Pensieve header: Normally Ordered Exponentials at 0-Co.

"NO" for “Normal Order”.

```
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := Simp@𝔼[Q1 + Q2]
```

```
Simp[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ⋀ Q1 == Q2];
```

```
NO[u_i_, c_j_, k_][ω_. 𝔼[Q_]] :=
  Simp@Module[{α = ∂_{c_j} Q, β = ∂_{u_i} Q, Θ = Q /. c_j | u_i → 0}, ω 𝔼[ℯ^(-α) β u_k + α c_k + Θ]];
NO[w_i_, c_j_, k_][ω_. 𝔼[Q_]] :=
  Simp@Module[{α = ∂_{c_j} Q, β = ∂_{w_i} Q, Θ = Q /. c_j | w_i → 0}, ω 𝔼[ℯ^α β w_k + α c_k + Θ]];
NO[w_i_, u_j_, k_][ω_. 𝔼[Q_]] :=
  Simp@Module[{α = ∂_{w_i} Q /. u_j → 0, β = ∂_{u_j} Q /. w_i → 0, δ = ∂_{w_i,u_j} Q, Θ = Q /. w_i | u_j → 0, ν},
    ν = (1 + b_k δ)^-1;
    ν ω 𝔼[-b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + Θ]
   ];
```

```
m[i_, j_, k_][Z_] :=
 Simp@Module[{a}, (Z /. b_{i|j} → b_k // NO[w_i, c_j, a] // NO[u_i, c_a, a] // NO[w_a, u_j, a]) /.
    {c_i → c_k, w_j → w_k, y_a :→ y_k}]
```

```
σ[i_, j_][ℰ_] := ℰ /. {b_i → b_j, c_i → c_j, u_i → u_j, w_i → w_j}
```

$$\texttt{Rp}[i\_,j\_] := \mathbb{E}\left[b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right];$$

$$\texttt{Rm}[i\_,j\_] := \mathbb{E}\left[-b_i c_j + \frac{e^{-b_i}-1}{b_i} u_i w_j\right];$$

```
NO[u1, c1, 1][𝔼[b2 c1 + w2 u1 + u3 w4]]
```

$$\mathbb{E}\left[b_2 c_1 + e^{-b_2} u_1 w_2 + u_3 w_4\right]$$

```
Q0 = 𝔼[Sum[β_i c_i, {i, 3}] + Sum[α_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1\beta_1 + c_2\beta_2 + c_3\beta_3 + u_1 w_1 \alpha_{1,1} + u_1 w_2 \alpha_{1,2} + u_1 w_3 \alpha_{1,3} + u_2 w_1 \alpha_{2,1} + u_2 w_2 \alpha_{2,2} + u_2 w_3 \alpha_{2,3} + u_3 w_1 \alpha_{3,1} + u_3 w_2 \alpha_{3,2} + u_3 w_3 \alpha_{3,3}]$$

```
Q0 // m[1, 2, 1]
```

$$\frac{1}{1+e^{\beta_2} b_1 \alpha_{2,1}} \mathbb{E}\left[c_1\beta_1 + c_1\beta_2 + c_3\beta_3 + e^{-\beta_2} u_1 (w_1 \alpha_{1,2} + w_3 \alpha_{1,3}) + \frac{e^{\beta_2} u_1 w_1 \alpha_{2,1}}{1+e^{\beta_2} b_1 \alpha_{2,1}} + \frac{u_1 (w_1 \alpha_{2,2} + w_3 \alpha_{2,3})}{1+e^{\beta_2} b_1 \alpha_{2,1}} + \frac{w_1 (u_1 \alpha_{1,1} + e^{\beta_2} u_3 \alpha_{3,1})}{1+e^{\beta_2} b_1 \alpha_{2,1}} - \frac{b_1 (w_1 \alpha_{2,2} + w_3 \alpha_{2,3}) (u_1 \alpha_{1,1} + e^{\beta_2} u_3 \alpha_{3,1})}{1+e^{\beta_2} b_1 \alpha_{2,1}} + u_3 w_1 \alpha_{3,2} + u_3 w_3 \alpha_{3,3}\right]$$

```
Q0 // NO[w1, u2, 3]
```

$$\frac{1}{1+b_3\,\alpha_{2,1}}\,\mathbb{E}\Big[c_1\,\beta_1+c_2\,\beta_2+c_3\,\beta_3+u_1\,w_2\,\alpha_{1,2}+u_1\,w_3\,\alpha_{1,3}+\frac{u_3\,w_3\,\alpha_{2,1}}{1+b_3\,\alpha_{2,1}}+\frac{u_3\,(w_2\,\alpha_{2,2}+w_3\,\alpha_{2,3})}{1+b_3\,\alpha_{2,1}}+$$
$$\frac{w_3\,(u_1\,\alpha_{1,1}+u_3\,\alpha_{3,1})}{1+b_3\,\alpha_{2,1}}-\frac{b_3\,(w_2\,\alpha_{2,2}+w_3\,\alpha_{2,3})\,(u_1\,\alpha_{1,1}+u_3\,\alpha_{3,1})}{1+b_3\,\alpha_{2,1}}+u_3\,w_2\,\alpha_{3,2}+u_3\,w_3\,\alpha_{3,3}\Big]$$

```
t1 = Q0 // m[1, 2, 1] // m[1, 3, 1]
```

$$\mathbb{E}\Big[\Big(e^{-\beta_2-\beta_3}\,\Big(-e^{\beta_2+\beta_3}\,c_1\,(\beta_1+\beta_2+\beta_3)\,\Big(1-e^{\beta_2+\beta_3}\,b_1^2\,(\alpha_{2,2}\,\alpha_{3,1}-\alpha_{2,1}\,\alpha_{3,2})+b_1\,\Big(e^{\beta_2}\,\alpha_{2,1}+e^{\beta_3}\,\Big(e^{\beta_2}\,\alpha_{3,1}+\alpha_{3,2}\Big)\Big)\Big)-$$
$$u_1\,w_1\,\Big(\alpha_{1,3}+e^{2\,\beta_2+\beta_3}\,\alpha_{2,1}+e^{\beta_2}\,b_1\,\alpha_{1,3}\,\alpha_{2,1}+e^{\beta_2+\beta_3}\,\alpha_{2,2}+e^{\beta_2}\,\alpha_{2,3}+e^{2\,(\beta_2+\beta_3)}\,\alpha_{3,1}+$$
$$e^{\beta_2+\beta_3}\,b_1\,\alpha_{1,3}\,\alpha_{3,1}-e^{2\,(\beta_2+\beta_3)}\,b_1\,\alpha_{2,2}\,\alpha_{3,1}-e^{\beta_2+\beta_3}\,b_1^2\,\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1}+e^{\beta_2+2\,\beta_3}\,\alpha_{3,2}+$$
$$e^{\beta_3}\,b_1\,\alpha_{1,3}\,\alpha_{3,2}+e^{2\,(\beta_2+\beta_3)}\,b_1\,\alpha_{2,1}\,\alpha_{3,2}+e^{\beta_2+\beta_3}\,b_1^2\,\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2}+e^{\beta_2+\beta_3}\,b_1\,\alpha_{2,3}\,\alpha_{3,2}+e^{\beta_2+\beta_3}\,\alpha_{3,3}-$$
$$e^{\beta_2+\beta_3}\,b_1\,\alpha_{2,2}\,\alpha_{3,3}-e^{\beta_3}\,\alpha_{1,2}\,\Big(-1+b_1\,\Big(-e^{\beta_2}\,\alpha_{2,1}+\alpha_{3,3}\Big)-e^{\beta_2}\,b_1^2\,(\alpha_{2,3}\,\alpha_{3,1}-\alpha_{2,1}\,\alpha_{3,3})\Big)+$$
$$e^{\beta_2}\,\alpha_{1,1}\,\Big(e^{\beta_3}-b_1\,\Big(e^{\beta_3}\,\alpha_{2,2}+\alpha_{2,3}+e^{\beta_3}\,\alpha_{3,3}\Big)-e^{\beta_3}\,b_1^2\,(\alpha_{2,3}\,\alpha_{3,2}-\alpha_{2,2}\,\alpha_{3,3})\Big)\Big)\Big)\Big)\Big/$$
$$\Big(-1+e^{\beta_2+\beta_3}\,b_1^2\,(\alpha_{2,2}\,\alpha_{3,1}-\alpha_{2,1}\,\alpha_{3,2})-b_1\,\Big(e^{\beta_2}\,\alpha_{2,1}+e^{\beta_3}\,\Big(e^{\beta_2}\,\alpha_{3,1}+\alpha_{3,2}\Big)\Big)\Big)\Big]\Big/\Big(1+$$
$$e^{\beta_3}\,b_1\,\Big(e^{\beta_2}\,\alpha_{3,1}+\alpha_{3,2}\Big)+$$
$$e^{\beta_2}\,b_1\,\Big(-e^{\beta_3}\,b_1\,\alpha_{2,2}\,\alpha_{3,1}+\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)\Big)$$

```
t2 = Q0 // m[2, 3, 2] // m[1, 2, 1]
```

$$\mathbb{E}\Big[\Big(e^{-\beta_2-\beta_3}\,\Big(e^{\beta_2+\beta_3}\,c_1\,\beta_1\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}+e^{\beta_2}\,b_1\,\Big(-e^{\beta_3}\,\Big(-1+b_1\,\alpha_{2,2}\Big)\,\alpha_{3,1}+\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)\Big)+$$
$$e^{\beta_2+\beta_3}\,c_1\,(\beta_2+\beta_3)\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}+e^{\beta_2}\,b_1\,\Big(-e^{\beta_3}\,\Big(-1+b_1\,\alpha_{2,2}\Big)\,\alpha_{3,1}+\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)\Big)+$$
$$u_1\,w_1\,\Big(e^{\beta_2+\beta_3}$$
$$\Big(-e^{\beta_3}\,\Big(-e^{\beta_2}+b_1\,\Big(\alpha_{1,2}+e^{\beta_2}\,\alpha_{2,2}\Big)\Big)\,\alpha_{3,1}+\alpha_{1,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)+e^{\beta_2}\,\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)+$$
$$\alpha_{1,3}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}+e^{\beta_2}\,b_1\,\Big(-e^{\beta_3}\,\Big(-1+b_1\,\alpha_{2,2}\Big)\,\alpha_{3,1}+\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)\Big)-$$
$$e^{\beta_2}\,\Big(-1+b_1\,\alpha_{1,1}\Big)\,\Big(\alpha_{2,3}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)+e^{\beta_3}\,\Big(e^{\beta_3}\,\alpha_{3,2}+\alpha_{3,3}\Big)-e^{\beta_3}\,\alpha_{2,2}\,\Big(-1+b_1\,\alpha_{3,3}\Big)\Big)+$$
$$e^{\beta_3}\,\alpha_{1,2}\,\Big(1-b_1\,\alpha_{3,3}-e^{\beta_2}\,b_1\,\Big(-\Big(e^{\beta_3}+b_1\,\alpha_{2,3}\Big)\,\alpha_{3,1}+\alpha_{2,1}\,\Big(-1+b_1\,\alpha_{3,3}\Big)\Big)\Big)\Big)\Big)\Big)\Big/$$
$$\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}+e^{\beta_2}\,b_1\,\Big(-e^{\beta_3}\,\Big(-1+b_1\,\alpha_{2,2}\Big)\,\alpha_{3,1}+\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)\Big)\Big]\Big/\Big(1+$$
$$e^{\beta_3}\,b_1\,\alpha_{3,2}+$$
$$e^{\beta_2}\,b_1\,\Big(-e^{\beta_3}\,\Big(-1+b_1\,\alpha_{2,2}\Big)\,\alpha_{3,1}+\alpha_{2,1}\,\Big(1+e^{\beta_3}\,b_1\,\alpha_{3,2}\Big)\Big)\Big)$$

```
t1 ≡ t2
```

```
True
```

```
Rp[1, 2] Rm[4, 3]
```

$$\mathbb{E}\Big[b_1\,c_2-b_4\,c_3+\frac{\Big(-1+e^{b_1}\Big)\,u_1\,w_2}{b_1}+\frac{\Big(-1+e^{-b_4}\Big)\,u_4\,w_3}{b_4}\Big]$$

```
Rp[1, 2] Rm[4, 3] // m[1, 4, 1] // m[2, 3, 2]
```

$$\mathbb{E}[0]$$

```
Rm[1, 2] Rp[4, 3] // m[4, 1, 4] // m[2, 3, 2]
```

$$\mathbb{E}[0]$$

```
t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, a] // m[1, 6, b] // m[2, 4, c]
```

$$\mathbb{E}\left[b_a\ (c_b + c_c) + \frac{\left(-1 + e^{b_a}\right)\ u_a\ (w_b + w_c)}{b_a} + \frac{b_b^2\ c_c + \left(-1 + e^{b_b}\right)\ u_b\ w_c}{b_b}\right]$$

```
t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, a] // m[2, 5, b] // m[4, 6, c]
```

$$\mathbb{E}\left[\frac{1}{b_a\ b_b}\left(b_a^2\ b_b\ (c_b + c_c) + \left(-1 + e^{b_a}\right)\ b_b\ u_a\ (w_b + w_c) + b_a\ \left(b_b^2\ c_c + \left(-1 + e^{b_b}\right)\ u_b\ w_c\right)\right)\right]$$

```
t1 ≡ t2
```

```
True
```

```
t3 = Rm[12, 1] Rm[2, 7] Rm[8, 3] Rm[4, 11] Rp[16, 5] Rp[6, 13] Rp[14, 9] Rp[10, 15]
```

$$\mathbb{E}\left[-b_{12}\ c_1 - b_8\ c_3 + b_{16}\ c_5 - b_2\ c_7 + b_{14}\ c_9 - b_4\ c_{11} + b_6\ c_{13} + b_{10}\ c_{15} + \frac{\left(-1 + e^{-b_{12}}\right)\ u_{12}\ w_1}{b_{12}} + \frac{\left(-1 + e^{-b_8}\right)\ u_8\ w_3}{b_8} + \frac{\left(-1 + e^{b_{16}}\right)\ u_{16}\ w_5}{b_{16}} + \frac{\left(-1 + e^{-b_2}\right)\ u_2\ w_7}{b_2} + \frac{\left(-1 + e^{b_{14}}\right)\ u_{14}\ w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right)\ u_4\ w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right)\ u_6\ w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right)\ u_{10}\ w_{15}}{b_{10}}\right]$$

```
t3 // m[1, 2, 1]
```

$$\mathbb{E}\left[-b_{12}\ c_1 - b_8\ c_3 + b_{16}\ c_5 - b_1\ c_7 + b_{14}\ c_9 - b_4\ c_{11} + b_6\ c_{13} + b_{10}\ c_{15} + \frac{\left(-1 + e^{-b_{12}}\right)\ u_{12}\ w_1}{b_{12}} + \frac{\left(-1 + e^{-b_8}\right)\ u_8\ w_3}{b_8} + \frac{\left(-1 + e^{b_{16}}\right)\ u_{16}\ w_5}{b_{16}} + \frac{\left(-1 + e^{-b_1}\right)\ u_1\ w_7}{b_1} - \frac{\left(-1 + e^{-b_1}\right)\ \left(-1 + e^{-b_{12}}\right)\ u_{12}\ w_7}{b_{12}} + \frac{\left(-1 + e^{b_{14}}\right)\ u_{14}\ w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right)\ u_4\ w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right)\ u_6\ w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right)\ u_{10}\ w_{15}}{b_{10}}\right]$$

```
t3 // m[1, 2, 1] // m[1, 3, 1]
```

$$\mathbb{E}\left[-b_8\ c_1 - b_{12}\ c_1 + b_{16}\ c_5 - b_1\ c_7 + b_{14}\ c_9 - b_4\ c_{11} + b_6\ c_{13} + b_{10}\ c_{15} + \frac{\left(-1 + e^{-b_8}\right)\ u_8\ w_1}{b_8} + \frac{e^{-b_8}\ \left(-1 + e^{-b_{12}}\right)\ u_{12}\ w_1}{b_{12}} + \frac{\left(-1 + e^{b_{16}}\right)\ u_{16}\ w_5}{b_{16}} + \frac{e^{b_8}\ \left(-1 + e^{-b_1}\right)\ u_1\ w_7}{b_1} - \frac{\left(-1 + e^{-b_1}\right)\ \left(-1 + e^{-b_{12}}\right)\ u_{12}\ w_7}{b_{12}} + \frac{\left(-1 + e^{b_{14}}\right)\ u_{14}\ w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right)\ u_4\ w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right)\ u_6\ w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right)\ u_{10}\ w_{15}}{b_{10}}\right]$$

```
t4 = t3 /. b_ → b;
Do[t4 = (t4 // m[1, kk, 1]) /. b_ → b, {kk, 2, 16}]; Simp@t4
```

$$-\frac{e^{3\,b}\ \mathbb{E}[0]}{1 - 4\ e^{b} + 8\ e^{2\,b} - 11\ e^{3\,b} + 8\ e^{4\,b} - 4\ e^{5\,b} + e^{6\,b}}$$

```
Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 4, 4] // m[4, 5, 5] // m[5, 2, 2] // m[2, 3, 3] //
  m[3, 6, 6] // m[6, 1, 1]
```

$$\frac{\mathbb{E}\left[3\ b_1\ c_1 + \frac{\left(1 - e^{-3\ b_1}\right)\ u_1\ w_1}{b_1}\right]}{1 + e^{b_1}\ \left(-1 + e^{b_1}\right)}$$