

Pensieve header: Finding the YB element for NOE-1t.

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SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
<< NOE-1t.m
```

```
AddRule[ff_, rule_] := (
  Print["As ", e0, ", adding ", rule];
  done = False; EC = EC ∪ {rule}
);
MF[ϕ_, v_] := Module[{t = ϕ, t1}, If[Simplify[t] === 0, 1,
  While[{t1 = Simplify[t /. v → 0]} === 0, t = D[t, v]]; t1
]];
```

```
RunMess := (
  done = False; While[! done, done = True;
  E0 = DeleteCases[Simplify[E0 /. EC], 0] // SortBy[LeafCount];
  Print["Length[E0]==", Length[E0],
  "; Length[EC]==", Length[EC], "; {gn,hn}==", {gn, hn}];
  For[k = 1, k ≤ Length@E0, ++k,
  e1 = Factor[e0 = E0[[k]];
  If[Head[e1] != Times, e2 = e1,
  E0[[k]] = e2 = Select[e1, ! FreeQ[#, f[_], _] | g[_] | h[[]] &]];
  If[e2 == 1, Print["Panic at ", e0, "! No solutions."]; Break[]];
  If[! FreeQ[e2, f[_], _] ∧ (FreeQ[e2, x] ∨ FreeQ[e2, y] ∨ FreeQ[e2, z]),
  {ff} = Cases[e2, f[_], _], {0, ∞}, 1];
  {{sol}} = Solve[e2 == 0, ff];
  rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
  AddRule[ff, rule]; Break[]
];
If[! FreeQ[e2, g[_]] ∧ (FreeQ[e2, y | z] ∨ FreeQ[e2, x | z] ∨ FreeQ[e2, z | y]),
{gg} = Cases[e2, g[_], ∞, 1];
{{sol}} = Solve[e2 == 0, gg];
rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
AddRule[gg, rule]; Break[]
];
If[Head[e2] === Plus,
s = List@@Collect[e2, f[_], _], Factor]; s1 = Select[s, FreeQ[f[_], _]];
sxy = Cases[s, a_. * f_[x, y]];
sxz = Cases[s, a_. * f_[x, z]]; syz = Cases[s, a_. * f_[y, z]];
Which[
sxy == {} ∧ sxz != {} ∧ syz != {}, (
{ff} = Cases[sxz, a_. * fk[x, z] ⇒ fk[x, z], {1}, 1];
mf = MF[First@sxz /. f_[x, z] → 1, x]; mf *= MF[First@syz /. f_[y, z] → 1, y];
s1 = Plus@@Simplify[s1/mf];
sxz = Plus@@Simplify[sxz/mf]; syz = Plus@@Simplify[syz/mf];
If[FreeQ[sxz, y] ∧ FreeQ[syz, x] ∧ FreeQ[s1, x | y] ∧
Simplify[(sxz /. x → y) + syz == 0],
{{sol}} = Solve[sxz == g++gn[z], ff];
```

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rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
AddRule[ff, rule]; Break[]
]
),
syz = {} ∧ sxy != {} ∧ sxz != {}, (
{ff} = Cases[sxy, a_. * fk[x, y] ⇒ fk[x, y], {1}, 1];
mf = MF[First@sxy /. f_[x, y] → 1, y]; mf *= MF[First@sxz /. f_[x, z] → 1, z];
s1 = Plus @@ Simplify[s1/mf];
sxy = Plus @@ Simplify[sxy/mf]; sxz = Plus @@ Simplify[sxz/mf];
If[FreeQ[sxy, z] ∧ FreeQ[sxz, y] ∧ FreeQ[s1, y | z] ∧
Simplify[(sxz /. z → y) + sxy == 0],
{{sol}} = Solve[sxy == g++gn[x], ff];
rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
AddRule[ff, rule]; Break[]
]
),
sxy != {} ∧ sxz != {} ∧ syz != {}, (
kk = Union@Cases[e2, a_. * fk[x, y] ⇒ k, ∞];
If[Length[kk] == 1,
{kk} = kk;
{{sol}} = Solve[e2 == 0, fkk[x, y]];
sol = fkk[x, y] /. sol;
e3 = D[sol, z] // Factor;
If[FreeQ[e3, f[_], _],
If[Head[e3] === Times,
e3 = Select[e3, ! FreeQ[#, f(0,1)[_], _] | g[_] | h[_]] &];
s = Collect[e3, f(0,1)[_], _], Factor];
s1 = Select[s, FreeQ[f(0,1)[_], _]];
pxz = Coefficient[s, fkk(0,1)[x, z]];
pyz = Coefficient[s, fkk(0,1)[y, z]];
mf = MF[pxz, x]; mf *= MF[pyz, y];
{s1, pxz, pyz} = Simplify[{s1, pxz, pyz}/mf];
If[FreeQ[pxz, y] ∧ FreeQ[pyz, x] ∧
FreeQ[s1, x | y] ∧ Simplify[(pyz /. y → x) + pxz == 0],
rule = (fkk[x_, z_] → g++gn[z] / pxz + g++gn[x]) /. Rule → RuleDelayed;
AddRule[fkk[x, z], rule]; Break[]
]
]
]
);
If[FreeQ[e2, f[_], _] ∧ ! FreeQ[e2, g[_]],
s = List @@ Collect[e2, g[_], Factor]; s1 = Select[s, FreeQ[g[_]]];
sx = Cases[s, a_. * g[x]]; sy = Cases[s, a_. * g[y]]; sz = Cases[s, a_. * g[z]];
Which[
FreeQ[e2, x] ∧ sy != {} ∧ sz != {}, (
{gg} = Cases[sy, a_. * gk[y] ⇒ gk[y], {1}, 1];
mf = MF[First@sy /. g_[y] → 1, y]; mf *= MF[First@sz /. g_[z] → 1, z];

```

```

s1 = Plus @@ Simplify[s1/mf];
sy = Plus @@ Simplify[sy/mf]; sz = Plus @@ Simplify[sz/mf];
If[FreeQ[sx, y] & FreeQ[sz, y] & FreeQ[s1, y | z] &
  Simplify[(sz /. z -> y) + sy == 0],
  {{sol}} = Solve[sy == h++hn[], gg];
  rule = ((gg /. {x -> x_, y -> y_, z -> z_}) -> (gg /. sol)) /. Rule -> RuleDelayed;
  AddRule[gg, rule]; Break[]
]
),
FreeQ[e2, z] & sy != {} & sx != {}, (
  {gg} = Cases[sy, a_. * gk[y] -> gk[y], {1}, 1];
  mf = MF[First@sy /. g_[y] -> 1, y]; mf *= MF[First@sx /. g_[x] -> 1, x];
  s1 = Plus @@ Simplify[s1/mf];
  sy = Plus @@ Simplify[sy/mf]; sx = Plus @@ Simplify[sx/mf];
  If[FreeQ[sz, y] & FreeQ[sx, y] & FreeQ[s1, y | x] &
    Simplify[(sx /. x -> y) + sy == 0],
    {{sol}} = Solve[sy == h++hn[], gg];
    rule = ((gg /. {x -> x_, y -> y_, z -> z_}) -> (gg /. sol)) /. Rule -> RuleDelayed;
    AddRule[gg, rule]; Break[]
  ]
)
] (* Which *)
] (* If *)
] (* If *)
] (* For *)
]; (* While *)
E0 = Union[DeleteCases[Simplify[E0 //. EC], 0]] // SortBy[LeafCount];
)

```

```

Rp[i_, j_] :=
E[1, bi cj, ui wj, ci f1[ti, tj] + cj f2[ti, tj] + ci2 f3[ti, tj] + ci cj f4[ti, tj] + cj2 f5[ti, tj] +
ui wi f6[ti, tj] + ui wj f7[ti, tj] + uj wi f8[ti, tj] + uj wj f9[ti, tj] + ci ui wi f10[ti, tj] +
ci ui wj f11[ti, tj] + ci uj wi f12[ti, tj] + ci uj wj f13[ti, tj] + cj ui wi f14[ti, tj] +
cj ui wj f15[ti, tj] + cj uj wi f16[ti, tj] + cj uj wj f17[ti, tj] + ui2 wi2 f18[ti, tj] +
ui2 wi wj f19[ti, tj] + ui2 wj2 f20[ti, tj] + ui uj wi2 f21[ti, tj] + ui uj wi wj f22[ti, tj] +
ui uj wj2 f23[ti, tj] + uj2 wi2 f24[ti, tj] + uj2 wi wj f25[ti, tj] + uj2 wj2 f26[ti, tj]]

```

```

Short[t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, x] // m[1, 6, y] // m[2, 4, z]]
E[1, <<1>>, <<1>>,
  cx f1[tx, ty] - ux wz f1[tx, ty] + cx f1[tx, tz] + <<331>> + uz2 wz2 f26[tx, tz] + uz2 wz2 f26[ty, tz]]

Short[t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, x] // m[2, 5, y] // m[4, 6, z]]
E[1, bx cy + bx cz + by cz, ux wy + ux wz + uy wz, 2 cy ux wz + ux2 wy wz + <<466>> + uz2 wz2 f26[ty, tz]]

Take[t1, 3] == Take[t2, 3]
True

```

Short [t3 = t1[[4]] - t2[[4]]]

$$-2 c_y u_x w_z - u_x^2 w_y w_z - \frac{1}{2} u_x^2 w_z^2 + \ll 710 \gg + 2 t_y t_z u_x u_z w_z^2 f_{26}[t_y, t_z]$$

Short [Errors = CoefficientRules[Expand[t3], {Cx, Cy, Cz, ux, uy, uz, wx, wy, wz}] /.

{(_ -> c_) :-> c} /. {tx -> x, ty -> y, tz -> z}]

{2 f3[x, z] + f10[x, z] - x f10[x, z], <<56>>, f8[x, z] - x f8[x, z]}

EC = {}; **E0 = Errors;** **gn = hn = 0;**

RunMess

Length[E0]==58; Length[EC]==0; {gn,hn}=={0, 0}

As -(-1+x) f8[x, z], adding f8[x_, z_] :-> 0

Length[E0]==56; Length[EC]==1; {gn,hn}=={0, 0}

As -(-1+x) f12[x, z], adding f12[x_, z_] :-> 0

Length[E0]==51; Length[EC]==2; {gn,hn}=={0, 0}

As -(-1+x) f16[x, z], adding f16[x_, z_] :-> 0

Length[E0]==45; Length[EC]==3; {gn,hn}=={0, 0}

As -2(-1+x) f21[x, z], adding f21[x_, z_] :-> 0

Length[E0]==41; Length[EC]==4; {gn,hn}=={0, 0}

As -2(-1+x) f24[x, z], adding f24[x_, z_] :-> 0

Length[E0]==31; Length[EC]==5; {gn,hn}=={0, 0}

As -(-1+x) f25[x, z], adding f25[x_, z_] :-> 0

Length[E0]==26; Length[EC]==6; {gn,hn}=={0, 0}

As f10[x, z] - 2(-1+x) f18[x, z], adding f10[x_, z_] :-> 2(-f18[x, z] + x f18[x, z])

Length[E0]==24; Length[EC]==7; {gn,hn}=={0, 0}

As f17[x, z] - 2(-1+z) f26[x, z], adding f17[x_, z_] :-> 2(-f26[x, z] + z f26[x, z])

Length[E0]==22; Length[EC]==8; {gn,hn}=={0, 0}

As 2 f3[x, z] - 2(-1+x)^2 f18[x, z], adding f3[x_, z_] :-> f18[x, z] - 2 x f18[x, z] + x^2 f18[x, z]

Length[E0]==19; Length[EC]==9; {gn,hn}=={0, 0}

As -2 f5[x, y] + 2(-1+y)^2 f26[x, y], adding f5[x_, y_] :-> f26[x, y] - 2 y f26[x, y] + y^2 f26[x, y]

Length[E0]==16; Length[EC]==10; {gn,hn}=={0, 0}

As -(-1+y) (f4[x, z] - f4[y, z] + f15[x, z] - f15[y, z]), adding f4[x_, z_] :-> -f15[x, z] - g1[z]

Length[E0]==15; Length[EC]==11; {gn,hn}=={1, 0}

As -(-1+y) (f13[x, z] - f13[y, z] + f23[x, z] - f23[y, z]), adding f13[x_, z_] :-> -f23[x, z] - g2[z]

Length[E0]==14; Length[EC]==12; {gn,hn}=={2, 0}

As -(-1+x) f14[x, z] + (-1+y) f14[y, z] - f15[x, z] + f15[y, z], adding f14[x_, z_] :-> $\frac{-f15[x, z] + g3[z]}{-1+x}$

Length[E0]==13; Length[EC]==13; {gn,hn}=={3, 0}

As $-(-1+x) f_{22}[x, z] + (-1+y) f_{22}[y, z] - f_{23}[x, z] + f_{23}[y, z]$, adding $f_{22}[x_, z_] \Rightarrow \frac{-f_{23}[x, z] + g_4[z]}{-1+x}$

Length[E0]==12; Length[EC]==14; {gn,hn}=={4, 0}

As $(-1+y) (f_{11}[x, y] - f_{11}[x, z] + 2(-1+x)^2 (f_{18}[x, y] - f_{18}[x, z]))$, adding $f_{11}[x_, y_] \Rightarrow -2 f_{18}[x, y] + 4 x f_{18}[x, y] - 2 x^2 f_{18}[x, y] - g_5[x]$

Length[E0]==11; Length[EC]==15; {gn,hn}=={5, 0}

As $(-1+y) (2(-1+x) f_{18}[x, y] - 2(-1+x) f_{18}[x, z] + f_{19}[x, y] - f_{19}[x, z])$, adding $f_{18}[x_, y_] \Rightarrow \frac{-f_{19}[x, y] - 4 g_6[x] + 8 x g_6[x] - 4 x^2 g_6[x]}{2(-1+x)}$

Length[E0]==10; Length[EC]==16; {gn,hn}=={6, 0}

As $-2+y f_{15}[y, z] + (y-yz) f_{23}[y, z] + g_1[y] + y g_1[z] + y g_2[z] - yz g_2[z] + y g_3[y] + g_5[y] - y g_5[y]$, adding $f_{15}[y_, z_] \Rightarrow \frac{1}{y}$

$(2-y f_{23}[y, z] + yz f_{23}[y, z] - g_1[y] - y g_1[z] - y g_2[z] + yz g_2[z] - y g_3[y] - g_5[y] + y g_5[y])$

Length[E0]==8; Length[EC]==17; {gn,hn}=={6, 0}

As $\frac{1}{-1+x}$

$(-g_1[y] + g_1[z] - g_2[y] + y g_2[y] + g_2[z] - z g_2[z] - g_3[y] + g_3[z] - g_4[y] + y g_4[y] + g_4[z] - z g_4[z])$, adding $g_1[y_] \Rightarrow -g_2[y] + y g_2[y] - g_3[y] - g_4[y] + y g_4[y] - h_1[]$

Length[E0]==7; Length[EC]==18; {gn,hn}=={6, 1}

As $-f_2[x, y] + f_2[x, z] - f_9[x, y] + y f_9[x, y] + f_9[x, z] - z f_9[x, z] + f_{26}[x, y] - 2y f_{26}[x, y] + y^2 f_{26}[x, y] - f_{26}[x, z] + 2z f_{26}[x, z] - z^2 f_{26}[x, z]$, adding $f_2[x_, y_] \Rightarrow -f_9[x, y] + y f_9[x, y] + f_{26}[x, y] - 2y f_{26}[x, y] + y^2 f_{26}[x, y] + g_7[x]$

Length[E0]==6; Length[EC]==19; {gn,hn}=={7, 1}

As $\frac{1}{-1+y} (2 - 2(-1+y) g_3[y] + 2(-1+y)^2 g_4[y] - 2 g_5[y] + 2y g_5[y] - 8 g_6[y] + 32y g_6[y] - 48y^2 g_6[y] + 32y^3 g_6[y] - 8y^4 g_6[y] + h_1[] + y h_1[])$, adding $g_3[y_] \Rightarrow \frac{1}{2(-1+y)} (2 + 2g_4[y] - 4y g_4[y] + 2y^2 g_4[y] - 2g_5[y] + 2y g_5[y] - 8g_6[y] + 32y g_6[y] - 48y^2 g_6[y] + 32y^3 g_6[y] - 8y^4 g_6[y] + h_1[] + y h_1[])$

Length[E0]==5; Length[EC]==20; {gn,hn}=={7, 1}

As $-1 + (-1+x) (-1+y) f_{19}[x, y] + 2(-1+y) f_{20}[x, y] + g_5[x] - y g_5[x] + g_5[y] - y g_5[y] + 4 g_6[y] - 16y g_6[y] + 24y^2 g_6[y] - 16y^3 g_6[y] + 4y^4 g_6[y] - y h_1[]$, adding $f_{19}[x_, y_] \Rightarrow (1 + 2 f_{20}[x, y] - 2y f_{20}[x, y] - g_5[x] + y g_5[x] - g_5[y] + y g_5[y] - 4 g_6[y] + 16y g_6[y] - 24y^2 g_6[y] + 16y^3 g_6[y] - 4y^4 g_6[y] + y h_1[]) / ((-1+x) (-1+y))$

Length[E0]==2; Length[EC]==21; {gn,hn}=={7, 1}

eq1 = E0[[1]]

$2 f_1[x, z] - 2 f_1[y, z] + 2 f_6[x, z] - 2 x f_6[x, z] - 2 f_6[y, z] + 2 y f_6[y, z] - 2 f_{20}[x, z] + 2 f_{20}[y, z] + g_5[x] - g_5[y] - 4 g_6[x] + 12 x g_6[x] - 12 x^2 g_6[x] + 4 x^3 g_6[x] + 4 g_6[y] - 12 y g_6[y] + 12 y^2 g_6[y] - 4 y^3 g_6[y]$

Select[eq1, FreeQ[x]] + Select[eq1, FreeQ[y]] - eq1

0

Solve[**Select**[**eq1**, **FreeQ**[**y**]] == **g_{++gn}**[**z**], **f₁**[**x**, **z**]]

$$\left\{ \left\{ f_1[x, z] \rightarrow \frac{1}{2} \left(-2 f_6[x, z] + 2 x f_6[x, z] + 2 f_{20}[x, z] - g_5[x] + 4 g_6[x] - 12 x g_6[x] + 12 x^2 g_6[x] - 4 x^3 g_6[x] + g_8[z] \right) \right\} \right\}$$

$$\text{rule1} = \left(f_1[x_, z_] \Rightarrow \frac{1}{2} \left(-2 f_6[x, z] + 2 x f_6[x, z] + 2 f_{20}[x, z] - g_5[x] + 4 g_6[x] - 12 x g_6[x] + 12 x^2 g_6[x] - 4 x^3 g_6[x] + g_8[z] \right) \right)$$

$$f_1[x_, z_] \Rightarrow \frac{1}{2} \left(-2 f_6[x, z] + 2 x f_6[x, z] + 2 f_{20}[x, z] - g_5[x] + 4 g_6[x] - 12 x g_6[x] + 12 x^2 g_6[x] - 4 x^3 g_6[x] + g_8[z] \right)$$

E0 /. rule1

$$\begin{aligned} & \{0, -1 + 2 y f_6[x, y] - 2 x y f_6[x, y] - 2 y^2 f_6[x, y] + 2 x y^2 f_6[x, y] - 2 f_6[y, z] + \\ & 2 y f_6[y, z] + 2 f_7[x, y] - 4 y f_7[x, y] + 2 y^2 f_7[x, y] - 2 f_7[x, z] + 4 y f_7[x, z] - \\ & 2 y^2 f_7[x, z] + 2 f_7[y, z] - 4 y f_7[y, z] + 2 y^2 f_7[y, z] - 2 f_{20}[x, y] + 2 y f_{20}[x, y] + \\ & 2 f_{20}[x, z] - 4 y f_{20}[x, z] + 2 y^2 f_{20}[x, z] + 4 y f_{20}[y, z] - 2 y^2 f_{20}[y, z] + y g_5[x] - \\ & y^2 g_5[x] - g_5[y] + y g_5[y] - y^2 g_5[y] - 4 y g_6[x] + 12 x y g_6[x] - 12 x^2 y g_6[x] + \\ & 4 x^3 y g_6[x] + 4 y^2 g_6[x] - 12 x y^2 g_6[x] + 12 x^2 y^2 g_6[x] - 4 x^3 y^2 g_6[x] + 4 g_6[y] - \\ & 8 y g_6[y] - 4 y^2 g_6[y] + 20 y^3 g_6[y] - 16 y^4 g_6[y] + 4 y^5 g_6[y] + 2 y g_7[y] - 2 y^2 g_7[y] - \\ & (-1 + y) \left(-2 f_6[x, y] + 2 x f_6[x, y] + 2 f_{20}[x, y] - g_5[x] + 4 g_6[x] - 12 x g_6[x] + 12 x^2 g_6[x] - \right. \\ & \quad \left. 4 x^3 g_6[x] + g_8[y] \right) + g_8[z] - (-1 + y)^2 \left(-2 f_6[x, z] + 2 x f_6[x, z] + \right. \\ & \quad \left. 2 f_{20}[x, z] - g_5[x] + 4 g_6[x] - 12 x g_6[x] + 12 x^2 g_6[x] - 4 x^3 g_6[x] + g_8[z] \right) - \\ & 2 y \left(-2 f_6[y, z] + 2 y f_6[y, z] + 2 f_{20}[y, z] - g_5[y] + 4 g_6[y] - 12 y g_6[y] + \right. \\ & \quad \left. 12 y^2 g_6[y] - 4 y^3 g_6[y] + g_8[z] \right) + y^2 \left(-2 f_6[y, z] + 2 y f_6[y, z] + 2 f_{20}[y, z] - \right. \\ & \quad \left. g_5[y] + 4 g_6[y] - 12 y g_6[y] + 12 y^2 g_6[y] - 4 y^3 g_6[y] + g_8[z] \right) - y h_1[] \} \end{aligned}$$

AppendTo[EC, rule1]

$$\{f_2[x_, y_] \Rightarrow -f_9[x, y] + y f_9[x, y] + f_{26}[x, y] - 2 y f_{26}[x, y] + y^2 f_{26}[x, y] + g_7[x],$$

$$f_3[x_, z_] \Rightarrow f_{18}[x, z] - 2 x f_{18}[x, z] + x^2 f_{18}[x, z],$$

$$f_4[x_, z_] \Rightarrow -f_{15}[x, z] - g_1[z], f_5[x_, y_] \Rightarrow f_{26}[x, y] - 2 y f_{26}[x, y] + y^2 f_{26}[x, y],$$

$$f_8[x_, z_] \Rightarrow 0, f_{10}[x_, z_] \Rightarrow 2(-f_{18}[x, z] + x f_{18}[x, z]),$$

$$f_{11}[x_, y_] \Rightarrow -2 f_{18}[x, y] + 4 x f_{18}[x, y] - 2 x^2 f_{18}[x, y] - g_5[x], f_{12}[x_, z_] \Rightarrow 0,$$

$$f_{13}[x_, z_] \Rightarrow -f_{23}[x, z] - g_2[z], f_{14}[x_, z_] \Rightarrow \frac{-f_{15}[x, z] + g_3[z]}{-1 + x}, f_{15}[y_, z_] \Rightarrow \frac{1}{y}$$

$$(2 - y f_{23}[y, z] + y z f_{23}[y, z] - g_1[y] - y g_1[z] - y g_2[z] + y z g_2[z] - y g_3[y] - g_5[y] + y g_5[y]),$$

$$f_{16}[x_, z_] \Rightarrow 0, f_{17}[x_, z_] \Rightarrow 2(-f_{26}[x, z] + z f_{26}[x, z]),$$

$$f_{18}[x_, y_] \Rightarrow \frac{-f_{19}[x, y] - 4 g_6[x] + 8 x g_6[x] - 4 x^2 g_6[x]}{2(-1 + x)},$$

$$f_{19}[x_, y_] \Rightarrow (1 + 2 f_{20}[x, y] - 2 y f_{20}[x, y] - g_5[x] + y g_5[x] - g_5[y] + y g_5[y] - 4 g_6[y] +$$

$$16 y g_6[y] - 24 y^2 g_6[y] + 16 y^3 g_6[y] - 4 y^4 g_6[y] + y h_1[]) / ((-1 + x)(-1 + y)),$$

$$f_{21}[x_, z_] \Rightarrow 0, f_{22}[x_, z_] \Rightarrow \frac{-f_{23}[x, z] + g_4[z]}{-1 + x}, f_{24}[x_, z_] \Rightarrow 0, f_{25}[x_, z_] \Rightarrow 0,$$

$$g_1[y_] \Rightarrow -g_2[y] + y g_2[y] - g_3[y] - g_4[y] + y g_4[y] - h_1[],$$

$$g_3[y_] \Rightarrow \frac{1}{2(-1 + y)}(2 + 2 g_4[y] - 4 y g_4[y] + 2 y^2 g_4[y] - 2 g_5[y] + 2 y g_5[y] -$$

$$8 g_6[y] + 32 y g_6[y] - 48 y^2 g_6[y] + 32 y^3 g_6[y] - 8 y^4 g_6[y] + h_1[] + y h_1[]),$$

$$f_1[x_, z_] \Rightarrow \frac{1}{2}(-2 f_6[x, z] + 2 x f_6[x, z] + 2 f_{20}[x, z] - g_5[x] +$$

$$4 g_6[x] - 12 x g_6[x] + 12 x^2 g_6[x] - 4 x^3 g_6[x] + g_8[z])\}$$
RunMess

```
Length[E0]==1; Length[EC]==22; {gn,hn}=={8, 1}
```

eq2 = E0[[1]]

$$-1 + 2(-1 + x)(-1 + y)^2 f_6[x, y] - 2(-1 + x)(-1 + y)^2 f_6[x, z] - 2 f_6[y, z] + 6 y f_6[y, z] -$$

$$6 y^2 f_6[y, z] + 2 y^3 f_6[y, z] + 2 f_7[x, y] - 4 y f_7[x, y] + 2 y^2 f_7[x, y] - 2 f_7[x, z] + 4 y f_7[x, z] -$$

$$2 y^2 f_7[x, z] + 2 f_7[y, z] - 4 y f_7[y, z] + 2 y^2 f_7[y, z] - g_5[y] + 3 y g_5[y] - 2 y^2 g_5[y] + 4 g_6[y] -$$

$$16 y g_6[y] + 24 y^2 g_6[y] - 16 y^3 g_6[y] + 4 y^4 g_6[y] + 2 y g_7[y] - 2 y^2 g_7[y] + g_8[y] - y g_8[y] - y h_1[]$$

```
eq3 = Collect[ $\frac{\text{eq2}}{2(-1 + y)^2}$ , f_[_]], FullSimplify]
```

$$(-1 + x) f_6[x, y] + (1 - x) f_6[x, z] + (-1 + y) f_6[y, z] + f_7[x, y] - f_7[x, z] + f_7[y, z] + \frac{1}{2(-1 + y)^2}$$

$$(-1 + (-1 + (3 - 2 y) y) g_5[y] + 4(-1 + y)^4 g_6[y] + g_8[y] - y(2(-1 + y) g_7[y] + g_8[y] + h_1[]))$$

```
eq4 = Collect[D[eq3, x], f_[_]], Simplify]
```

$$f_6[x, y] - f_6[x, z] + (-1 + x) f_6^{(1,0)}[x, y] - (-1 + x) f_6^{(1,0)}[x, z] + f_7^{(1,0)}[x, y] - f_7^{(1,0)}[x, z]$$

```
(eq4 /. f_[x, y] | f_^{(1,0)}[x, y] -> 0) + (eq4 /. f_[x, z] | f_^{(1,0)}[x, z] -> 0) == eq4 // Simplify
```

```
True
```

`FreeQ[(eq4 /. f_[x, y] | f_(1,0)[x, y] → 0), y]`

True

`FreeQ[(eq4 /. f_[x, z] | f_(1,0)[x, z] → 0), z]`

True

`eq5 = Collect[(-eq4 /. f_[x, y] | f_(1,0)[x, y] → 0), f6[x, z] | f(1,0)6[x, z], Simplify]`

`f6[x, z] + (-1 + x) f(1,0)6[x, z] + f(1,0)7[x, z]`

`D[(x - 1) f6[x, z] + f7[x, z], x] - eq5`

0

`Solve[(x - 1) f6[x, z] + f7[x, z] == g++gn[x] + g++gn[z], f7[x, z]]`

`{ {f7[x, z] → f6[x, z] - x f6[x, z] + g9[x] + g10[z] }`

`eq6 = Simplify[eq2 /. f7[x_, z_] => f6[x, z] - x f6[x, z] + g9[x] + g10[z]]`

`-1 + (-1 + 3 y - 2 y2) g5[y] + 4 (-1 + y)4 g6[y] + 2 y g7[y] - 2 y2 g7[y] + g8[y] - y g8[y] + 2 g9[y] - 4 y g9[y] + 2 y2 g9[y] + 2 g10[y] - 4 y g10[y] + 2 y2 g10[y] - y h1[]`

`First@Solve[eq6 == 0, g10[y]]`

`{g10[y] →`

$$\frac{1}{2(1 - 2y + y^2)} \left(1 + g_5[y] - 3y g_5[y] + 2y^2 g_5[y] - 4g_6[y] + 16y g_6[y] - 24y^2 g_6[y] + 16y^3 g_6[y] - 4y^4 g_6[y] - 2y g_7[y] + 2y^2 g_7[y] - g_8[y] + y g_8[y] - 2g_9[y] + 4y g_9[y] - 2y^2 g_9[y] + y h_1[] \right)$$

`Simplify[E0 /. f7[x_, z_] => f6[x, z] - x f6[x, z] + g9[x] + g10[z] /. g10[y_] =>`

$$\frac{1}{2(1 - 2y + y^2)} \left(1 + g_5[y] - 3y g_5[y] + 2y^2 g_5[y] - 4g_6[y] + 16y g_6[y] - 24y^2 g_6[y] + 16y^3 g_6[y] - 4y^4 g_6[y] - 2y g_7[y] + 2y^2 g_7[y] - g_8[y] + y g_8[y] - 2g_9[y] + 4y g_9[y] - 2y^2 g_9[y] + y h_1[] \right)$$

`{0}`

$$\begin{aligned}
 & \frac{1}{2} c_1 \left(-2 f_6[t_1, t_2] + 2 t_1 f_6[t_1, t_2] + 2 f_{20}[t_1, t_2] - g_5[t_1] + \right. \\
 & \quad \left. 4 g_6[t_1] - 12 t_1 g_6[t_1] + 12 t_1^2 g_6[t_1] - 4 t_1^3 g_6[t_1] + g_8[t_2] \right) + \\
 & \frac{1}{(-1+t_1)(-1+t_2)} u_1^2 w_1 w_2 \left(1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + \right. \\
 & \quad \left. t_2 g_5[t_2] - 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[] \right) + \\
 & \frac{1}{2(-1+t_1)} u_1^2 w_1^2 \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \\
 & \quad \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \\
 & \quad \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) + \\
 & u_1 w_2 \left(f_6[t_1, t_2] - t_1 f_6[t_1, t_2] + g_9[t_1] + \frac{1}{2(1-2t_2+t_2^2)} (1 + g_5[t_2] - 3 t_2 g_5[t_2] + \right. \\
 & \quad 2 t_2^2 g_5[t_2] - 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] - 2 t_2 g_7[t_2] + \\
 & \quad \left. 2 t_2^2 g_7[t_2] - g_8[t_2] + t_2 g_8[t_2] - 2 g_9[t_2] + 4 t_2 g_9[t_2] - 2 t_2^2 g_9[t_2] + t_2 h_1[]) \right) + \\
 & 2 c_1 u_1 w_1 \left(-\frac{1}{2(-1+t_1)} \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \right. \\
 & \quad \left. \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \right. \\
 & \quad \left. \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) \right) + \\
 & \frac{1}{2(-1+t_1)} t_1 \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \\
 & \quad \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \\
 & \quad \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) \Big) + \\
 & c_1 u_1 w_2 \left(-g_5[t_1] - \frac{1}{-1+t_1} \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \right. \\
 & \quad \left. \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \right. \\
 & \quad \left. \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) \right) + \\
 & \frac{1}{-1+t_1} 2 t_1 \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \\
 & \quad \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \\
 & \quad \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) - \\
 & \frac{1}{-1+t_1} t_1^2 \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \\
 & \quad \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \\
 & \quad \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) \Big) +
 \end{aligned}$$

$$\begin{aligned}
 & c_1^2 \left(\frac{1}{2(-1+t_1)} \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \right. \\
 & \quad \left. \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \right. \\
 & \quad \left. \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) \right) - \\
 & \frac{1}{-1+t_1} t_1 \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \\
 & \quad \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \\
 & \quad \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) + \\
 & \frac{1}{2(-1+t_1)} t_1^2 \left(-4 g_6[t_1] + 8 t_1 g_6[t_1] - 4 t_1^2 g_6[t_1] - \frac{1}{(-1+t_1)(-1+t_2)} \right. \\
 & \quad \left. (1 + 2 f_{20}[t_1, t_2] - 2 t_2 f_{20}[t_1, t_2] - g_5[t_1] + t_2 g_5[t_1] - g_5[t_2] + t_2 g_5[t_2] - \right. \\
 & \quad \left. 4 g_6[t_2] + 16 t_2 g_6[t_2] - 24 t_2^2 g_6[t_2] + 16 t_2^3 g_6[t_2] - 4 t_2^4 g_6[t_2] + t_2 h_1[]) \right) \Big) + \\
 & \frac{1}{t_1} c_2 u_1 w_2 \left(2 - t_1 f_{23}[t_1, t_2] + t_1 t_2 f_{23}[t_1, t_2] + g_2[t_1] - t_1 g_2[t_1] - t_1 g_2[t_2] + \right. \\
 & \quad t_1 t_2 g_2[t_2] + g_4[t_1] - t_1 g_4[t_1] - g_5[t_1] + t_1 g_5[t_1] + h_1[] + \\
 & \quad \frac{1}{2(-1+t_1)} (2 + 2 g_4[t_1] - 4 t_1 g_4[t_1] + 2 t_1^2 g_4[t_1] - 2 g_5[t_1] + 2 t_1 g_5[t_1] - 8 g_6[t_1] + \\
 & \quad 32 t_1 g_6[t_1] - 48 t_1^2 g_6[t_1] + 32 t_1^3 g_6[t_1] - 8 t_1^4 g_6[t_1] + h_1[] + t_1 h_1[]) - \\
 & \quad \frac{1}{2(-1+t_1)} t_1 (2 + 2 g_4[t_1] - 4 t_1 g_4[t_1] + 2 t_1^2 g_4[t_1] - 2 g_5[t_1] + 2 t_1 g_5[t_1] - \\
 & \quad 8 g_6[t_1] + 32 t_1 g_6[t_1] - 48 t_1^2 g_6[t_1] + 32 t_1^3 g_6[t_1] - 8 t_1^4 g_6[t_1] + h_1[] + t_1 h_1[]) - \\
 & \quad t_1 \left(-g_2[t_2] + t_2 g_2[t_2] - g_4[t_2] + t_2 g_4[t_2] - h_1[] - \frac{1}{2(-1+t_2)} \right. \\
 & \quad \left. (2 + 2 g_4[t_2] - 4 t_2 g_4[t_2] + 2 t_2^2 g_4[t_2] - 2 g_5[t_2] + 2 t_2 g_5[t_2] - 8 g_6[t_2] + \right. \\
 & \quad \left. 32 t_2 g_6[t_2] - 48 t_2^2 g_6[t_2] + 32 t_2^3 g_6[t_2] - 8 t_2^4 g_6[t_2] + h_1[] + t_2 h_1[]) \right) \Big) + \\
 & \frac{1}{-1+t_1} c_2 u_1 w_1 \left(\frac{1}{2(-1+t_2)} (2 + 2 g_4[t_2] - 4 t_2 g_4[t_2] + 2 t_2^2 g_4[t_2] - 2 g_5[t_2] + 2 t_2 g_5[t_2] - \right. \\
 & \quad \left. 8 g_6[t_2] + 32 t_2 g_6[t_2] - 48 t_2^2 g_6[t_2] + 32 t_2^3 g_6[t_2] - 8 t_2^4 g_6[t_2] + h_1[] + t_2 h_1[]) - \right. \\
 & \quad \frac{1}{t_1} \left(2 - t_1 f_{23}[t_1, t_2] + t_1 t_2 f_{23}[t_1, t_2] + g_2[t_1] - t_1 g_2[t_1] - t_1 g_2[t_2] + \right. \\
 & \quad t_1 t_2 g_2[t_2] + g_4[t_1] - t_1 g_4[t_1] - g_5[t_1] + t_1 g_5[t_1] + h_1[] + \\
 & \quad \frac{1}{2(-1+t_1)} (2 + 2 g_4[t_1] - 4 t_1 g_4[t_1] + 2 t_1^2 g_4[t_1] - 2 g_5[t_1] + 2 t_1 g_5[t_1] - 8 g_6[t_1] + \\
 & \quad 32 t_1 g_6[t_1] - 48 t_1^2 g_6[t_1] + 32 t_1^3 g_6[t_1] - 8 t_1^4 g_6[t_1] + h_1[] + t_1 h_1[]) - \\
 & \quad \frac{1}{2(-1+t_1)} t_1 (2 + 2 g_4[t_1] - 4 t_1 g_4[t_1] + 2 t_1^2 g_4[t_1] - 2 g_5[t_1] + 2 t_1 g_5[t_1] - \\
 & \quad \left. 8 g_6[t_1] + 32 t_1 g_6[t_1] - 48 t_1^2 g_6[t_1] + 32 t_1^3 g_6[t_1] - 8 t_1^4 g_6[t_1] + h_1[] + t_1 h_1[]) \right) -
 \end{aligned}$$

$$\begin{aligned}
 & t_1 \left(-g_2[t_2] + t_2 g_2[t_2] - g_4[t_2] + t_2 g_4[t_2] - h_1[] - \frac{1}{2(-1+t_2)} \right. \\
 & \quad \left. (2 + 2 g_4[t_2] - 4 t_2 g_4[t_2] + 2 t_2^2 g_4[t_2] - 2 g_5[t_2] + 2 t_2 g_5[t_2] - 8 g_6[t_2] + 32 \right. \\
 & \quad \left. t_2 g_6[t_2] - 48 t_2^2 g_6[t_2] + 32 t_2^3 g_6[t_2] - 8 t_2^4 g_6[t_2] + h_1[] + t_2 h_1[]) \right) \Bigg) + \\
 c_1 c_2 & \left(g_2[t_2] - t_2 g_2[t_2] + g_4[t_2] - t_2 g_4[t_2] + h_1[] + \frac{1}{2(-1+t_2)} \right. \\
 & \quad \left. (2 + 2 g_4[t_2] - 4 t_2 g_4[t_2] + 2 t_2^2 g_4[t_2] - 2 g_5[t_2] + 2 t_2 g_5[t_2] - 8 g_6[t_2] + \right. \\
 & \quad \left. 32 t_2 g_6[t_2] - 48 t_2^2 g_6[t_2] + 32 t_2^3 g_6[t_2] - 8 t_2^4 g_6[t_2] + h_1[] + t_2 h_1[]) - \right. \\
 & \quad \left. \frac{1}{t_1} \left(2 - t_1 f_{23}[t_1, t_2] + t_1 t_2 f_{23}[t_1, t_2] + g_2[t_1] - t_1 g_2[t_1] - t_1 g_2[t_2] + \right. \right. \\
 & \quad \left. \left. t_1 t_2 g_2[t_2] + g_4[t_1] - t_1 g_4[t_1] - g_5[t_1] + t_1 g_5[t_1] + h_1[] + \right. \right. \\
 & \quad \left. \left. \frac{1}{2(-1+t_1)} (2 + 2 g_4[t_1] - 4 t_1 g_4[t_1] + 2 t_1^2 g_4[t_1] - 2 g_5[t_1] + 2 t_1 g_5[t_1] - 8 g_6[t_1] + \right. \right. \\
 & \quad \left. \left. 32 t_1 g_6[t_1] - 48 t_1^2 g_6[t_1] + 32 t_1^3 g_6[t_1] - 8 t_1^4 g_6[t_1] + h_1[] + t_1 h_1[]) - \right. \right. \\
 & \quad \left. \left. \frac{1}{2(-1+t_1)} t_1 (2 + 2 g_4[t_1] - 4 t_1 g_4[t_1] + 2 t_1^2 g_4[t_1] - 2 g_5[t_1] + 2 t_1 g_5[t_1] - \right. \right. \\
 & \quad \left. \left. 8 g_6[t_1] + 32 t_1 g_6[t_1] - 48 t_1^2 g_6[t_1] + 32 t_1^3 g_6[t_1] - 8 t_1^4 g_6[t_1] + h_1[] + t_1 h_1[]) - \right. \right. \\
 & \quad \left. \left. t_1 \left(-g_2[t_2] + t_2 g_2[t_2] - g_4[t_2] + t_2 g_4[t_2] - h_1[] - \frac{1}{2(-1+t_2)} \right. \right. \\
 & \quad \left. \left. (2 + 2 g_4[t_2] - 4 t_2 g_4[t_2] + 2 t_2^2 g_4[t_2] - 2 g_5[t_2] + 2 t_2 g_5[t_2] - 8 g_6[t_2] + \right. \right. \\
 & \quad \left. \left. 32 t_2 g_6[t_2] - 48 t_2^2 g_6[t_2] + 32 t_2^3 g_6[t_2] - 8 t_2^4 g_6[t_2] + h_1[] + t_2 h_1[]) \right) \right) \Bigg) \Bigg]
 \end{aligned}$$