2+3T_i^3+12a_iT_i^3+12a_i^2T_i^3-7t_iT_i^3-28a_it_iT_i^3-28a_i^2t_iT_i^3+14t_iT_i^4+56a_it_iT_i^4+ 56a_i^2t_iT_i^4-3T_i^5-12a_iT_i^5-12a_i^2T_i^5-7t_iT_i^5-28a_it_iT_i^5-28a_i^2t_iT_i^5+3T_i^6+12a_iT_i^6+ 12a_i^2T_i^6-t_iT_i^6-4a_it_iT_i^6-4a_i^2t_iT_i^6-T_i^7-4a_iT_i^7-4a_i^2T_i^7+t_iT_i^7+4a_it_iT_i^7+4a_i^2t_iT_i^7+ 8a_iT_ix_iy_i+8a_it_iT_ix_iy_i-16a_iT_i^2x_iy_i+8a_iT_i^3x_iy_i-56a_it_iT_i^3x_iy_i+8a_iT_i^4x_iy_i+ 56a_it_iT_i^4x_iy_i-16a_iT_i^5x_iy_i+8a_iT_i^6x_iy_i-8a_it_iT_i^6x_iy_i+4T_ix_i^2y_i^2+4t_iT_ix_i^2y_i^2- 4T_i^2x_i^2y_i^2+4t_iT_i^2x_i^2y_i^2-24t_iT_i^3x_i^2y_i^2+4T_i^4x_i^2y_i^2+4t_iT_i^4x_i^2y_i^2-4T_i^5x_i^2y_i^2+4t_iT_i^5x_i^2y_i^2\right)\epsilon^2\right)/ \left(2t_i-10t_iT_i+26t_iT_i^2-44t_iT_i^3+52t_iT_i^4-44t_iT_i^5+26t_iT_i^6-10t_iT_i^7+2t_iT_i^8\right)+O[\epsilon]^3$

In[∘]:= Coefficient[(Z31[[3]] /. {T → T_i, x → x_i, y → y_i, a → a_i}) - ϵ cw - cw2, ϵ^2 y_i^2 x_i^2] // Together // Factor

Out[∘]= 0

In[∘]:= $\text{logkink}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \left(\hbar b_i/2 + \hbar a_i b_i - \frac{\hbar^2 b_i x_i y_i}{-1+B_i} + \epsilon\left(\hbar a_i/2 - \hbar^2 a_i(-1+B_i+\hbar b_i B_i) x_i y_i \big/ (-1+B_i)^2 + \hbar^3 (-2+\hbar b_i + 2 B_i + \hbar b_i B_i) x_i^2 y_i^2 \big/ \left(2(-1+B_i)^3\right)\right)\right)\right]$ // b2t_i // CF

In[∘]:= logkink_i

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \left(\frac{t_i}{2} + a_i t_i + \frac{t_i x_i y_i}{1-T_i}\right) + \left(a_i + a_i^2 - \frac{2 a_i x_i y_i}{-1+T_i} - \frac{2 a_i t_i T_i x_i y_i}{1-2T_i+T_i^2} + \frac{x_i^2 y_i^2}{1-2T_i+T_i^2} + \frac{t_i (1+T_i) x_i^2 y_i^2}{-2+6T_i-6T_i^2+2T_i^3}\right)\epsilon + \left(-\frac{3 a_i^2 T_i x_i y_i}{1-2T_i+T_i^2} + \frac{a_i^2 t_i (-3T_i - 3T_i^2) x_i y_i}{-2+6T_i-6T_i^2+2T_i^3} + \frac{a_i (1+5T_i) x_i^2 y_i^2}{-2+6T_i-6T_i^2+2T_i^3} + \frac{a_i t_i (2T_i+T_i^2) x_i^2 y_i^2}{1-4T_i+6T_i^2-4T_i^3+T_i^4}\right)\epsilon^2 + O[\epsilon]^3\right]$$

In[∘]:= $w_{i\_} := \text{logkink}_i\, \mathbb{E}_{\{\}\to\{j\}}\left[0, 0, \hbar^{-2}\frac{(1-T_j)}{t_j}\right]$ // $\text{tm}_{i,j\to i}$

In[∘]:= w_i

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \left(\frac{1}{2}\times(1-T_i) + a_i(1-T_i) + x_i y_i\right) + \frac{1}{2t_i - 4t_i T_i + 2t_i T_i^2}\left(2a_i + 2a_i^2 - 6a_i T_i - 6a_i^2 T_i + 6a_i T_i^2 + 6a_i^2 T_i^2 - 2a_i T_i^3 - 2a_i^2 T_i^3 + 4a_i x_i y_i - 8a_i T_i x_i y_i - 4a_i t_i T_i x_i y_i + 4a_i T_i^2 x_i y_i + 4a_i t_i T_i^2 x_i y_i + 2x_i^2 y_i^2 - t_i x_i^2 y_i^2 - 2T_i x_i^2 y_i^2 - t_i T_i x_i^2 y_i^2\right)\epsilon + \frac{1}{-2t_i + 6t_i T_i - 6t_i T_i^2 + 2t_i T_i^3}\left(6a_i^2 T_i x_i y_i - 3a_i^2 t_i T_i x_i y_i - 12a_i^2 T_i^2 x_i y_i + 6a_i^2 T_i^3 x_i y_i + 3a_i^2 t_i T_i^3 x_i y_i + a_i x_i^2 y_i^2 + 4a_i T_i x_i^2 y_i^2 - 4a_i t_i T_i x_i^2 y_i^2 - 5a_i T_i^2 x_i^2 y_i^2 - 2a_i t_i T_i^2 x_i^2 y_i^2\right)\epsilon^2 + O[\epsilon]^3\right]$$

In[∘]:= (logkink_i // $\text{tm}_{i,j\to k}$) ≡ (logkink_i // $\text{tm}_{j,i\to k}$)

In[∘]:=
```
w11 = (𝔼_{{}→{2}}[0, 0, ϵ + O[ϵ]^3] w_1)⟦3⟧ /. {u_1 → u}
w21 = (𝔼_{{}→{2}}[0, 0, ϵ^2 + O[ϵ]^3] w_1)⟦3⟧ /. {u_1 → u}
w22 = ((w_1 w_2 // tm_{1,2→1}) (𝔼_{{}→{1}}[0, 0, ϵ^2 + O[ϵ]^3]))⟦3⟧ /. {u_1 → u}
```

Out[∘]= $\left(\frac{1-T}{2}+a\,(1-T)+x\,y\right)\epsilon+\frac{1}{2t-4tT+2tT^2}\left(2a+2a^2-6aT-6a^2T+6aT^2+6a^2T^2-2aT^3-2a^2T^3+4axy-8aTxy-4atTxy+4aT^2xy+4atT^2xy+2x^2y^2-tx^2y^2-2Tx^2y^2-tTx^2y^2\right)\epsilon^2+O[\epsilon]^3$

Out[∘]= $\left(\frac{1-T}{2}+a\,(1-T)+x\,y\right)\epsilon^2+O[\epsilon]^3$

Out[∘]= $\left(\frac{1}{4}\times\left(1-2T+T^2\right)+a\left(1-2T+T^2\right)+a^2\left(1-2T+T^2\right)+a\,(2-2T)\,x\,y+x^2y^2\right)\epsilon^2+O[\epsilon]^3$

In[∘]:=
```
Z31 = Z@Knot[3, 1];
```

Get: ParentDirectory[File] in $Path is not a string.

KnotTheory: Loading precomputed data in PD4Knots`.

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In[∘]:=
```
w11
```

Out[∘]= $\left(\frac{1-T}{2}+a\,(1-T)+x\,y\right)\epsilon+\frac{1}{2t-4tT+2tT^2}\left(2a+2a^2-6aT-6a^2T+6aT^2+6a^2T^2-2aT^3-2a^2T^3+4axy-8aTxy-4atTxy+4aT^2xy+4atT^2xy+2x^2y^2-tx^2y^2-2Tx^2y^2-tTx^2y^2\right)\epsilon^2+O[\epsilon]^3$

In[∘]:=
```
Z31⟦3⟧
```

Out[∘]= $\frac{T}{1-T+T^2}+\left(\frac{a\left(-2T+2T^3\right)}{1-2T+3T^2-2T^3+T^4}+\frac{-2T+3T^2-2T^3+T^4}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T-2T^2\right)xy}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+\left(\frac{a^2\left(2T+2T^2-12T^3+2T^4+2T^5\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{a\left(4T-4T^2-14T^3+16T^4-10T^5+4T^6\right)}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\frac{4T-11T^2+6T^3-2T^5+4T^6-2T^7+T^8}{2-10T+30T^2-60T^3+90T^4-102T^5+90T^6-60T^7+30T^8-10T^9+2T^{10}}+\frac{a\left(4T+12T^2-12T^3-8T^4\right)xy}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(2T-2T^2-4T^3-4T^4-2T^5+2T^6\right)xy}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\frac{\left(3T+9T^2+3T^3\right)x^2y^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}\right)\epsilon^2+O[\epsilon]^3$

In[∘]:=
```
ZZ = Z31⟦3⟧ - Coefficient[Z31⟦3⟧, y x ϵ] w11 // CF
```

In[∘]:=
```
ZZZ = (ZZ - Coefficient[ZZ, y^2 x^2 ϵ^2] w22) // CF
ZZZZ = ZZZ - Coefficient[ZZZ, y x ϵ^2] w21 // CF
```

Out[∘]=
$$\frac{T}{1-T+T^2}+\frac{\left(-T+2T^2-2T^3+2T^4-T^5\right)\epsilon}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\left(\left(-T+3tT+4atT+3T^2-10tT^2-12atT^2-5T^3+11tT^3+8atT^3+4T^4-14tT^4-4atT^4+18tT^5-4T^6-10tT^6+4atT^6+5T^7+3tT^7-8atT^7-3T^8+2tT^8+12atT^8+T^9-tT^9-4atT^9+4tTxy-8tT^2xy-4tT^4xy-4tT^5xy-8tT^7xy+4tT^8xy\right)\epsilon^2\right)/\left(2t-10tT+30tT^2-60tT^3+90tT^4-102tT^5+90tT^6-60tT^7+30tT^8-10tT^9+2tT^{10}\right)+O[\epsilon]^3$$

Out[∘]=
$$\frac{T}{1-T+T^2}+\frac{\left(-T+2T^2-2T^3+2T^4-T^5\right)\epsilon}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\left(\left(-T+tT+3T^2-4tT^2-5T^3+7tT^3+4T^4-12tT^4+18tT^5-4T^6-12tT^6+5T^7+7tT^7-3T^8-4tT^8+T^9+tT^9\right)\epsilon^2\right)/\left(2t-10tT+30tT^2-60tT^3+90tT^4-102tT^5+90tT^6-60tT^7+30tT^8-10tT^9+2tT^{10}\right)+O[\epsilon]^3$$

In[∘]:=
```
Factor /@ ZZZZ
```

Out[∘]=
$$\frac{T}{1-T+T^2}-\frac{(-1+T)^2T\left(1+T^2\right)\epsilon}{\left(1-T+T^2\right)^3}+\frac{1}{2t\left(1-T+T^2\right)^5}T\left(-1+t+3T-4tT-5T^2+7tT^2+4T^3-12tT^3+18tT^4-4T^5-12tT^5+5T^6+7tT^6-3T^7-4tT^7+T^8+tT^8\right)\epsilon^2+O[\epsilon]^3$$

Use the quantum Casimir wc defined below (generalizes to all orders) to express Z as in Theorem 48 of PG.pdf

Below we tabulate rho2 = $\rho_{2,0}$ in this expansion

$$Z=\Delta^{-1}\text{Exp}\left(\epsilon\left(\frac{\rho_{1,0}}{\Delta^2}+\frac{\rho_{1,1}}{\Delta}w\right)+\epsilon^2\left(\frac{\rho_{2,0}}{\Delta^3}+\frac{\rho_{2,1}}{\Delta^2}w+\frac{\rho_{2,2}}{\Delta}w^2\right)+\ldots\right)$$

$$\text{wc}_{i\_}:=\mathbb{E}_{\{\}\to\{i\}}\left[0,0,y_i\,e^{\epsilon a_i}x_i+\frac{e^{\epsilon(a_i+1)}+e^{-\epsilon a_i}T-(T+1)(e^{\epsilon}+1)/2}{e^{\epsilon}-1}+O[\epsilon]^2\right]$$

$$\text{wc11}=\left(\mathbb{E}_{\{\}\to\{2\}}\left[0,0,\epsilon+O[\epsilon]^3\right]\text{wc}_1\right)[[3]]\ /.\ \{u\_1\to u\}\ (*\epsilon\ \text{wc}*)$$

$$\text{wc21}=\left(\mathbb{E}_{\{\}\to\{2\}}\left[0,0,\epsilon^2+O[\epsilon]^3\right]\text{wc}_1\right)[[3]]\ /.\ \{u\_1\to u\}\ (*\epsilon^2\ \text{wc}*)$$

$$\text{wc22}=\left((\text{wc}_1\,\text{wc}_2\ //\ \text{km}_{1,2\to1})\left(\mathbb{E}_{\{\}\to\{1\}}\left[0,0,\epsilon^2+O[\epsilon]^3\right]\right)\right)[[3]]\ /.\ \{u\_1\to u\}\ (*\epsilon^2\ \text{wc}^2*)$$

Out[∘]=
$$\left(1+a+\frac{1}{2}\times(-1-T)-aT+xy\right)\epsilon+\left(\frac{1}{2}\left(a+a^2+aT+a^2T\right)+axy\right)\epsilon^2+O[\epsilon]^3$$

Out[∘]=
$$\left(1+a+\frac{1}{2}\times(-1-T)-aT+xy\right)\epsilon^2+O[\epsilon]^3$$

Out[∘]=
$$\left(\frac{1}{4}\times\left(1-2T+T^2\right)+a\left(1-2T+T^2\right)+a^2\left(1-2T+T^2\right)+a(2-2T)xy+x^2y^2\right)\epsilon^2+O[\epsilon]^3$$

```
(*Tabulate Z up to $k=2, SLOW :(  *)
Zs = Z /@ AllKnots[{3, 6}]

ExtractRhos[Z_] := Module[{Z0 = Z[[3]], Z1, Z2, Z3, Z4, Alex, rho2},
  Alex = (Normal[Z0] /. ϵ → 0)^-1;
  Z1 = Z0 * Alex // CF;
  Z2 = Z1 - Coefficient[Z1, y x ϵ] wc11 // CF;
  Z3 = (Z2 - Coefficient[Z2, y^2 x^2 ϵ^2] wc22) // CF;
  Z4 = Z3 - Coefficient[Z3, y x ϵ^2] wc21 // CF;
  Factor /@ Z4 // Echo;
  rho2 = 4 ((-1 + T)^2/T)^-1 Alex^4 (Z4[[3, 3]] - Z4[[3, 2]]^2/2) // Together // Expand;
  Last@Last@CoefficientRules[rho2, T^-1]]
```

In[∘]:= `Last@Last@CoefficientRules[-9 - 1/T^3 - 1/T^2 - 4/T - 4 T - T^2 - T^3, {T^-1}]`

Out[∘]= $-9-4T-T^2-T^3$

In[∘]:= `ExtractRhos[Zs[[1]]]`

» $1-\frac{(-1+T)^2(1+T^2)\epsilon}{(1-T+T^2)^2}+\frac{(-1+T)^2(1-5T+2T^2-17T^3+2T^4-5T^5+T^6)\epsilon^2}{4(1-T+T^2)^4}+O[\epsilon]^3$

Out[∘]= $-9-4T-T^2-T^3$

In[∘]:= `ExtractRhos[Zs[[2]]]`

» $1-\frac{((-1+T)^2(1-2T-9T^2-2T^3+T^4))\epsilon^2}{4(1-3T+T^2)^3}+O[\epsilon]^3$

Out[∘]= $-23-\frac{1}{T^3}+\frac{5}{T^2}+\frac{2}{T}+2T+5T^2-T^3$

In[∘]:= `QuietEcho[ExtractRhos /@ Zs] // Column`

Out[∘]=
$-9-4T-T^2-T^3$
$-23+2T+5T^2-T^3$
$-61-4T-21T^2-22T^3-7T^5-T^6-2T^7$
$-102+16T+4T^2-14T^3$
$-298+108T+36T^2-14T^3$
$-529+326T-25T^2-82T^3+62T^4-33T^5+13T^6-2T^7$
$-671+498T-155T^2-68T^3+96T^4-49T^5+13T^6-2T^7$