

Pensieve header: Verifying the commutation relations of frakg_1.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
<< OneCo.m
```

Testing g_1

$a_{1,2}$

$b_1 U[C_2] - \in U[C_1, C_2] + U[u_1, w_2]$

Table[x → B[U@x, a_{1,2}], {x, {c₁, c₂, u₁, u₂, w₁, w₂, c₃, u₃, w₃}}] // Column

$c_1 \rightarrow U[u_1, w_2]$

$c_2 \rightarrow -U[u_1, w_2]$

$u_1 \rightarrow \in U[C_2, u_1]$

$u_2 \rightarrow b_2 U[u_1] - b_1 U[u_2] + \in U[c_1, u_2] - 2 \in U[C_2, u_1]$

$w_1 \rightarrow -b_1 U[w_2] + 2 \in U[C_1, w_2] - \in U[C_2, w_1]$

$w_2 \rightarrow b_1 U[w_2] - \in U[C_1, w_2]$

$c_3 \rightarrow \emptyset$

$u_3 \rightarrow \emptyset$

$w_3 \rightarrow \emptyset$

B[a_{1,3}, a_{2,3}] + B[a_{1,2}, a_{2,3}] + B[a_{1,2}, a_{1,3}]

\emptyset

{t1 = B[a_{1,2}, a_{1,3}], t2 = $\in U@c_3 ** a_{1,2} - \in U@c_2 ** a_{1,3}$, t1 - t2} // Expand // Column

$-\in U[C_2, u_1, w_3] + \in U[C_3, u_1, w_2]$

$-\in U[C_2, u_1, w_3] + \in U[C_3, u_1, w_2]$

\emptyset

{t1 = B[a_{1,3}, a_{2,3}], t2 = $bc_2 ** a_{1,3} - bc_1 ** a_{2,3}$, t1 - t2} // Expand // Column

$b_2 U[u_1, w_3] - b_1 U[u_2, w_3] + \in U[C_1, u_2, w_3] - \in U[C_2, u_1, w_3]$

$b_2 U[u_1, w_3] - b_1 U[u_2, w_3] + \in U[C_1, u_2, w_3] - \in U[C_2, u_1, w_3]$

\emptyset

{t1 = $a_{1,3} ** a_{2,4} - a_{1,4} ** a_{2,3}$,

t2 = $a_{1,3} ** bc_2 ** U@c_4 - a_{2,3} ** bc_1 ** U@c_4 - a_{1,4} ** bc_2 ** U@c_3 + a_{2,4} ** bc_1 ** U@c_3$,

t1 - t2

} // Column

$-b_2 U[C_3, u_1, w_4] + b_1 U[C_3, u_2, w_4] + b_2 U[C_4, u_1, w_3] - b_1 U[C_4, u_2, w_3] -$

$\in U[C_1, c_3, u_2, w_4] + \in U[C_1, c_4, u_2, w_3] + \in U[C_2, c_3, u_1, w_4] - \in U[C_2, c_4, u_1, w_3]$

$-b_2 U[C_3, u_1, w_4] + b_1 U[C_3, u_2, w_4] + b_2 U[C_4, u_1, w_3] - b_1 U[C_4, u_2, w_3] -$

$\in U[C_1, c_3, u_2, w_4] + \in U[C_1, c_4, u_2, w_3] + \in U[C_2, c_3, u_1, w_4] - \in U[C_2, c_4, u_1, w_3]$

\emptyset

LBasis[2]

```
{U[], ∈ U[], U[c1], U[c2], U[u1], U[u2], U[w1], U[w2], ∈ U[c1], ∈ U[c2], ∈ U[u1], ∈ U[u2],
 ∈ U[w1], ∈ U[w2], U[u1, w1], U[u1, w2], U[u2, w1], U[u2, w2], ∈ U[c1, c1], ∈ U[c1, c2],
 ∈ U[c1, u1], ∈ U[c1, u2], ∈ U[c1, w1], ∈ U[c1, w2], ∈ U[c2, c2], ∈ U[c2, u1], ∈ U[c2, u2],
 ∈ U[c2, w1], ∈ U[c2, w2], ∈ U[u1, w1], ∈ U[u1, w2], ∈ U[u2, w1], ∈ U[u2, w2], ∈ U[c1, u1, w1],
 ∈ U[c1, u1, w2], ∈ U[c1, u2, w1], ∈ U[c1, u2, w2], ∈ U[c2, u1, w1], ∈ U[c2, u1, w2],
 ∈ U[c2, u2, w1], ∈ U[c2, u2, w2], ∈ U[u1, u1, w1], ∈ U[u1, u1, w2], ∈ U[u1, u2, w1],
 ∈ U[u1, u2, w2], ∈ U[u1, w1, w1], ∈ U[u1, w1, w2], ∈ U[u1, w2, w2], ∈ U[u2, u2, w1],
 ∈ U[u2, u2, w2], ∈ U[u2, w1, w1], ∈ U[u2, w1, w2], ∈ U[u2, w2, w2], ∈ U[u1, u1, w1, w1],
 ∈ U[u1, u1, w1, w2], ∈ U[u1, u1, w2, w2], ∈ U[u1, u2, w1, w1], ∈ U[u1, u2, w1, w2],
 ∈ U[u1, u2, w2, w2], ∈ U[u2, u2, w1, w1], ∈ U[u2, u2, w1, w2], ∈ U[u2, u2, w2, w2]}
```

```
bas = LBasis[2];
```

```
Table[B[x, y] + B[y, x], {x, bas}, {y, bas}] // Flatten // Union
```

```
{0}
```

```
bas = LBasis[2]; Timing[
```

```
Table[
```

```
{x, y, z} = xyz;
```

```
Simp[B[B[x, y], z] + B[B[y, z], x] + B[B[z, x], y]],
```

```
{xyz, Subsets[bas, {3}]}]
```

```
// Flatten // Union
```

```
]
```

```
{31., {0}}
```

Infinitesimal Spinner Relations

$$s_{i_} := \frac{b_i}{2} U[] - \in U[c_i]$$

```
a1,2
```

$$b_1 U[c_2] - \in U[c_1, c_2] + U[u_1, w_2]$$

```
B[a1,2, s1]
```

$$\in U[u_1, w_2]$$

```
B[a1,2, s2]
```

$$- \in U[u_1, w_2]$$

```
B[a1,2, s1 + s2] // Simplify
```

```
0
```

```
B[a1,3, a2,3]
```

$$b_2 U[u_1, w_3] - b_1 U[u_2, w_3] + \in U[c_1, u_2, w_3] - \in U[c_2, u_1, w_3]$$

```
Simplify[b1 UU[u1, w3] - b1 UU[u1, w3] + ∈ UU[c1, u1, w3] - ∈ UU[u1, c1, w3]]
```

$$\in U[u_1, w_3]$$

Testing bi-local exponential relations in g_0

1. The Yang-Baxter Element.
2. c Relations.

$B[U[c], U[u, w]]$

\emptyset

```
With[{n = 7}, Collect[
  UExp[β U@u, n] ** UExp[α U@c, n] - UExp[α U@c, n] ** UExp[e-α β U@u, n],
  U, Series[#, {α, 0, n}] &
]]
O[α]8
```

```
With[{n = 7}, Collect[
  UExp[β U@w, n] ** UExp[α U@c, n] - UExp[α U@c, n] ** UExp[eα β U@w, n],
  U, Series[#, {α, 0, n}] &
]]
O[α]8
```

3. u, w, e^u, e^w .

```
With[{n = 10},
  Simp[B[U@w, UExp[γ U@u, n]] + (b γ U[] - 2 γ ε U[c] + ε γ2 U[u]) ** UExp[γ U@u, n]]]
  (  $\frac{b \gamma^{11}}{3628800} + \frac{\gamma^{11} \epsilon}{362880}$  ) U[u, u, u, u, u, u, u, u, u, u] -
   $\frac{\gamma^{11} \epsilon U[c, u, u, u, u, u, u, u, u, u]}{1814400} + \frac{\gamma^{12} \epsilon U[u, u, u, u, u, u, u, u, u, u]}{3628800}$ 
```

```
With[{n = 7},
  Simp[B[U@u, UExp[γ U@w, n]] - (b γ U[] - 2 γ ε U[c] - ε γ2 U[w]) ** UExp[γ U@w, n]]]
  (  $-\frac{b \gamma^8}{5040} + \frac{\gamma^8 \epsilon}{720}$  ) U[w, w, w, w, w, w, w] +
   $\frac{\gamma^8 \epsilon U[c, w, w, w, w, w, w, w, w, w]}{2520} + \frac{\gamma^9 \epsilon U[w, w, w, w, w, w, w, w, w]}{5040}$ 
```

4. M_{uw} Relations.

$$\text{Muw}[\gamma, n] := \text{Expand}\left[\sum_{k=0}^n \frac{\gamma^k}{k!} \text{UU}[u^k, w^k]\right]$$

$\text{Muw}[\gamma, 5]$

$$U[] + \gamma U[u, w] + \frac{1}{2} \gamma^2 U[u, u, w, w] + \frac{1}{6} \gamma^3 U[u, u, u, w, w, w] +$$

$$\frac{1}{24} \gamma^4 U[u, u, u, u, w, w, w, w] + \frac{1}{120} \gamma^5 U[u, u, u, u, u, w, w, w, w, w]$$


```
With[{n = 10, v = (1 + b δ)^-1}, t1 = Simp[
  -Mwu[δ, n] + (v (1 - v^2 b δ^2 ε) U[] + 2 v^2 δ ε U[c]) ** Muw[v δ, n] +
  (-2 v^4 b δ^3 ε U[u] + 2 v^3 δ^2 ε U[c, u]) ** Muw[v δ, n] ** U[w] -
  
$$\frac{b v^5 \delta^4 \epsilon}{2} U[u, u] ** Muw[v \delta, n] ** U[w, w]$$

] // ToDegree[
  n]
```

0

```
Coefficient[t1, U[u, u, w, w]] /. ħ → 1
```

$$\frac{1}{2} b \delta^4 \epsilon - \frac{5}{2} b^2 \delta^5 \epsilon + \frac{15}{2} b^3 \delta^6 \epsilon - \frac{35}{2} b^4 \delta^7 \epsilon + 35 b^5 \delta^8 \epsilon - 63 b^6 \delta^9 \epsilon$$

```
Series[ $\frac{b \delta^4 \epsilon}{2 (1 + b \delta)^5}$ , {b, 0, 7}] // Normal
```

$$\frac{1}{2} b \delta^4 \epsilon - \frac{5}{2} b^2 \delta^5 \epsilon + \frac{15}{2} b^3 \delta^6 \epsilon - \frac{35}{2} b^4 \delta^7 \epsilon + 35 b^5 \delta^8 \epsilon - 63 b^6 \delta^9 \epsilon + 105 b^7 \delta^{10} \epsilon$$

```
Simplify[ $\left(\frac{1}{1 + b \delta} + \frac{-b \delta^2 \epsilon}{(1 + b \delta)^3}\right)$ ]
```

$$\frac{1}{1 + b \delta} - \frac{b \delta^2 \epsilon}{(1 + b \delta)^3}$$

7. Hard core uw relations.

```
With[{n = 3}, Simp[
  UExp[α ħ U@w, n] ** UExp[β U@u, n] - e^{-b ħ α β} UExp[β U@u, n] ** UExp[α ħ U@w, n]
] // ToDegree[n]
```

```
n = 14; With[{v = (1 + b δ)^-1},
```

```
LHS = Expand[e^{b ħ v α β} Sum[
  
$$\frac{(\alpha \hbar)^{n_1} \delta^{n_2} \beta^{n_3}}{n_1! n_2! n_3!} U[u^{n_1+n_2}, u^{n_2+n_3}],$$

  {n1, 0, n}, {n2, 0, n - n1}, {n3, 0, n - n1 - n2}
]];
```

```
RHS = Expand[v Sum[
  
$$\frac{(v \beta)^{n_1} (v \delta)^{n_2} (v \alpha \hbar)^{n_3}}{n_1! n_2! n_3!} U[u^{n_1+n_2}, w^{n_2+n_3}],$$

  {n1, 0, n}, {n2, 0, n - n1}, {n3, 0, n - n1 - n2}
]];
```

]

t3 = With[{ $v = (1 + b \delta)^{-1}$ }, **Series**[

$$\frac{1}{2} b \delta^4 v^5 \epsilon,$$

{b, 0, 12}]] // Normal // Simplify

$$\frac{1}{2} b \delta^4 (1 - 5 b \delta + 15 b^2 \delta^2 - 35 b^3 \delta^3 + 70 b^4 \delta^4 - 126 b^5 \delta^5 + 210 b^6 \delta^6 - 330 b^7 \delta^7 + 495 b^8 \delta^8 - 715 b^9 \delta^9 + 1001 b^{10} \delta^{10} - 1365 b^{11} \delta^{11}) \epsilon$$

Series[t2 - t3, {b, 0, 8}]

$$\frac{6435}{8} \alpha^4 \beta^4 \delta^7 \epsilon b^8 + 0 [b]^9$$

Simplify[$\beta \delta^2 (1 + 2 b \delta) v^4 \epsilon$]

$$\beta \delta^2 (1 + 2 b \delta) \epsilon v^4$$

Logos and testing

$$\begin{aligned} \text{Logos}[M_-, \gamma_-] := & -\frac{1}{2} b \gamma (2 \delta^2 + 4 \alpha \beta \delta \gamma + \alpha^2 \beta^2 \gamma^2) M + \\ & 2 (\delta + \alpha \beta \gamma) U[c] \otimes M - \beta (1 + 2 b \delta) \gamma^2 (2 \delta + \alpha \beta \gamma) U[u] \otimes M + \alpha \gamma^2 (2 \delta + \alpha \beta \gamma) M \otimes U[w] + \\ & 2 \beta \delta \gamma U[c, u] \otimes M + 2 \alpha \delta \gamma U[c] \otimes M \otimes U[w] - \frac{1}{2} \beta^2 \delta (2 + 3 b \delta) \gamma^3 U[u, u] \otimes M + \\ & \frac{1}{2} \alpha^2 \delta (2 + b \delta) \gamma^3 M \otimes U[w, w] - 2 b \delta^2 \gamma^2 (\delta + \alpha \beta \gamma) U[u] \otimes M \otimes U[w] + 2 \delta^2 \gamma U[c, u] \otimes M \otimes U[w] - \\ & \beta \delta^2 (1 + 2 b \delta) \gamma^3 U[u, u] \otimes M \otimes U[w] + \alpha \delta^2 \gamma^3 U[u] \otimes M \otimes U[w, w] - \frac{1}{2} b \delta^4 \gamma^3 U[u, u] \otimes M \otimes U[w, w]; \end{aligned}$$

With[{n = 10, \gamma = (1 + b \delta)^{-1}],

```
LHS = Expand[Sum[
  (\alpha)^n1 (\delta)^n2 (\beta)^n3
  UU[w^n1+n2, u^n2+n3],
  {n1, 0, n}, {n2, 0, n - n1}, {n3, 0, n - n1 - n2}
]];
RHS = Expand[\gamma e^{-b \gamma \alpha \beta} Sum[
  (\gamma \beta)^n1 (\gamma \delta)^n2 (\gamma \alpha)^n3
  UU[u^n1+n2, w^n2+n3],
  {n1, 0, n}, {n2, 0, n - n1}, {n3, 0, n - n1 - n2}
]];
t1 = Simp[-LHS + RHS + \epsilon \gamma Logos[RHS, \gamma] /. TensorProduct -> NonCommutativeMultiply];
t1 /. \epsilon -> 0
```

$$\begin{aligned} & \left(-1 + b \alpha \beta - \frac{1}{2} b^2 \alpha^2 \beta^2 + \frac{1}{6} b^3 \alpha^3 \beta^3 - \frac{1}{24} b^4 \alpha^4 \beta^4 + \frac{1}{120} b^5 \alpha^5 \beta^5 + b \delta - 2 b^2 \alpha \beta \delta + \frac{3}{2} b^3 \alpha^2 \beta^2 \delta - \right. \\ & \quad \left. \frac{2}{3} b^4 \alpha^3 \beta^3 \delta + \frac{5}{24} b^5 \alpha^4 \beta^4 \delta - b^2 \delta^2 + \dots 18 \dots + b^5 \delta^5 - 6 b^6 \alpha \beta \delta^5 + \frac{21}{2} b^7 \alpha^2 \beta^2 \delta^5 - b^6 \delta^6 + \right. \\ & \quad \left. 7 b^7 \alpha \beta \delta^6 - 14 b^8 \alpha^2 \beta^2 \delta^6 + b^7 \delta^7 - 8 b^8 \alpha \beta \delta^7 - b^8 \delta^8 + 9 b^9 \alpha \beta \delta^8 + b^9 \delta^9 - b^{10} \delta^{10} + \frac{\epsilon^{-1} b \delta}{1 + b \delta} \right) U[] + \\ & \quad \dots 119 \dots + \left(\dots 1 \dots \right) U[u, \dots 18 \dots, w] \end{aligned}$$

large output | show less | show more | show all | set size limit...

$$O[\hbar]^{11} + (t1 /. \epsilon \to 0 /. a_. x_U \Rightarrow (a /. \{b \to \hbar b, \alpha \to \hbar \alpha\}) \hbar^{\text{Count}[x, u]} x)$$

$$O[\hbar]^{11} + (t1 /. a_. x_U \Rightarrow (a /. \{b \to \hbar b, \alpha \to \hbar \alpha, \epsilon \to \hbar \epsilon\}) \hbar^{\text{Count}[x, u]} x)$$

logos = Logos[1, \gamma] /. TensorProduct | U -> Times

$$\begin{aligned} & 2 c w \alpha \delta \gamma + 2 c u \beta \delta \gamma + 2 c u w \delta^2 \gamma + u w^2 \alpha \delta^2 \gamma^3 - \frac{1}{2} b u^2 w^2 \delta^4 \gamma^3 + \frac{1}{2} w^2 \alpha^2 \delta (2 + b \delta) \gamma^3 - \\ & u^2 w \beta \delta^2 (1 + 2 b \delta) \gamma^3 - \frac{1}{2} u^2 \beta^2 \delta (2 + 3 b \delta) \gamma^3 + 2 c (\delta + \alpha \beta \gamma) - 2 b u w \delta^2 \gamma^2 (\delta + \alpha \beta \gamma) + \\ & w \alpha \gamma^2 (2 \delta + \alpha \beta \gamma) - u \beta (1 + 2 b \delta) \gamma^2 (2 \delta + \alpha \beta \gamma) - \frac{1}{2} b \gamma (2 \delta^2 + 4 \alpha \beta \delta \gamma + \alpha^2 \beta^2 \gamma^2) \end{aligned}$$

$\Delta = (\text{Collect}[\text{Simplify}[\text{logos}], \{c, u, w\}, \text{C}[\text{Simplify}[\#]] \&] // \text{Expand}) /. \text{C}[x_] \Rightarrow x$

$$2 c w \alpha \delta v + 2 c u \beta \delta v + 2 c u w \delta^2 v + u w^2 \alpha \delta^2 v^3 - \frac{1}{2} b u^2 w^2 \delta^4 v^3 + \frac{1}{2} w^2 \alpha^2 \delta (2 + b \delta) v^3 - u^2 w \beta \delta^2 (1 + 2 b \delta) v^3 - \frac{1}{2} u^2 \beta^2 \delta (2 + 3 b \delta) v^3 + 2 c (\delta + \alpha \beta v) - 2 b u w \delta^2 v^2 (\delta + \alpha \beta v) + w \alpha v^2 (2 \delta + \alpha \beta v) - u \beta (1 + 2 b \delta) v^2 (2 \delta + \alpha \beta v) - \frac{1}{2} b v (2 \delta^2 + 4 \alpha \beta \delta v + \alpha^2 \beta^2 v^2)$$

$\Delta // \text{TeXForm}$

$$-\frac{1}{2} b \nu \left(\alpha^2 \beta^2 \nu^2 + 4 \alpha \beta \nu \delta + \delta^2 \right) - \frac{1}{2} \beta^2 \delta \nu^3 u^2 (3 b \delta + 2) - \frac{1}{2} b \delta^4 \nu^3 u^2 w^2 - \beta \delta^2 \nu^3 u^2 w (2 b \delta + 1) - \beta \nu^2 u (2 b \delta + 1) (\alpha \beta \nu + \delta) - 2 b \delta^2 \nu^2 u w (\alpha \beta \nu + \delta) + \frac{1}{2} \alpha^2 \delta \nu^3 w^2 (b \delta + 2) + 2 c (\alpha \beta \nu + \delta) + 2 \beta c \delta \nu u + 2 c \delta^2 \nu u w + 2 \alpha c \delta \nu w + \alpha \delta^2 \nu^3 u w^2 + \alpha \nu^2 w (\alpha \beta \nu + \delta)$$

$$2 c w \alpha \delta v + 2 c u \beta \delta v + 2 c u w \delta^2 v + u w^2 \alpha \delta^2 v^3 - \frac{1}{2} b u^2 w^2 \delta^4 v^3 + \frac{1}{2} w^2 \alpha^2 \delta (2 + b \delta) v^3 - u^2 w \beta \delta^2 (1 + 2 b \delta) v^3 - \frac{1}{2} u^2 \beta^2 \delta (2 + 3 b \delta) v^3 + 2 c (\delta + \alpha \beta v) - 2 b u w \delta^2 v^2 (\delta + \alpha \beta v) + w \alpha v^2 (2 \delta + \alpha \beta v) - u \beta (1 + 2 b \delta) v^2 (2 \delta + \alpha \beta v) - \frac{1}{2} b v (2 \delta^2 + 4 \alpha \beta \delta v + \alpha^2 \beta^2 v^2) /. \{\delta \rightarrow 0, v \rightarrow 1\}$$

$$2 c \alpha \beta + w \alpha^2 \beta - u \alpha \beta^2 - \frac{1}{2} b \alpha^2 \beta^2$$