Pensieve header: Normally Ordered Exponentials at 1-Co.

"NO" for "Normal Order".

```
DP[P_, x_ → D_α_, y_ → D_β_][f_] :=
 Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :> c D[f, {α, m}, {β, n}]]
```

```
ϵ /: ϵ^n_ /; n ≥ 1 := 0;
Λ[b_, c_, u_, w_, α_, β_, δ_, ν_] := 2 c w α δ ν + 2 c u β δ ν + 2 c u w δ^2 ν + u w^2 α δ^2 ν^3 -
    1/2 b u^2 w^2 δ^4 ν^3 + 1/2 w^2 α^2 δ (2 + b δ) ν^3 - u^2 w β δ^2 (1 + 2 b δ) ν^3 - 1/2 u^2 β^2 δ (2 + 3 b δ) ν^3 + 2 c (δ + α β ν) -
    2 b u w δ^2 ν^2 (δ + α β ν) + w α ν^2 (2 δ + α β ν) - u β (1 + 2 b δ) ν^2 (2 δ + α β ν) - 1/2 b ν (2 δ^2 + 4 α β δ ν + α^2 β^2 ν^2);
```

```
Simp[P_. 𝔼[Q_]] := Together[P] 𝔼[Together[Q]];
(*P1_.𝔼[Q1_] ≡ P2_.𝔼[Q2_] := Simplify[P1==P2⋀Q1==Q2];*)
P1_. 𝔼[Q1_] ≡ P2_. 𝔼[Q2_] := Together[P1 - P2] == 0 ⋀ Together[Q1 - Q2] == 0;
```

```
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := Simp@𝔼[Q1 + Q2]
```

```
NO[u_i_, c_j_, k_][P_. 𝔼[Q_]] := Simp@Module[{q(*,α,β,Θ*)},
    q = ℯ^-α β u_k + α c_k + Θ;
    ℯ^-q DP[P, c_j → D_α, u_i → D_β][ℯ^q] 𝔼[q] /. {α → Together[∂_c_j Q], β → ∂_u_i Q, Θ → (Q /. c_j | u_i → 0)}
   ];
```

```
NO[w_i_, c_j_, k_][P_. 𝔼[Q_]] := Simp@Module[{q(*,α,β,Θ*)},
    q = ℯ^α β w_k + α c_k + Θ;
    ℯ^-q DP[P, c_j → D_α, w_i → D_β][ℯ^q] 𝔼[q] /. {α → Together[∂_c_j Q], β → ∂_w_i Q, Θ → (Q /. c_j | w_i → 0)}
   ];
```

```
NO[w_i_, u_j_, k_][P_. 𝔼[Q_]] := Simp@Module[
    {α0 = ∂_w_i Q /. u_j → 0, β0 = ∂_u_j Q /. w_i → 0, δ0 = ∂_w_i,u_j Q, Θ0 = Q /. w_i | u_j → 0, q(*,α,β,δ, Θ, ν*)},
    q = -b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + Θ;
    ℯ^-q DP[P, w_i → D_α, u_j → D_β][ν (1 + ϵ ν Λ[b_k, c_k, u_k, w_k, α, β, δ, ν]) ℯ^q] 𝔼[q] /.
     {α → α0, β → β0, δ → δ0, Θ → Θ0, ν -> (1 + b_k δ0)^-1}
   ];
```

```
m[i_, j_, kk_][Z_] := Simp@Module[{x, y},
   Z // ReplaceAll[b_i|j → b_kk] // NO[w_i, c_j, x] // NO[w_x, u_j, y] // ReplaceAll[{c_x|y → c_x, w_j → w_y}] //
     NO[u_i, c_x, x] // ReplaceAll[{c_i|x → c_kk, u_x|y → u_kk, w_y → w_kk, b_x|y → b_kk}]
  ]
```

```
Rp[i_, j_] := 𝔼[b_i c_j + (ℯ^b_i - 1)/b_i u_i w_j]; Rm[i_, j_] := 𝔼[-b_i c_j + (ℯ^-b_i - 1)/b_i u_i w_j];
```

```
Q0 = 𝔼[u_1 w_1 + u_2 w_3];
t1 = Q0 // m[1, 2, 4]
```

$$\frac{1}{2}\left(2 - 4\,\epsilon\,u_4\,w_3 + 4\,\epsilon\,c_4\,u_4\,w_3 - 2\,\epsilon\,u_4^2\,w_3^2 + 3\,\epsilon\,b_4\,u_4^2\,w_3^2 - 2\,\epsilon\,u_4^2\,w_3\,w_4\right)\mathbb{E}\left[u_4\,w_3 - b_4\,u_4\,w_3 + u_4\,w_4\right]$$

```
NO[u_1, c_1, 1][𝔼[b_2 c_1 + w_2 u_1 + u_3 w_4]]
```

$$\mathbb{E}\left[e^{-b_2}\left(e^{b_2}\,b_2\,c_1 + u_1\,w_2 + e^{b_2}\,u_3\,w_4\right)\right]$$

```
Q0 = 𝔼[u1 w1 + u3 w2];

SeedRandom[0];
Q0 = 𝔼[Sum[RandomInteger[{-2, 2}] ci, {i, 3}] + Sum[RandomInteger[{-2, 2}] ui wj, {i, 3}, {j, 3}]]
```

$$\mathbb{E}\left[-c_1-2c_2+2c_3-2u_1w_1+u_3w_1+2u_1w_2-u_2w_2+u_3w_2-u_1w_3-2u_2w_3+2u_3w_3\right]$$

```
Q0 = 𝔼[Sum[βi ci, {i, 3}] + Sum[αi,j ui wj, {i, 3}, {j, 3}]]
```

$$\mathbb{E}\left[c_1\beta_1+c_2\beta_2+c_3\beta_3+u_1w_1\alpha_{1,1}+u_1w_2\alpha_{1,2}+u_1w_3\alpha_{1,3}+u_2w_1\alpha_{2,1}+u_2w_2\alpha_{2,2}+u_2w_3\alpha_{2,3}+u_3w_1\alpha_{3,1}+u_3w_2\alpha_{3,2}+u_3w_3\alpha_{3,3}\right]$$

```
Q0 = 𝔼[-2 b1 c1 - b2 c1 + 2 b3 c1 + b2 c2 - b3 c2 + 2 b1 c3 + b2 c3 - b3 c3 - u3 w1 + 2 u2 w2 + u3 w2 + u1 w3 - u2 w3]
```

$$\mathbb{E}\left[-2b_1c_1-b_2c_1+2b_3c_1+b_2c_2-b_3c_2+2b_1c_3+b_2c_3-b_3c_3-u_3w_1+2u_2w_2+u_3w_2+u_1w_3-u_2w_3\right]$$

```
t1 = Q0 // m[1, 2, 4] // m[4, 3, 5]
t2 = Q0 // m[2, 3, 4] // m[1, 4, 5]
t3 = (t1 ≡ t2)
```

$$\frac{1}{2\left(1+2e^{2b_5}b_5^2\right)^5}e^{-2b_5}\left(2e^{2b_5}+16e^{4b_5}b_5^2+8e^{6b_5}\in b_5^2-16e^{6b_5}\in b_5^3+48e^{6b_5}b_5^4+32e^{8b_5}\in b_5^4-64e^{8b_5}\in b_5^5+64e^{8b_5}b_5^6+32e^{10b_5}\in b_5^6-64e^{10b_5}\in b_5^7+32e^{10b_5}b_5^8+16e^{4b_5}\in b_5c_5+96e^{6b_5}\in b_5^3c_5+192e^{8b_5}\in b_5^5c_5+128e^{10b_5}\in b_5^7c_5+8e^{2b_5}\in b_5u_5w_5-28e^{4b_5}\in b_5u_5w_5-16e^{6b_5}\in b_5u_5w_5-8e^{4b_5}\in b_5^2u_5w_5+48e^{6b_5}\in b_5^2u_5w_5+32e^{4b_5}\in b_5^3u_5w_5-192e^{6b_5}\in b_5^3u_5w_5-32e^{8b_5}\in b_5^3u_5w_5+32e^{6b_5}\in b_5^4u_5w_5+64e^{8b_5}\in b_5^4u_5w_5-32e^{6b_5}\in b_5^5u_5w_5-272e^{8b_5}\in b_5^5u_5w_5+96e^{8b_5}\in b_5^6u_5w_5-64e^{10b_5}\in b_5^6u_5w_5-128e^{8b_5}\in b_5^7u_5w_5+4e^{2b_5}\in c_5u_5w_5-8e^{4b_5}\in c_5u_5w_5-16e^{2b_5}\in b_5c_5u_5w_5+32e^{4b_5}\in b_5c_5u_5w_5+8e^{4b_5}\in b_5^2c_5u_5w_5-16e^{6b_5}\in b_5^2c_5u_5w_5-64e^{4b_5}\in b_5^3c_5u_5w_5+128e^{6b_5}\in b_5^3c_5u_5w_5-16e^{6b_5}\in b_5^4c_5u_5w_5+32e^{8b_5}\in b_5^4c_5u_5w_5-64e^{6b_5}\in b_5^5c_5u_5w_5+128e^{8b_5}\in b_5^5c_5u_5w_5-32e^{8b_5}\in b_5^6c_5u_5w_5+64e^{10b_5}\in b_5^6c_5u_5w_5-2\in u_5^2w_5^2+6e^{4b_5}\in u_5^2w_5^2+4e^{6b_5}\in u_5^2w_5^2+4\in b_5u_5^2w_5^2+7e^{2b_5}\in b_5u_5^2w_5^2-20e^{4b_5}\in b_5u_5^2w_5^2-20e^{6b_5}\in b_5u_5^2w_5^2-12e^{2b_5}\in b_5^2u_5^2w_5^2+6e^{4b_5}\in b_5^2u_5^2w_5^2+36e^{6b_5}\in b_5^2u_5^2w_5^2+32e^{2b_5}\in b_5^3u_5^2w_5^2-12e^{4b_5}\in b_5^3u_5^2w_5^2-104e^{6b_5}\in b_5^3u_5^2w_5^2-24e^{4b_5}\in b_5^4u_5^2w_5^2+80e^{6b_5}\in b_5^4u_5^2w_5^2-64e^{8b_5}\in b_5^4u_5^2w_5^2+64e^{4b_5}\in b_5^5u_5^2w_5^2-140e^{6b_5}\in b_5^5u_5^2w_5^2+32e^{8b_5}\in b_5^5u_5^2w_5^2-16e^{10b_5}\in b_5^5u_5^2w_5^2+32e^{6b_5}\in b_5^6u_5^2w_5^2-64e^{8b_5}\in b_5^6u_5^2w_5^2\right)$$

$$\mathbb{E}\left[\frac{1}{1+2e^{2b_5}b_5^2}\left(b_5c_5+2e^{2b_5}b_5^3c_5+2u_5w_5-b_5u_5w_5+2e^{2b_5}b_5u_5w_5+2b_5^2u_5w_5\right)\right]$$

$$\frac{1}{2\left(1+2e^{2b_5}b_5^2\right)^5}e^{-2b_5}\left(2e^{2b_5}+16e^{4b_5}b_5^2+8e^{6b_5}\in b_5^2-16e^{6b_5}\in b_5^3+48e^{6b_5}b_5^4+32e^{8b_5}\in b_5^4-64e^{8b_5}\in b_5^5+64e^{8b_5}b_5^6+32e^{10b_5}\in b_5^6-64e^{10b_5}\in b_5^7+32e^{10b_5}b_5^8+16e^{4b_5}\in b_5c_5+96e^{6b_5}\in b_5^3c_5+192e^{8b_5}\in b_5^5c_5+128e^{10b_5}\in b_5^7c_5+8e^{2b_5}\in b_5u_5w_5-28e^{4b_5}\in b_5u_5w_5-16e^{6b_5}\in b_5u_5w_5-8e^{4b_5}\in b_5^2u_5w_5+48e^{6b_5}\in b_5^2u_5w_5+32e^{4b_5}\in b_5^3u_5w_5-192e^{6b_5}\in b_5^3u_5w_5-32e^{8b_5}\in b_5^3u_5w_5+32e^{6b_5}\in b_5^4u_5w_5+64e^{8b_5}\in b_5^4u_5w_5-32e^{6b_5}\in b_5^5u_5w_5-272e^{8b_5}\in b_5^5u_5w_5+96e^{8b_5}\in b_5^6u_5w_5-64e^{10b_5}\in b_5^6u_5w_5-128e^{8b_5}\in b_5^7u_5w_5+4e^{2b_5}\in c_5u_5w_5-8e^{4b_5}\in c_5u_5w_5-16e^{2b_5}\in b_5c_5u_5w_5+32e^{4b_5}\in b_5c_5u_5w_5+8e^{4b_5}\in b_5^2c_5u_5w_5-16e^{6b_5}\in b_5^2c_5u_5w_5-64e^{4b_5}\in b_5^3c_5u_5w_5+128e^{6b_5}\in b_5^3c_5u_5w_5-16e^{6b_5}\in b_5^4c_5u_5w_5+32e^{8b_5}\in b_5^4c_5u_5w_5-64e^{6b_5}\in b_5^5c_5u_5w_5+128e^{8b_5}\in b_5^5c_5u_5w_5-32e^{8b_5}\in b_5^6c_5u_5w_5+64e^{10b_5}\in b_5^6c_5u_5w_5-2\in u_5^2w_5^2+6e^{4b_5}\in u_5^2w_5^2+4e^{6b_5}\in u_5^2w_5^2+4\in b_5u_5^2w_5^2+7e^{2b_5}\in b_5u_5^2w_5^2-20e^{4b_5}\in b_5u_5^2w_5^2-20e^{6b_5}\in b_5u_5^2w_5^2-12e^{2b_5}\in b_5^2u_5^2w_5^2+6e^{4b_5}\in b_5^2u_5^2w_5^2+36e^{6b_5}\in b_5^2u_5^2w_5^2+32e^{2b_5}\in b_5^3u_5^2w_5^2-12e^{4b_5}\in b_5^3u_5^2w_5^2-104e^{6b_5}\in b_5^3u_5^2w_5^2-24e^{4b_5}\in b_5^4u_5^2w_5^2+80e^{6b_5}\in b_5^4u_5^2w_5^2-64e^{8b_5}\in b_5^4u_5^2w_5^2+64e^{4b_5}\in b_5^5u_5^2w_5^2-140e^{6b_5}\in b_5^5u_5^2w_5^2+32e^{8b_5}\in b_5^5u_5^2w_5^2-16e^{10b_5}\in b_5^5u_5^2w_5^2+32e^{6b_5}\in b_5^6u_5^2w_5^2-64e^{8b_5}\in b_5^6u_5^2w_5^2\right)$$

$$\mathbb{E}\left[\frac{1}{1+2e^{2b_5}b_5^2}\left(b_5c_5+2e^{2b_5}b_5^3c_5+2u_5w_5-b_5u_5w_5+2e^{2b_5}b_5u_5w_5+2b_5^2u_5w_5\right)\right]$$

True

```
Rp[1, 2] Rm[4, 3]
```

$$\mathbb{E}\left[b_1c_2-b_4c_3+\frac{\left(-1+e^{b_1}\right)u_1w_2}{b_1}+\frac{\left(-1+e^{-b_4}\right)u_4w_3}{b_4}\right]$$

```
Rp[1, 2] Rm[4, 3] // m[1, 4, 1] // m[2, 3, 2]
```

$$\mathbb{E}[0]$$

```
Rm[1, 2] Rp[4, 3] // m[4, 1, 4] // m[2, 3, 2]
```

$$\mathbb{E}[0]$$

```
t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, a] // m[1, 6, b] // m[2, 4, c]
```

$$\mathbb{E}\left[\frac{1}{b_a\, b_b}\left(b_a^2\, b_b\, (c_b + c_c) + \left(-1 + e^{b_a}\right) b_b\, u_a\, (w_b + w_c) + b_a\left(b_b^2\, c_c + \left(-1 + e^{b_b}\right) u_b\, w_c\right)\right)\right]$$

```
t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, a] // m[2, 5, b] // m[4, 6, c]
```

$$\mathbb{E}\left[\frac{1}{b_a\, b_b}\left(b_a^2\, b_b\, (c_b + c_c) + \left(-1 + e^{b_a}\right) b_b\, u_a\, (w_b + w_c) + b_a\left(b_b^2\, c_c + \left(-1 + e^{b_b}\right) u_b\, w_c\right)\right)\right]$$

```
t1 ≡ t2
```

True

```
t3 = Rm[12, 1] Rm[2, 7] Rm[8, 3] Rm[4, 11] Rp[16, 5] Rp[6, 13] Rp[14, 9] Rp[10, 15]
```

$$\mathbb{E}\left[-b_{12}\, c_1 - b_8\, c_3 + b_{16}\, c_5 - b_2\, c_7 + b_{14}\, c_9 - b_4\, c_{11} + b_6\, c_{13} + b_{10}\, c_{15} + \frac{\left(-1 + e^{-b_{12}}\right) u_{12}\, w_1}{b_{12}} + \frac{\left(-1 + e^{-b_8}\right) u_8\, w_3}{b_8} + \frac{\left(-1 + e^{b_{16}}\right) u_{16}\, w_5}{b_{16}} + \frac{\left(-1 + e^{-b_2}\right) u_2\, w_7}{b_2} + \frac{\left(-1 + e^{b_{14}}\right) u_{14}\, w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right) u_4\, w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right) u_6\, w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right) u_{10}\, w_{15}}{b_{10}}\right]$$

```
t3 // m[1, 2, 1]
```

$$\mathbb{E}\left[-b_{12}\, c_1 - b_8\, c_3 + b_{16}\, c_5 - b_1\, c_7 + b_{14}\, c_9 - b_4\, c_{11} + b_6\, c_{13} + b_{10}\, c_{15} + \frac{\left(-1 + e^{-b_{12}}\right) u_{12}\, w_1}{b_{12}} + \frac{\left(-1 + e^{-b_8}\right) u_8\, w_3}{b_8} + \frac{\left(-1 + e^{b_{16}}\right) u_{16}\, w_5}{b_{16}} + \frac{\left(-1 + e^{-b_1}\right) u_1\, w_7}{b_1} - \frac{\left(-1 + e^{-b_1}\right)\left(-1 + e^{-b_{12}}\right) u_{12}\, w_7}{b_{12}} + \frac{\left(-1 + e^{b_{14}}\right) u_{14}\, w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right) u_4\, w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right) u_6\, w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right) u_{10}\, w_{15}}{b_{10}}\right]$$

```
t3 // m[1, 2, 1] // m[1, 3, 1]
```

$$\mathbb{E}\left[-b_8\, c_1 - b_{12}\, c_1 + b_{16}\, c_5 - b_1\, c_7 + b_{14}\, c_9 - b_4\, c_{11} + b_6\, c_{13} + b_{10}\, c_{15} + \frac{\left(-1 + e^{-b_8}\right) u_8\, w_1}{b_8} + \frac{e^{-b_8}\left(-1 + e^{-b_{12}}\right) u_{12}\, w_1}{b_{12}} + \frac{\left(-1 + e^{b_{16}}\right) u_{16}\, w_5}{b_{16}} + \frac{e^{b_8}\left(-1 + e^{-b_1}\right) u_1\, w_7}{b_1} - \frac{\left(-1 + e^{-b_1}\right)\left(-1 + e^{-b_{12}}\right) u_{12}\, w_7}{b_{12}} + \frac{\left(-1 + e^{b_{14}}\right) u_{14}\, w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right) u_4\, w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right) u_6\, w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right) u_{10}\, w_{15}}{b_{10}}\right]$$

```
Do[t3 = t3 // m[1, kk, 1], {kk, 2, 16}]; t3
```

$$-\frac{e^{3\, b_1}\, \mathbb{E}[0]}{1 - 4\, e^{b_1} + 8\, e^{2\, b_1} - 11\, e^{3\, b_1} + 8\, e^{4\, b_1} - 4\, e^{5\, b_1} + e^{6\, b_1}}$$

```
Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 4, 4] // m[4, 5, 5] // m[5, 2, 2] // m[2, 3, 3] // m[3, 6, 6] // m[6, 1, 1]
```

$$\frac{\mathbb{E}\left[3\, b_1\, c_1 + \frac{\left(1 - e^{-3\, b_1}\right) u_1\, w_1}{b_1}\right]}{1 + e^{b_1}\left(-1 + e^{b_1}\right)}$$