```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
<< VdVAlgebraAt1.m
```

```
t2 UU[c2, u1]
```

$t_2\, U[c_2, u_1]$

```
t2 UU[c2, u1] // m[1, 2]
```

$-t_1\, U[u_1] + t_1\, U[c_1, u_1]$

```
(BLBasis[4] // m[1, 2] // m[1, 3]) - (BLBasis[4] // m[2, 3] // m[1, 2])
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
U[] /. a_. * x_U :> {a, x}
```

```
{1, U[]}
```

```
U[] /. a_. + x_U :> {a, x}
```

```
{0, U[]}
```

```
(LBasis[2] // Δ[2, 2, 3] // Δ[3, 3, 4]) - (LBasis[2] // Δ[2, 2, 4] // Δ[2, 2, 3])
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(LBasis[3] // m[2, 3, 2] // Δ[2, 2, 3]) -
 (LBasis[3] // Δ[2, 2, 4] // Δ[3, 3, 5] // m[2, 3, 2] // m[4, 5, 3])
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
t1 U[] // S[1]
```

$$\frac{U[]}{t_1}$$

```
U[u2] // S[1]
```

$U[u_2]$

```
U[] ** U[]
```

$U[]$

```
U[u1] // S[1]
```

$$-\frac{2\,U[u_1]}{t_1}+\frac{U[c_1,\,u_1]}{t_1}$$

```
U[u1] // S[1] // S[1]
```

$2\,U[u_1]+U[c_1,\,u_1]-U[c_1,\,c_1,\,u_1]$

```
Simp[(# // S[3] // S[2]) - (# // S[2] // S[3])] & /@ LBasis[3]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
tt = U[u3, w2]
```

$U[u_3,\,w_2]$

```
tt // m[2, 3, 2]
```

$(1-t_2)\,U[]-2\,\epsilon\,t_2\,U[c_2]+(1+\epsilon)\,U[u_2,\,w_2]$

```
tt // m[2, 3, 2] // S[2]
```

$$\left(1-\frac{1}{t_2}\right)U[]+\left(1+\epsilon-\frac{1}{t_2}+\frac{\epsilon}{t_2}\right)U[c_2]+\left(-1+\epsilon+\frac{1}{t_2}+\frac{\epsilon}{t_2}\right)U[c_2,\,c_2]-2\,\epsilon\,U[c_2,\,c_2,\,c_2]+\left(-\frac{1}{t_2}-\frac{2\,\epsilon}{t_2}\right)U[c_2,\,u_2,\,w_2]+\left(\frac{1}{t_2}+\frac{2\,\epsilon}{t_2}\right)U[c_2,\,c_2,\,u_2,\,w_2]$$

```
tt // S[2] // S[3]
```

$$-\frac{2\,U[c_2,\,u_3,\,w_2]}{t_3}+\frac{U[c_2,\,c_3,\,u_3,\,w_2]}{t_3}$$

```
tt // S[2] // S[3] // m[3, 2, 2]
```

$$\frac{2\,U[u_2,\,w_2]}{t_2}-\frac{3\,U[c_2,\,u_2,\,w_2]}{t_2}+\frac{U[c_2,\,c_2,\,u_2,\,w_2]}{t_2}$$

```
Simp[(# // m[2, 3, 2] // S[2]) - (# // S[2] // S[3] // m[3, 2, 2])] & /@ LBasis[3]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
Simp[(# // S[2] // Δ[2, 2, 3]) - (# // Δ[2, 3, 2] // S[2] // S[3])] & /@ LBasis[2]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Simp[(# // Δ[2, 2, 3] // S[2] // m[2, 3, 2])] & /@ LBasis[2]
```

{U[], ∈ U[], U[c_1], 0, U[u_1], 0, U[w_1], 0, ∈ U[c_1], 0, ∈ U[u_1], 0, ∈ U[w_1], 0, U[u_1, w_1], 0, 0, 0, ∈ U[c_1, c_1], 0, ∈ U[c_1, u_1], 0, ∈ U[c_1, w_1], 0, 0, 0, 0, 0, 0, ∈ U[u_1, w_1], 0, 0, 0, ∈ U[c_1, u_1, w_1], 0, 0, 0, 0, 0, 0, 0, ∈ U[u_1, u_1, w_1], 0, 0, 0, ∈ U[u_1, w_1, w_1], 0, 0, 0, 0, 0, 0, 0, ∈ U[u_1, u_1, w_1, w_1], 0, 0, 0, 0, 0, 0, 0, 0}

```
Simp[(# // Δ[2, 2, 3] // S[3] // m[2, 3, 2])] & /@ LBasis[2]
```

{U[], ∈ U[], U[c_1], 0, U[u_1], 0, U[w_1], 0, ∈ U[c_1], 0, ∈ U[u_1], 0, ∈ U[w_1], 0, U[u_1, w_1], 0, 0, 0, ∈ U[c_1, c_1], 0, ∈ U[c_1, u_1], 0, ∈ U[c_1, w_1], 0, 0, 0, 0, 0, 0, ∈ U[u_1, w_1], 0, 0, 0, ∈ U[c_1, u_1, w_1], 0, 0, 0, 0, 0, 0, 0, ∈ U[u_1, u_1, w_1], 0, 0, 0, ∈ U[u_1, w_1, w_1], 0, 0, 0, 0, 0, 0, 0, ∈ U[u_1, u_1, w_1, w_1], 0, 0, 0, 0, 0, 0, 0, 0}

```
R[1, 2, 1]
```

U[] + $\hbar$ b_1 U[c_2] + ∈ $\hbar$ U[c_1, c_2] + $\hbar$ U[u_1, w_2]

```
R[1, 2, 1] // S[1] // S[2] // ToDegree[1]
```

U[] + $\hbar$ b_1 U[c_2] + ∈ $\hbar$ U[c_1, c_2] + $\hbar$ U[u_1, w_2]

```
With[{d = 4}, (R[1, 2, d] // S[1] // S[2] // ToDegree[d]) - R[1, 2, d]]
```

0

```
With[{d = 4}, (R[1, 2, d] // S[1]) ** R[1, 2, d] // ToDegree[d]]
```

U[]

```
With[{d = 4}, (R[1, 2, d] // S[2]) ** R[1, 2, d] // ToDegree[d]]
```

U[] + ∈ $\hbar^2$ U[u_1, w_2]

```
LBasis[2] // CoUnit[2]
```

{U[], ∈ U[], U[c_1], 0, U[u_1], 0, U[w_1], 0, ∈ U[c_1], 0, ∈ U[u_1], 0, ∈ U[w_1], 0, U[u_1, w_1], 0, 0, 0, ∈ U[c_1, c_1], 0, ∈ U[c_1, u_1], 0, ∈ U[c_1, w_1], 0, 0, 0, 0, 0, 0, ∈ U[u_1, w_1], 0, 0, 0, ∈ U[c_1, u_1, w_1], 0, 0, 0, 0, 0, 0, 0, ∈ U[u_1, u_1, w_1], 0, 0, 0, ∈ U[u_1, w_1, w_1], 0, 0, 0, 0, 0, 0, 0, ∈ U[u_1, u_1, w_1, w_1], 0, 0, 0, 0, 0, 0, 0, 0}

```
Simp[(# // Δ[2, 2, 3] // S[3] // m[2, 3, 2]) - # // CoUnit[2]] & /@ LBasis[2]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
{R[1, 2, 5] // CoUnit[1], R[1, 2, 5] // CoUnit[2]}
```

{U[], U[]}

```
With[{d = 4}, ToDegree[d][(R[1, 3, d] // Δ[1, 1, 2]) - (R[1, 3, d] ** R[2, 3, d])]]
```

0

```
With[{d = 4}, ToDegree[d][(R[1, 2, d] // Δ[2, 2, 3]) - (R[1, 3, d] ** R[1, 2, d])]]
```

0

```
With[{d = 4},
 ToDegree[d][R[1, 2, d] ** Δ[1, 1, 2][#] - Δ[1, 2, 1][#] ** R[1, 2, d]] & /@ LBasis[1]]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
With[{d = 3},
 Dru = R[1, 2, d] // S[2] // m[2, 1, 1];
 {Dru, Dru // S[1]} // ToDegree[d]
]
```

$$\Big\{\left(1+\hbar\, b_1+\frac{1}{2}\hbar^2 b_1^2+\frac{1}{6}\hbar^3 b_1^3\right) U[\,]+\left(\hbar\,(2\,\epsilon-b_1)+\hbar^2\left(2\,\epsilon\, b_1-b_1^2\right)+\hbar^3\left(\epsilon\, b_1^2-\frac{b_1^3}{2}\right)\right) U[c_1]+$$
$$\left(-\epsilon\,\hbar+\hbar^2\left(-3\,\epsilon\, b_1+\frac{b_1^2}{2}\right)+\frac{1}{2}\hbar^3\left(-5\,\epsilon\, b_1^2+b_1^3\right)\right) U[c_1, c_1]+$$
$$\left(-\hbar-\hbar^2 b_1-\frac{1}{2}\hbar^3 b_1^2\right) U[u_1, w_1]+\left(\epsilon\,\hbar^2 b_1+\hbar^3\left(2\,\epsilon\, b_1^2-\frac{b_1^3}{6}\right)\right) U[c_1, c_1, c_1]+$$
$$\left(\hbar^2\,(-2\,\epsilon+b_1)+\hbar^3\left(-2\,\epsilon\, b_1+b_1^2\right)\right) U[c_1, u_1, w_1]-\frac{1}{2}\epsilon\,\hbar^3 b_1^2\, U[c_1, c_1, c_1, c_1]+$$
$$\left(\epsilon\,\hbar^2+\hbar^3\left(3\,\epsilon\, b_1-\frac{b_1^2}{2}\right)\right) U[c_1, c_1, u_1, w_1]+\left(\frac{\hbar^2}{2}+\frac{1}{4}\hbar^3\,(\epsilon+2\, b_1)\right) U[u_1, u_1, w_1, w_1]-$$
$$\epsilon\,\hbar^3 b_1\, U[c_1, c_1, c_1, u_1, w_1]+\hbar^3\left(\epsilon-\frac{b_1}{2}\right) U[c_1, u_1, u_1, w_1, w_1]-$$
$$\frac{1}{2}\epsilon\,\hbar^3\, U[c_1, c_1, u_1, u_1, w_1, w_1]-\frac{1}{6}\hbar^3\, U[u_1, u_1, u_1, w_1, w_1, w_1],$$
$$U[\,]-\hbar\, b_1\, U[c_1]+\left(-\epsilon\,\hbar+\frac{1}{2}\hbar^2 b_1^2\right) U[c_1, c_1]-\hbar\, U[u_1, w_1]+\left(\epsilon\,\hbar^2 b_1-\frac{1}{6}\hbar^3 b_1^3\right) U[c_1, c_1, c_1]+$$
$$\hbar^2 b_1\, U[c_1, u_1, w_1]-\frac{1}{2}\epsilon\,\hbar^3 b_1^2\, U[c_1, c_1, c_1, c_1]+\left(\epsilon\,\hbar^2-\frac{1}{2}\hbar^3 b_1^2\right) U[c_1, c_1, u_1, w_1]+$$
$$\left(\frac{\hbar^2}{2}+\frac{\epsilon\,\hbar^3}{4}\right) U[u_1, u_1, w_1, w_1]-\epsilon\,\hbar^3 b_1\, U[c_1, c_1, c_1, u_1, w_1]-\frac{1}{2}\hbar^3 b_1\, U[c_1, u_1, u_1, w_1, w_1]-$$
$$\frac{1}{2}\epsilon\,\hbar^3\, U[c_1, c_1, u_1, u_1, w_1, w_1]-\frac{1}{6}\hbar^3\, U[u_1, u_1, u_1, w_1, w_1, w_1]\Big\}$$

```
ToDegree[3][{t1^-1 Dru, Dru // S[1]} /. ϵ → 0] // Column
```

$$U[\,]-\hbar\, b_1\, U[c_1]+\tfrac{1}{2}\hbar^2 b_1^2\, U[c_1, c_1]-\hbar\, U[u_1, w_1]-\tfrac{1}{6}\hbar^3 b_1^3\, U[c_1, c_1, c_1]+$$
$$\hbar^2 b_1\, U[c_1, u_1, w_1]-\tfrac{1}{2}\hbar^3 b_1^2\, U[c_1, c_1, u_1, w_1]+\tfrac{1}{2}\hbar^2\, U[u_1, u_1, w_1, w_1]-$$
$$\tfrac{1}{2}\hbar^3 b_1\, U[c_1, u_1, u_1, w_1, w_1]-\tfrac{1}{6}\hbar^3\, U[u_1, u_1, u_1, w_1, w_1, w_1]$$
$$U[\,]-\hbar\, b_1\, U[c_1]+\tfrac{1}{2}\hbar^2 b_1^2\, U[c_1, c_1]-\hbar\, U[u_1, w_1]-\tfrac{1}{6}\hbar^3 b_1^3\, U[c_1, c_1, c_1]+$$
$$\hbar^2 b_1\, U[c_1, u_1, w_1]-\tfrac{1}{2}\hbar^3 b_1^2\, U[c_1, c_1, u_1, w_1]+\tfrac{1}{2}\hbar^2\, U[u_1, u_1, w_1, w_1]-$$
$$\tfrac{1}{2}\hbar^3 b_1\, U[c_1, u_1, u_1, w_1, w_1]-\tfrac{1}{6}\hbar^3\, U[u_1, u_1, u_1, w_1, w_1, w_1]$$

```
With[{d = 3},
 Dru = ToDegree[d][R[1, 2, d] // S[2] // m[2, 1, 1]];
 Drv2 = ToDegree[d][(Dru // S[1]) ** Dru];
 ToDegree[d][Drv2 - Dru ** (Dru // S[1])]
]
```

```
0
```

```
With[{d = 3},
 Dru = ToDegree[d][R[1, 2, d] // S[2] // m[2, 1, 1]];
 Drv2 = ToDegree[d][(Dru // S[1]) ** Dru];
 ToDegree[d][Drv2 ** # - # ** Drv2] & /@ LBasis[1]
]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```