

Pensieve header: Verifying the commutation relations of frakg_0.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
<< OneCo.m;
ϵ = 0;
```

Testing g₀

```
a1,2
```

```
b1 U[c2] + U[u1, w2]
```

```
Table[x → B[U@x, a1,2], {x, {c1, c2, u1, u2, w1, w2, c3, u3, w3}}] // Column
```

```
c1 → U[u1, w2]
c2 → -U[u1, w2]
u1 → 0
u2 → b2 U[u1] - b1 U[u2]
w1 → -b1 U[w2]
w2 → b1 U[w2]
c3 → 0
u3 → 0
w3 → 0
```

```
B[a1,3, a2,3] + B[a1,2, a2,3] + B[a1,2, a1,3]
```

```
0
```

```
{t1 = B[a1,2, a1,3], t2 = ϵ U@c3 ** a1,2 - ϵ U@c2 ** a1,3, t1 - t2} // Expand // Column
```

```
0
0
0
```

```
{t1 = B[a1,3, a2,3], t2 = bc2 ** a1,3 - bc1 ** a2,3, t1 - t2} // Expand // Column
```

```
b2 U[u1, w3] - b1 U[u2, w3]
b2 U[u1, w3] - b1 U[u2, w3]
0
```

```
{t1 = a1,3 ** a2,4 - a1,4 ** a2,3,
```

```
  t2 = a1,3 ** bc2 ** U@c4 - a2,3 ** bc1 ** U@c4 - a1,4 ** bc2 ** U@c3 + a2,4 ** bc1 ** U@c3,
```

```
  t1 - t2
```

```
} // Column
```

```
-b2 U[c3, u1, w4] + b1 U[c3, u2, w4] + b2 U[c4, u1, w3] - b1 U[c4, u2, w3]
-b2 U[c3, u1, w4] + b1 U[c3, u2, w4] + b2 U[c4, u1, w3] - b1 U[c4, u2, w3]
0
```

```
LBasis[2]
```

```
{U[], U[c1], U[c2], U[u1], U[u2], U[w1], U[w2], U[u1, w1], U[u1, w2], U[u2, w1], U[u2, w2]}
```

```
bas = LBasis[2]; Table[B[x, y] + B[y, x], {x, bas}, {y, bas}] // Flatten // Union
```

```
{0}
```

```

bas = LBasis[2]; Timing[
  Table[
    {x, y, z} = xyz;
    Simp[B[B[x, y], z] + B[B[y, z], x] + B[B[z, x], y]],
    {xyz, Subsets[bas, {3}]}
  ] // Flatten // Union
]
{0.078125, {0}}

(# -> Ea[tau, i, j][U@#] - AutoEa[tau, i, j][U@#]) & /@ {c_i, c_j, u_i, u_j, w_i, w_j}
{c_i -> 0, c_j -> 0, u_i -> 0, u_j -> 0, w_i -> 0, w_j -> 0}

bas = U/@Flatten@Table[y_k, {y, {c, u, w}}, {k, 3}];
Table[{x, y} -> Expand[Ea[tau, 1, 2][B[x, y]] - B[Ea[tau, 1, 2][x], Ea[tau, 1, 2][y]]],
  {x, bas}, {y, bas}]
{{{U[c_1], U[c_1]} -> 0, {U[c_1], U[c_2]} -> 0, {U[c_1], U[c_3]} -> 0,
  {U[c_1], U[u_1]} -> 0, {U[c_1], U[u_2]} -> 0, {U[c_1], U[u_3]} -> 0,
  {U[c_1], U[w_1]} -> 0, {U[c_1], U[w_2]} -> 0, {U[c_1], U[w_3]} -> 0},
{{U[c_2], U[c_1]} -> 0, {U[c_2], U[c_2]} -> 0, {U[c_2], U[c_3]} -> 0,
  {U[c_2], U[u_1]} -> 0, {U[c_2], U[u_2]} -> 0, {U[c_2], U[u_3]} -> 0,
  {U[c_2], U[w_1]} -> 0, {U[c_2], U[w_2]} -> 0, {U[c_2], U[w_3]} -> 0},
{{U[c_3], U[c_1]} -> 0, {U[c_3], U[c_2]} -> 0, {U[c_3], U[c_3]} -> 0,
  {U[c_3], U[u_1]} -> 0, {U[c_3], U[u_2]} -> 0, {U[c_3], U[u_3]} -> 0,
  {U[c_3], U[w_1]} -> 0, {U[c_3], U[w_2]} -> 0, {U[c_3], U[w_3]} -> 0},
{{U[u_1], U[c_1]} -> 0, {U[u_1], U[c_2]} -> 0, {U[u_1], U[c_3]} -> 0,
  {U[u_1], U[u_1]} -> 0, {U[u_1], U[u_2]} -> 0, {U[u_1], U[u_3]} -> 0,
  {U[u_1], U[w_1]} -> 0, {U[u_1], U[w_2]} -> 0, {U[u_1], U[w_3]} -> 0},
{{U[u_2], U[c_1]} -> 0, {U[u_2], U[c_2]} -> 0, {U[u_2], U[c_3]} -> 0,
  {U[u_2], U[u_1]} -> 0, {U[u_2], U[u_2]} -> 0, {U[u_2], U[u_3]} -> 0,
  {U[u_2], U[w_1]} -> 0, {U[u_2], U[w_2]} -> 0, {U[u_2], U[w_3]} -> 0},
{{U[u_3], U[c_1]} -> 0, {U[u_3], U[c_2]} -> 0, {U[u_3], U[c_3]} -> 0,
  {U[u_3], U[u_1]} -> 0, {U[u_3], U[u_2]} -> 0, {U[u_3], U[u_3]} -> 0,
  {U[u_3], U[w_1]} -> 0, {U[u_3], U[w_2]} -> 0, {U[u_3], U[w_3]} -> 0},
{{U[w_1], U[c_1]} -> 0, {U[w_1], U[c_2]} -> 0, {U[w_1], U[c_3]} -> 0,
  {U[w_1], U[u_1]} -> 0, {U[w_1], U[u_2]} -> 0, {U[w_1], U[u_3]} -> 0,
  {U[w_1], U[w_1]} -> 0, {U[w_1], U[w_2]} -> 0, {U[w_1], U[w_3]} -> 0},
{{U[w_2], U[c_1]} -> 0, {U[w_2], U[c_2]} -> 0, {U[w_2], U[c_3]} -> 0,
  {U[w_2], U[u_1]} -> 0, {U[w_2], U[u_2]} -> 0, {U[w_2], U[u_3]} -> 0,
  {U[w_2], U[w_1]} -> 0, {U[w_2], U[w_2]} -> 0, {U[w_2], U[w_3]} -> 0},
{{U[w_3], U[c_1]} -> 0, {U[w_3], U[c_2]} -> 0, {U[w_3], U[c_3]} -> 0,
  {U[w_3], U[u_1]} -> 0, {U[w_3], U[u_2]} -> 0, {U[w_3], U[u_3]} -> 0,
  {U[w_3], U[w_1]} -> 0, {U[w_3], U[w_2]} -> 0, {U[w_3], U[w_3]} -> 0}}

```

Testing bi-local exponential relations in g_0

1. The Yang-Baxter Element.

$a_{1,2}$

$$b_1 U[c_2] + U[u_1, w_2]$$

`UExp[a1,2, 3] // ToDegree[3]`

$$U[] + (b_1 U[c_2] + U[u_1, w_2]) \hbar + \left(\frac{1}{2} b_1^2 U[c_2, c_2] + \frac{1}{2} b_1 U[u_1, w_2] + b_1 U[c_2, u_1, w_2] + \frac{1}{2} U[u_1, u_1, w_2, w_2] \right) \hbar^2 + \left(\frac{1}{6} b_1^2 U[u_1, w_2] + \frac{1}{6} b_1^3 U[c_2, c_2, c_2] + \frac{1}{2} b_1^2 U[c_2, u_1, w_2] + \frac{1}{2} b_1^2 U[c_2, c_2, u_1, w_2] + \frac{1}{2} b_1 U[u_1, u_1, w_2, w_2] + \frac{1}{2} b_1 U[c_2, u_1, u_1, w_2, w_2] + \frac{1}{6} U[u_1, u_1, u_1, w_2, w_2, w_2] \right) \hbar^3 + O[\hbar]^4$$

`With[{n = 10}, Simp[`

$$UExp[a_{1,2}, n] - UExp[b_1 U[c_2], n] ** UExp\left[\frac{e^{b_1} - 1}{b_1} U[u_1, w_2], n\right]$$

`] // ToDegree[n]`

$$O[\hbar]^{11}$$

Aside - testing R3:

`With[{n = 3}, Simp[`

$$(UExp[a_{1,2}, n] ** UExp[a_{1,3}, n] ** UExp[a_{2,3}, n]) - (UExp[a_{2,3}, n] ** UExp[a_{1,3}, n] ** UExp[a_{1,2}, n]) // ToDegree[n]$$

`]]`

$$O[\hbar]^5$$

2. c Relations.

`B[U[c], U[u, w]]`

0

`With[{n = 7}, Collect[`

$$UExp[\beta U@u, n] ** UExp[\alpha U@c, n] - UExp[\alpha U@c, n] ** UExp[e^{-\alpha} \beta U@u, n],$$

`_U, Series[#, {\alpha, 0, n}] &`

`]]`

$$O[\alpha]^8$$

```
With[{n = 7}, Collect[
  UExp[β U@w, n] ** UExp[α U@c, n] - UExp[α U@c, n] ** UExp[eα β U@w, n],
  _U, Series[#, {α, 0, n}] &
]]
O[α]8
```

3. $u, w, e^u, e^w.$

```
With[{n = 10}, Simp[B[U@w, UExp[γ U@u, n]] + b γ UExp[γ U@u, n]]
  b γ11 U[u, u, u, u, u, u, u, u, u, u]
  3 628 800
```

```
With[{n = 7}, Simp[B[U@u, UExp[γ U@w, n]] - b γ UExp[γ U@w, n]]
  b γ8 U[w, w, w, w, w, w, w]
  5040
```

4. M_{uw} Relations.

$$M_{uw}[\gamma, n] := \text{Expand}\left[\sum_{k=0}^n \frac{\gamma^k}{k!} (U@@\text{Table}[u, \{k\}]) ** (U@@\text{Table}[w, \{k\}])\right]$$

```
With[{n = 12}, Simplify[B[U[u], Muw[γ, n]] - b γ U[u] ** Muw[γ, n]]]
  1
  479 001 600 b γ13 U[u, u, u, u, u, u, u, u, u, u, u, u, u, w, w, w, w, w, w, w, w, w, w, w]
```

```
With[{n = 7}, Simplify[B[U[w], Muw[γ, n]] + b γ Muw[γ, n] ** U[w]]]
  1
  5040 b γ8 U[u, u, u, u, u, u, u, w, w, w, w, w, w, w, w]
```

```
With[{n = 7}, Simplify[
  (∂α Muw[γ[α], n]) -  $\frac{\gamma'[\alpha]}{1 - b \gamma[\alpha]} M_{uw}[\gamma[\alpha], n] ** U[u, w]$ 
]]
  (U[u, u, u, u, u, u, u, u, w, w, w, w, w, w, w, w] γ[α]7 γ'[α]) / (5040 (-1 + b γ[α]))
```

5. M_{wu} Relations.

$$M_{wu}[\gamma, n] := \text{Expand}\left[\sum_{k=0}^n \frac{\gamma^k}{k!} (U@@\text{Table}[w, \{k\}]) ** (U@@\text{Table}[u, \{k\}])\right]$$

```
With[{n = 7},
  {t1 = Simplify[B[U[u], Mwu[δ, n]] - b δ Mwu[δ, n] ** U[u]], t1 // ToDegree[n]}]
  {  $\frac{1}{5040} b \delta^8 (40 320 b^7 U[u] - 141 120 b^6 U[u, u, w] + 141 120 b^5 U[u, u, u, w, w] -$ 
    58 800 b4 U[u, u, u, u, w, w, w] + 11 760 b3 U[u, u, u, u, u, w, w, w, w] - 1176 b2
    U[u, u, u, u, u, u, w, w, w, w, w, w] + 56 b U[u, u, u, u, u, u, u, w, w, w, w, w, w] -
    U[u, u, u, u, u, u, u, u, w, w, w, w, w, w, w, w]), O[ħ]9 }
```

```
With[{n = 7}, {t1 = Simp[B[U[w], Mwu[δ, n]] + b δ U[w] ** Mwu[δ, n]], t1 // ToDegree[n]]]
{-8 b^8 δ^8 U[w] + 28 b^7 δ^8 U[u, w, w] - 28 b^6 δ^8 U[u, u, w, w, w] +  $\frac{35}{3}$  b^5 δ^8 U[u, u, u, w, w, w, w] -
 $\frac{7}{3}$  b^4 δ^8 U[u, u, u, u, w, w, w, w, w] +  $\frac{7}{30}$  b^3 δ^8 U[u, u, u, u, u, w, w, w, w, w, w] -
 $\frac{1}{90}$  b^2 δ^8 U[u, u, u, u, u, u, w, w, w, w, w, w, w, w] +  $\frac{1}{5040}$ 
b δ^8 U[u, u, u, u, u, u, u, w, w, w, w, w, w, w, w, w], O[ħ]^8}
```

```
With[{n = 7}, {t1 = Simplify[
  (∂αMwu[α δ, n]) -  $\frac{\delta}{1 + b \alpha \delta}$  Mwu[α δ, n] ** U[w] ** U[u]
], t1 // ToDegree[n]]]
{- $\frac{1}{5040 (1 + b \alpha \delta)}$  α^7 δ^8 (40 320 b^8 U[] - 322 560 b^7 U[u, w] + 564 480 b^6 U[u, u, w, w] -
376 320 b^5 U[u, u, u, w, w, w] + 117 600 b^4 U[u, u, u, u, w, w, w, w] - 18 816 b^3
U[u, u, u, u, u, w, w, w, w, w] + 1568 b^2 U[u, u, u, u, u, u, w, w, w, w, w, w] -
64 b U[u, u, u, u, u, u, u, w, w, w, w, w, w, w, w] +
U[u, u, u, u, u, u, u, u, w, w, w, w, w, w, w, w, w]), O[ħ]^8}
```

```
With[{n = 7}, Simplify[
  (∂αMwu[α δ, n]) -  $\frac{\delta}{1 + b \alpha \delta}$  Mwu[α δ, n] ** (U[u, w] - b U[])
]]
- $\frac{1}{5040 (1 + b \alpha \delta)}$  α^7 δ^8 (40 320 b^8 U[] - 322 560 b^7 U[u, w] + 564 480 b^6 U[u, u, w, w] -
376 320 b^5 U[u, u, u, w, w, w] + 117 600 b^4 U[u, u, u, u, w, w, w, w] -
18 816 b^3 U[u, u, u, u, u, w, w, w, w, w] + 1568 b^2 U[u, u, u, u, u, u, w, w, w, w, w, w] -
64 b U[u, u, u, u, u, u, u, w, w, w, w, w, w, w, w] +
U[u, u, u, u, u, u, u, u, w, w, w, w, w, w, w, w, w])
```

o. M_{wu} and M_{uw} .

$$\text{DSolve}\left[\frac{\delta}{1 + b \alpha \delta} == \frac{\gamma[\alpha]}{1 - b \gamma[\alpha]} \wedge \gamma[0] == 0, \gamma[\alpha], \alpha\right]$$

$$\left\{\left\{\gamma[\alpha] \rightarrow \frac{\alpha \delta}{1 + b \alpha \delta}\right\}\right\}$$

```

With[{n = 7}, {t1 = Simp[
  Mwu[α δ, n] - (Mwu[ $\frac{\alpha \delta}{1 + \alpha b \delta}$ , n] ** ( $\frac{1}{1 + \alpha b \delta}$  U[]))
], t1 // ToDegree[n]}]
{ (1 - b α δ + b2 α2 δ2 - b3 α3 δ3 + b4 α4 δ4 - b5 α5 δ5 + b6 α6 δ6 - b7 α7 δ7 -  $\frac{1}{1 + b \alpha \delta}$ ) U[] +
  (α δ - 2 b α2 δ2 + 3 b2 α3 δ3 - 4 b3 α4 δ4 + 5 b4 α5 δ5 - 6 b5 α6 δ6 + 7 b6 α7 δ7 -  $\frac{\alpha \delta}{(1 + b \alpha \delta)^2}$ ) U[u, w] +
  ( $\frac{\alpha^2 \delta^2}{2}$  -  $\frac{3}{2}$  b α3 δ3 + 3 b2 α4 δ4 - 5 b3 α5 δ5 +  $\frac{15}{2}$  b4 α6 δ6 -  $\frac{21}{2}$  b5 α7 δ7 -  $\frac{\alpha^2 \delta^2}{2 (1 + b \alpha \delta)^3}$ )
  U[u, u, w, w] +
  ( $\frac{\alpha^3 \delta^3}{6}$  -  $\frac{2}{3}$  b α4 δ4 +  $\frac{5}{3}$  b2 α5 δ5 -  $\frac{10}{3}$  b3 α6 δ6 +  $\frac{35}{6}$  b4 α7 δ7 -  $\frac{\alpha^3 \delta^3}{6 (1 + b \alpha \delta)^4}$ ) U[u, u, u, w, w, w] +
  ( $\frac{\alpha^4 \delta^4}{24}$  -  $\frac{5}{24}$  b α5 δ5 +  $\frac{5}{8}$  b2 α6 δ6 -  $\frac{35}{24}$  b3 α7 δ7 -  $\frac{\alpha^4 \delta^4}{24 (1 + b \alpha \delta)^5}$ ) U[u, u, u, u, w, w, w, w] +
  ( $\frac{\alpha^5 \delta^5}{120}$  -  $\frac{1}{20}$  b α6 δ6 +  $\frac{7}{40}$  b2 α7 δ7 -  $\frac{\alpha^5 \delta^5}{120 (1 + b \alpha \delta)^6}$ ) U[u, u, u, u, u, w, w, w, w, w] +
  ( $\frac{\alpha^6 \delta^6}{720}$  -  $\frac{7}{720}$  b α7 δ7 -  $\frac{\alpha^6 \delta^6}{720 (1 + b \alpha \delta)^7}$ ) U[u, u, u, u, u, u, w, w, w, w, w, w] +
  ( $\frac{\alpha^7 \delta^7}{5040}$  -  $\frac{\alpha^7 \delta^7}{5040 (1 + b \alpha \delta)^8}$ ) U[u, u, u, u, u, u, u, w, w, w, w, w, w, w, w], O[ħ]8 }

```

4. Hard core uw relations.

```

With[{n = 3}, Simp[
  UExp[α ħ U@w, n] ** UExp[β U@u, n] - e-b ħ α β UExp[β U@u, n] ** UExp[α ħ U@w, n]
] // ToDegree[n]

```

O[ħ]⁴

```

With[{n = 5, v = (1 + b δ)-1}, Simp[
  UExp[α ħ U@w, n] ** Mwu[δ, n] ** UExp[β U@u, n]
  - v e-b ħ v α β UExp[v β U@u, n] ** Mwu[v δ, n] ** UExp[v α ħ U@w, n]
] // ToDegree[n]

```

O[ħ]⁶