


```

EC = {}; E0 = Errors; gn = hn = 0;
AddRule[ff_, rule_] := (
  Print["As ", e0, ", adding ", rule];
  done = False; EC = EC ∪ {rule}
);
MF[ϕ_, v_] := Module[{t = ϕ, t1}, If[Simplify[t] === 0, 1,
  While[{t1 = Simplify[t /. v → 0]} === 0, t = D[t, v]]; t1
]];
done = False; While[!done, done = True;
  E0 = DeleteCases[Simplify[E0 //. EC], 0] // SortBy[LeafCount];
  Print["Length[E0]==", Length[E0],
    "; Length[EC]==", Length[EC], "; {gn,hn}==", {gn, hn}];
  For[k = 1, k ≤ Length@E0, ++k,
    e1 = Factor[e0 = E0[[k]];
    If[Head[e1] != Times, e2 = e1,
      E0[[k]] = e2 = Select[e1, !FreeQ[#, f[_], _] | g[_] | h[[]] &]];
    If[e2 == 1, Print["Panic at ", e0, "! No solutions."]; Break[]];
    (*Replace[e2, f_k[_,_]>0] := (
      Print["As ", e0, ", adding ", f_k[_,_]>0];
      done=False; EC=EC∪{f_k[_,_]>0};
      Break[]
    );*)
    If[!FreeQ[e2, f[_], _] ∧ (FreeQ[e2, x] ∨ FreeQ[e2, y] ∨ FreeQ[e2, z]),
      {ff} = Cases[e2, f[_], _], {0, ∞}, 1];
      {{sol}} = Solve[e2 == 0, ff];
      rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
      AddRule[ff, rule]; Break[]
    ];
    If[!FreeQ[e2, g[_]] ∧ (FreeQ[e2, y | z] ∨ FreeQ[e2, x | z] ∨ FreeQ[e2, z | y]),
      {gg} = Cases[e2, g[_], ∞, 1];
      {{sol}} = Solve[e2 == 0, gg];
      rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
      AddRule[gg, rule]; Break[]
    ];
    If[Head[e2] === Plus,
      s = List@@Collect[e2, f[_], _], Factor]; s1 = Select[s, FreeQ[f[_], _]];
      sxy = Cases[s, a_. * f[x, y]];
      sxz = Cases[s, a_. * f[x, z]]; syz = Cases[s, a_. * f[y, z]];
      Which[
        sxy == {} ∧ sxz != {} ∧ syz != {}, (
          {ff} = Cases[sxz, a_. * f_k[x, z] :=> f_k[x, z], {1}, 1];
          mf = MF[First@sxz /. f[x, z] → 1, x]; mf *= MF[First@syz /. f[y, z] → 1, y];

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s1 = Plus@@Simplify[s1/mf];
sxz = Plus@@Simplify[sxz/mf]; syz = Plus@@Simplify[syz/mf];
If[FreeQ[sxz, y] & FreeQ[syz, x] & FreeQ[s1, x | y] &
  Simplify[(sxz /. x -> y) + syz == 0],
  {{sol}} = Solve[sxz == g++gn[z], ff];
rule = ((ff /. {x -> x_, y -> y_, z -> z_}) -> (ff /. sol)) /. Rule -> RuleDelayed;
AddRule[ff, rule]; Break[]
]
),
syz == {} & sxy != {} & sxz != {}, (
  {ff} = Cases[sxy, a_. * f_k_[x, y] -> f_k[x, y], {1}, 1];
mf = MF[First@sxy /. f_[x, y] -> 1, y]; mf *= MF[First@sxz /. f_[x, z] -> 1, z];
s1 = Plus@@Simplify[s1/mf];
sxy = Plus@@Simplify[sxy/mf]; sxz = Plus@@Simplify[sxz/mf];
If[FreeQ[sxy, z] & FreeQ[sxz, y] & FreeQ[s1, y | z] &
  Simplify[(sxz /. z -> y) + sxy == 0],
  {{sol}} = Solve[sxy == g++gn[x], ff];
rule = ((ff /. {x -> x_, y -> y_, z -> z_}) -> (ff /. sol)) /. Rule -> RuleDelayed;
AddRule[ff, rule]; Break[]
]
),
sxy != {} & sxz != {} & syz != {}, (
  kk = Union@Cases[e2, a_. * f_k_[x, y] -> k, ∞];
If[Length[kk] == 1,
  {kk} = kk;
  {{sol}} = Solve[e2 == 0, f_kk[x, y]];
  sol = f_kk[x, y] /. sol;
  e3 = D[sol, z] // Factor;
  If[FreeQ[e3, f[_ , _]],
    If[Head[e3] === Times,
      e3 = Select[e3, !FreeQ[#, f_^(0,1)[_ , _] | g[_] | h[_]] &];
      s = Collect[e3, f_^(0,1)[_ , _], Factor];
      s1 = Select[s, FreeQ[f_^(0,1)[_ , _]]];
      pxz = Coefficient[s, f_kk^(0,1)[x, z]];
      pyz = Coefficient[s, f_kk^(0,1)[y, z]];
      mf = MF[pxz, x]; mf *= MF[pyz, y];
      {s1, pxz, pyz} = Simplify[{s1, pxz, pyz}/mf];
      If[FreeQ[pxz, y] & FreeQ[pyz, x] &
        FreeQ[s1, x | y] & Simplify[(pyz /. y -> x) + pxz == 0],
        rule = (f_kk[x_, z_] -> g++gn[z] / pxz + g++gn[x]) /. Rule -> RuleDelayed;
        AddRule[f_kk[x, z], rule]; Break[]
      ]
    ]
  ]
]
]

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]
)
];
If[FreeQ[e2, f[_ , _]] & ! FreeQ[e2, g[_]],
  s = List@@Collect[e2, g[_], Factor]; s1 = Select[s, FreeQ[g[_]]];
  sx = Cases[s, a_. * g_[x]]; sy = Cases[s, a_. * g_[y]]; sz = Cases[s, a_. * g_[z]];
  Which[
    FreeQ[e2, x] & sy != {} & sz != {}, (
      {gg} = Cases[sy, a_. * g_k[y] -> g_k[y], {1}, 1];
      mf = MF[First@sy /. g_[y] -> 1, y]; mf *= MF[First@sz /. g_[z] -> 1, z];
      s1 = Plus@@Simplify[s1/mf];
      sy = Plus@@Simplify[sy/mf]; sz = Plus@@Simplify[sz/mf];
      If[FreeQ[sz, y] & FreeQ[sz, y] & FreeQ[s1, y | z] &
        Simplify[(sz /. z -> y) + sy == 0],
        {{sol}} = Solve[sy == h++hn[], gg];
        rule = ((gg /. {x -> x_, y -> y_, z -> z_}) -> (gg /. sol)) /. Rule -> RuleDelayed;
        AddRule[gg, rule]; Break[]
      )
    ]
  )
] (* Which *)
] (* If *)
] (* If *)
] (* For *)
]; (* While *)
E0 = Union[DeleteCases[Simplify[E0 //. EC], 0]] // SortBy[LeafCount];

Length[E0]==134; Length[EC]==0; {gn,hn}=={0, 0}
As -2 (-1 + e^x) x f24[x, z], adding f24[x_, z_] -> 0
Length[E0]==120; Length[EC]==1; {gn,hn}=={0, 0}
As e^x (-1 + e^y) z f12[x, z], adding f12[x_, z_] -> 0
Length[E0]==112; Length[EC]==2; {gn,hn}=={0, 0}
As -2 (-1 + e^x) x f21[x, z], adding f21[x_, z_] -> 0
Length[E0]==97; Length[EC]==3; {gn,hn}=={0, 0}
As e^x (-1 + e^y) z f16[x, z], adding f16[x_, z_] -> 0
Length[E0]==89; Length[EC]==4; {gn,hn}=={0, 0}
As e^x (-1 + e^y) z f8[x, z], adding f8[x_, z_] -> 0
Length[E0]==81; Length[EC]==5; {gn,hn}=={0, 0}
As -2 e^x (-1 + e^y) y f25[x, y], adding f25[x_, y_] -> 0
Length[E0]==66; Length[EC]==6; {gn,hn}=={0, 0}
As (-1 + e^x) (f10[y, z] - 2 y f18[y, z]), adding f10[y_, z_] -> 2 y f18[y, z]
Length[E0]==64; Length[EC]==7; {gn,hn}=={0, 0}

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As -2 (-1 + e^x) (f3[x, z] - x^2 f18[x, z]), adding f3[x_, z_] :=> x^2 f18[x, z]
Length[E0]==54; Length[EC]==8; {gn,hn}=={0, 0}
As -e^x (-1 + e^y) (f17[x, z] - 2 z f26[x, z]), adding f17[x_, z_] :=> 2 z f26[x, z]
Length[E0]==52; Length[EC]==9; {gn,hn}=={0, 0}
As (-1 + e^y) x f23[x, z] - (-1 + e^x) y f23[y, z], adding f23[x_, z_] :=> - (1 + e^x) g1[z]
                                                                x
Length[E0]==51; Length[EC]==10; {gn,hn}=={1, 0}
As 2 e^x (-1 + e^y) (f5[x, y] - y^2 f26[x, y]), adding f5[x_, y_] :=> y^2 f26[x, y]
Length[E0]==41; Length[EC]==11; {gn,hn}=={1, 0}
As e^{-x-y} (-1 + e^y) z (f11[x, y] - f11[x, z]), adding f11[x_, y_] :=> -g2[x]
Length[E0]==40; Length[EC]==12; {gn,hn}=={2, 0}
As (-1 + e^y) x (f19[x, y] - f19[x, z]), adding f19[x_, y_] :=> -g3[x]
Length[E0]==38; Length[EC]==13; {gn,hn}=={3, 0}
As (-1 + e^x) (-f13[x, z] + f13[y, z] + x f22[x, z] - y f22[y, z])
, adding f13[x_, z_] :=> x f22[x, z] - g4[z]
Length[E0]==36; Length[EC]==14; {gn,hn}=={4, 0}
As -(-1 + e^x) (f4[x, z] - f4[y, z] - x f14[x, z] + y f14[y, z])
, adding f4[x_, z_] :=> x f14[x, z] - g5[z]
Length[E0]==32; Length[EC]==15; {gn,hn}=={5, 0}
As (1 + e^y) (y g4[y] - z g4[z] - g5[y] + g5[z]), adding g4[y_] :=> (g5[y] - h1[])
                                                                y
Length[E0]==30; Length[EC]==16; {gn,hn}=={5, 1}
As e^x (-1 + e^y) (f14[x, y] - f14[x, z] - y f22[x, y] + z f22[x, z])
, adding f14[x_, y_] :=> y f22[x, y] - g6[x]
Length[E0]==24; Length[EC]==17; {gn,hn}=={6, 1}
As -(-1 + e^x) (f1[x, z] - f1[y, z] - x f6[x, z] + y f6[y, z] - x^2 f18[x, z] + y^2 f18[y, z])
, adding f1[x_, z_] :=> x f6[x, z] + x^2 f18[x, z] - g7[z]
Length[E0]==22; Length[EC]==18; {gn,hn}=={7, 1}
As e^x (-1 + e^y) (f2[x, y] - f2[x, z] - y f9[x, y] + z f9[x, z] - y^2 f26[x, y] + z^2 f26[x, z])
, adding f2[x_, y_] :=> y f9[x, y] + y^2 f26[x, y] - g8[x]
Length[E0]==20; Length[EC]==19; {gn,hn}=={8, 1}
As -(-1 + e^y) x f7[x, y] + (-1 + e^y) x f7[x, z] - (-1 + e^x) (y f7[y, z] - y g2[y] + e^y g7[y] + g8[y])
, adding f7[x_, z_] :=> (1 - e^x) g9[z]
                                                                x
                                                                + g10[x]
Length[E0]==20; Length[EC]==20; {gn,hn}=={10, 1}
As (-1 + e^x) (y g2[y] - e^y g7[y] - g8[y] - g9[y] + e^y g9[y] - y g10[y])
, adding g2[y_] :=> (e^y g7[y] + g8[y] + g9[y] - e^y g9[y] + y g10[y])
                                                                y
Length[E0]==18; Length[EC]==21; {gn,hn}=={10, 1}

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As $-1 + e^{x+y} - \frac{1}{x} + \frac{e^x}{x} + \frac{e^y}{x} - \frac{e^{x+y}}{x} - \frac{1}{y} + \frac{e^x}{y} + \frac{e^y}{y} - \frac{e^{x+y}}{y} - (-1 + e^y) x f_{15}[x, y] + e^y g_5[y] - e^{x+y} g_5[y] +$
 $y g_6[y] - e^x y g_6[y] + (-1 + e^x) (e^y g_7[y] + g_8[y] + g_9[y] - e^y g_9[y] + y g_{10}[y]) + h_1[] - e^x h_1[]$
 , adding $f_{15}[x_, y_] := \frac{1}{(-1 + e^y) x^2 y}$

$(-x + e^x x + e^y x - e^{x+y} x - y + e^x y + e^y y - e^{x+y} y - x y + e^{x+y} x y + e^y x y g_5[y] - e^{x+y} x y g_5[y] +$
 $x y^2 g_6[y] - e^x x y^2 g_6[y] - e^y x y g_7[y] + e^{x+y} x y g_7[y] - x y g_8[y] + e^x x y g_8[y] - x y g_9[y] +$
 $e^x x y g_9[y] + e^y x y g_9[y] - e^{x+y} x y g_9[y] - x y^2 g_{10}[y] + e^x x y^2 g_{10}[y] + x y h_1[] - e^x x y h_1[])$

Length[E0]==12; Length[EC]==22; {gn,hn}=={10, 1}

As $\frac{1}{y^2} (-1 + e^x)$

$(2 - 2 e^y + y + e^y y + 2 (-1 + e^y) y^2 g_1[y] + 2 y^3 g_3[y] - 2 e^y y g_5[y] - 2 y^2 g_6[y] - y h_1[] + e^y y h_1[])$
 , adding $g_1[y_] := \frac{1}{2 (-1 + e^y) y^2}$

$(-2 + 2 e^y - y - e^y y - 2 y^3 g_3[y] + 2 e^y y g_5[y] + 2 y^2 g_6[y] + y h_1[] - e^y y h_1[])$

Length[E0]==8; Length[EC]==23; {gn,hn}=={10, 1}

As $-\frac{1}{x^3} e^{-x} y (-2 + 2 e^x + 2 e^y - 2 e^{x+y} - x - e^x x + e^y x + e^{x+y} x - 2 (-1 + e^y) x^3 f_{20}[x, y] +$

$x^3 g_3[x] - e^x x^3 g_3[x] - e^y x^3 g_3[x] + e^{x+y} x^3 g_3[x] + x y^2 g_3[y] - 2 e^x x y^2 g_3[y] +$
 $e^{2x} x y^2 g_3[y] - (-1 + e^x) (-1 + e^y) x (e^x g_7[x] + g_8[x] + g_9[x] - e^x g_9[x] + x g_{10}[x]) -$
 $x (e^y g_7[y] + g_8[y] + g_9[y] - e^y g_9[y] + y g_{10}[y]) +$
 $2 e^x x (e^y g_7[y] + g_8[y] + g_9[y] - e^y g_9[y] + y g_{10}[y]) -$
 $e^{2x} x (e^y g_7[y] + g_8[y] + g_9[y] - e^y g_9[y] + y g_{10}[y]) + x h_1[] - 2 e^x x h_1[] + e^{2x} x h_1[])$

, adding $f_{20}[x_, y_] := \frac{1}{2 (-1 + e^y) x^3} (-2 + 2 e^x + 2 e^y - 2 e^{x+y} - x - e^x x + e^y x + e^{x+y} x +$

$x^3 g_3[x] - e^x x^3 g_3[x] - e^y x^3 g_3[x] + e^{x+y} x^3 g_3[x] + x y^2 g_3[y] - 2 e^x x y^2 g_3[y] + e^{2x} x y^2 g_3[y] -$
 $e^x x g_7[x] + e^{2x} x g_7[x] + e^{x+y} x g_7[x] - e^{2x+y} x g_7[x] - e^y x g_7[y] + 2 e^{x+y} x g_7[y] - e^{2x+y} x g_7[y] -$
 $x g_8[x] + e^x x g_8[x] + e^y x g_8[x] - e^{x+y} x g_8[x] - x g_8[y] + 2 e^x x g_8[y] - e^{2x} x g_8[y] - x g_9[x] +$
 $2 e^x x g_9[x] - e^{2x} x g_9[x] + e^y x g_9[x] - 2 e^{x+y} x g_9[x] + e^{2x+y} x g_9[x] - x g_9[y] + 2 e^x x g_9[y] -$
 $e^{2x} x g_9[y] + e^y x g_9[y] - 2 e^{x+y} x g_9[y] + e^{2x+y} x g_9[y] - x^2 g_{10}[x] + e^x x^2 g_{10}[x] + e^y x^2 g_{10}[x] -$
 $e^{x+y} x^2 g_{10}[x] - x y g_{10}[y] + 2 e^x x y g_{10}[y] - e^{2x} x y g_{10}[y] + x h_1[] - 2 e^x x h_1[] + e^{2x} x h_1[])$

Length[E0]==0; Length[EC]==24; {gn,hn}=={10, 1}

EC

$$\begin{aligned}
&\{f_1[x_, z_] \Rightarrow x f_6[x, z] + x^2 f_{18}[x, z] - g_7[z], f_2[x_, y_] \Rightarrow y f_9[x, y] + y^2 f_{26}[x, y] - g_8[x], \\
&f_3[x_, z_] \Rightarrow x^2 f_{18}[x, z], f_4[x_, z_] \Rightarrow x f_{14}[x, z] - g_5[z], \\
&f_5[x_, y_] \Rightarrow y^2 f_{26}[x, y], f_7[x_, z_] \Rightarrow \frac{(1 - e^x) g_9[z]}{x} + g_{10}[x], f_8[x_, z_] \Rightarrow 0, \\
&f_{10}[y_, z_] \Rightarrow 2 y f_{18}[y, z], f_{11}[x_, y_] \Rightarrow -g_2[x], f_{12}[x_, z_] \Rightarrow 0, \\
&f_{13}[x_, z_] \Rightarrow x f_{22}[x, z] - g_4[z], f_{14}[x_, y_] \Rightarrow y f_{22}[x, y] - g_6[x], \\
&f_{15}[x_, y_] \Rightarrow \frac{1}{(-1 + e^y) x^2 y} \left(-x + e^x x + e^y x - e^{x+y} x - y + e^x y + e^y y - e^{x+y} y - \right. \\
&\quad x y + e^{x+y} x y + e^y x y g_5[y] - e^{x+y} x y g_5[y] + x y^2 g_6[y] - e^x x y^2 g_6[y] - \\
&\quad e^y x y g_7[y] + e^{x+y} x y g_7[y] - x y g_8[y] + e^x x y g_8[y] - x y g_9[y] + e^x x y g_9[y] + \\
&\quad \left. e^y x y g_9[y] - e^{x+y} x y g_9[y] - x y^2 g_{10}[y] + e^x x y^2 g_{10}[y] + x y h_1[] - e^x x y h_1[] \right), \\
&f_{16}[x_, z_] \Rightarrow 0, f_{17}[x_, z_] \Rightarrow 2 z f_{26}[x, z], f_{19}[x_, y_] \Rightarrow -g_3[x], \\
&f_{20}[x_, y_] \Rightarrow \frac{1}{2(-1 + e^y) x^3} \left(-2 + 2 e^x + 2 e^y - 2 e^{x+y} - x - e^x x + e^y x + e^{x+y} x + \right. \\
&\quad x^3 g_3[x] - e^x x^3 g_3[x] - e^y x^3 g_3[x] + e^{x+y} x^3 g_3[x] + x y^2 g_3[y] - 2 e^x x y^2 g_3[y] + \\
&\quad e^2 x x y^2 g_3[y] - e^x x g_7[x] + e^2 x x g_7[x] + e^{x+y} x g_7[x] - e^{2x+y} x g_7[x] - e^y x g_7[y] + \\
&\quad 2 e^{x+y} x g_7[y] - e^{2x+y} x g_7[y] - x g_8[x] + e^x x g_8[x] + e^y x g_8[x] - e^{x+y} x g_8[x] - \\
&\quad x g_8[y] + 2 e^x x g_8[y] - e^{2x} x g_8[y] - x g_9[x] + 2 e^x x g_9[x] - e^{2x} x g_9[x] + e^y x g_9[x] - \\
&\quad 2 e^{x+y} x g_9[x] + e^{2x+y} x g_9[x] - x g_9[y] + 2 e^x x g_9[y] - e^{2x} x g_9[y] + e^y x g_9[y] - \\
&\quad 2 e^{x+y} x g_9[y] + e^{2x+y} x g_9[y] - x^2 g_{10}[x] + e^x x^2 g_{10}[x] + e^y x^2 g_{10}[x] - e^{x+y} x^2 g_{10}[x] - \\
&\quad \left. x y g_{10}[y] + 2 e^x x y g_{10}[y] - e^{2x} x y g_{10}[y] + x h_1[] - 2 e^x x h_1[] + e^{2x} x h_1[] \right), \\
&f_{21}[x_, z_] \Rightarrow 0, f_{23}[x_, z_] \Rightarrow -\frac{(1 + e^x) g_1[z]}{x}, f_{24}[x_, z_] \Rightarrow 0, \\
&f_{25}[x_, y_] \Rightarrow 0, \\
&g_1[y_] \Rightarrow \frac{1}{2(-1 + e^y) y^2} \\
&\quad \left(-2 + 2 e^y - y - e^y y - 2 y^3 g_3[y] + 2 e^y y g_5[y] + 2 y^2 g_6[y] + y h_1[] - e^y y h_1[] \right), \\
&g_2[y_] \Rightarrow \frac{e^y g_7[y] + g_8[y] + g_9[y] - e^y g_9[y] + y g_{10}[y]}{y}, \\
&g_4[y_] \Rightarrow \frac{g_5[y] - h_1[]}{y} \}
\end{aligned}$$

E0 // Simplify // Column**Simplify[Errors // EC]**

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{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```