

Pensieve header: Finding the most general R using low algebra.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\V1\\OneCo-1606"];
<< OneCo.m
```

Finding R

```
bas = Cases[LBasis[{j, k}], e * _]
```

```
{e c_j, e c_k, e u_j, e u_k, e w_j, e w_k, e c_j^2, e c_j c_k, e c_k^2, e c_j u_j, e c_k u_j, e c_j u_k, e c_k u_k,
 e c_j w_j, e c_k w_j, e u_j w_j, e u_k w_j, e c_j w_k, e c_k w_k, e u_j w_k, e u_k w_k, e c_j u_j w_j, e c_k u_j w_j,
 e u_j^2 w_j, e c_j u_k w_j, e c_k u_k w_j, e u_j u_k w_j, e u_k^2 w_j, e u_j w_j^2, e u_k w_k^2, e c_j u_j w_k, e c_k u_j w_k,
 e u_j^2 w_k, e c_j u_k w_k, e c_k u_k w_k, e u_j u_k w_k, e u_k^2 w_k, e u_j w_j w_k, e u_k w_j w_k, e u_j w_k^2, e u_k w_k^2,
 e u_j^2 w_j^2, e u_j u_k w_j^2, e u_k^2 w_j^2, e u_j^2 w_j w_k, e u_j u_k w_j w_k, e u_k^2 w_j w_k, e u_j^2 w_k^2, e u_j u_k w_k^2, e u_k^2 w_k^2}
```

```
Sum[f_i[b_j, b_k] bas[[i]], {i, Length@bas}]
```

```
e c_j f_1[b_j, b_k] + e c_k f_2[b_j, b_k] + e u_j f_3[b_j, b_k] + e u_k f_4[b_j, b_k] + e w_j f_5[b_j, b_k] +
 e w_k f_6[b_j, b_k] + e c_j^2 f_7[b_j, b_k] + e c_j c_k f_8[b_j, b_k] + e c_k^2 f_9[b_j, b_k] + e c_j u_j f_10[b_j, b_k] +
 e c_k u_j f_11[b_j, b_k] + e c_j u_k f_12[b_j, b_k] + e c_k u_k f_13[b_j, b_k] + e c_j w_j f_14[b_j, b_k] +
 e c_k w_j f_15[b_j, b_k] + e u_j w_j f_16[b_j, b_k] + e u_k w_j f_17[b_j, b_k] + e c_j w_k f_18[b_j, b_k] +
 e c_k w_k f_19[b_j, b_k] + e u_j w_k f_20[b_j, b_k] + e u_k w_k f_21[b_j, b_k] + e c_j u_j w_j f_22[b_j, b_k] +
 e c_k u_j w_j f_23[b_j, b_k] + e u_j^2 w_j f_24[b_j, b_k] + e c_j u_k w_j f_25[b_j, b_k] + e c_k u_k w_j f_26[b_j, b_k] +
 e u_j u_k w_j f_27[b_j, b_k] + e u_k^2 w_j f_28[b_j, b_k] + e u_j w_j^2 f_29[b_j, b_k] + e u_k w_k^2 f_30[b_j, b_k] +
 e c_j u_j w_k f_31[b_j, b_k] + e c_k u_k w_k f_32[b_j, b_k] + e u_j^2 w_k f_33[b_j, b_k] + e c_j u_k w_k f_34[b_j, b_k] +
 e c_k u_k w_k f_35[b_j, b_k] + e u_j u_k w_k f_36[b_j, b_k] + e u_k^2 w_k f_37[b_j, b_k] + e u_j w_j w_k f_38[b_j, b_k] +
 e u_k w_j w_k f_39[b_j, b_k] + e u_j w_k^2 f_40[b_j, b_k] + e u_k w_k^2 f_41[b_j, b_k] + e u_j^2 w_j^2 f_42[b_j, b_k] +
 e u_j u_k w_j^2 f_43[b_j, b_k] + e u_k^2 w_j^2 f_44[b_j, b_k] + e u_j^2 w_j w_k f_45[b_j, b_k] + e u_j u_k w_j w_k f_46[b_j, b_k] +
 e u_k^2 w_j w_k f_47[b_j, b_k] + e u_j^2 w_k^2 f_48[b_j, b_k] + e u_j u_k w_k^2 f_49[b_j, b_k] + e u_k^2 w_k^2 f_50[b_j, b_k]
```

```
 $\rho_0[j_, k_] := e c_j f_1[b_j, b_k] + e c_k f_2[b_j, b_k] + e u_j f_3[b_j, b_k] + e u_k f_4[b_j, b_k] +
 e w_j f_5[b_j, b_k] + e w_k f_6[b_j, b_k] + e c_j^2 f_7[b_j, b_k] + e c_j c_k f_8[b_j, b_k] +
 e c_k^2 f_9[b_j, b_k] + e c_j u_j f_10[b_j, b_k] + e c_k u_j f_11[b_j, b_k] + e c_j u_k f_12[b_j, b_k] +
 e c_k u_k f_13[b_j, b_k] + e c_j w_j f_14[b_j, b_k] + e c_k w_j f_15[b_j, b_k] + e u_j w_j f_16[b_j, b_k] +
 e u_k w_j f_17[b_j, b_k] + e c_j w_k f_18[b_j, b_k] + e c_k w_k f_19[b_j, b_k] + e u_j w_k f_20[b_j, b_k] +
 e u_k w_k f_21[b_j, b_k] + e c_j u_j w_j f_22[b_j, b_k] + e c_k u_j w_j f_23[b_j, b_k] + e u_j^2 w_j f_24[b_j, b_k] +
 e c_j u_k w_j f_25[b_j, b_k] + e c_k u_k w_j f_26[b_j, b_k] + e u_j u_k w_j f_27[b_j, b_k] +
 e u_k^2 w_j f_28[b_j, b_k] + e u_j w_j^2 f_29[b_j, b_k] + e u_k w_k^2 f_30[b_j, b_k] + e c_j u_j w_k f_31[b_j, b_k] +
 e c_k u_k w_k f_32[b_j, b_k] + e u_j^2 w_k f_33[b_j, b_k] + e c_j u_k w_k f_34[b_j, b_k] + e c_k u_k w_k f_35[b_j, b_k] +
 e u_j u_k w_k f_36[b_j, b_k] + e u_k^2 w_k f_37[b_j, b_k] + e u_j w_j w_k f_38[b_j, b_k] + e u_k w_j w_k f_39[b_j, b_k] +
 e u_j w_k^2 f_40[b_j, b_k] + e u_k w_k^2 f_41[b_j, b_k] + e u_j^2 w_j^2 f_42[b_j, b_k] + e u_j u_k w_j^2 f_43[b_j, b_k] +
 e u_k^2 w_j^2 f_44[b_j, b_k] + e u_j^2 w_j w_k f_45[b_j, b_k] + e u_j u_k w_j w_k f_46[b_j, b_k] +
 e u_k^2 w_j w_k f_47[b_j, b_k] + e u_j^2 w_k^2 f_48[b_j, b_k] + e u_j u_k w_k^2 f_49[b_j, b_k] + e u_k^2 w_k^2 f_50[b_j, b_k]$ 
```

```
R[j_, k_][ $\mathcal{E}_-$ ] :=  $\mathcal{E}$  // R_0[j, k] // (# + b[\rho_0[j, k], #]) &
```

$c_1 // R[1, 2] // R[1, 3] // R[2, 3]$

$$C_1 - \frac{e^{b_1} (-1+e^{-b_1}) u_1 w_2}{b_1} + \frac{e^{b_1} (-1+e^{-b_1}+b_1) u_1 w_2}{b_1^2} - \frac{e^{b_1} (-1+e^{-b_1}+b_1) c_1 u_1 w_2}{b_1^2} + \frac{e^{b_1} (-1+e^{-b_1}+b_1) c_2 u_1 w_2}{b_1^2} +$$

$$\dots 1038 \dots + \frac{2 e^{b_1+b_2} (-1+e^{-b_1}) (-1+e^{-b_2}) \epsilon b_3^2 u_1 u_2 w_3^2 f_{50}[b_1, b_3]}{b_1 b_2} - \frac{2 e^{b_1+b_2} (-1+e^{-b_1}) \epsilon b_3 u_1 u_3 w_3^2 f_{50}[b_1, b_3]}{b_1} -$$

$$\frac{2 e^{b_1+b_2} (-1+e^{-b_1}) \epsilon b_3 u_1 u_3 w_3^2 f_{50}[b_2, b_3]}{b_1} - \frac{2 e^{b_1+b_2} (-1+e^{-b_1}) (-1+e^{-b_2}) \epsilon b_3 u_1 u_3 w_3^2 f_{50}[b_2, b_3]}{b_1}$$

large output | **show less** | **show more** | **show all** | **set size limit...**

```

R3[ε_] :=
  Expand[(ε // R[1, 2] // R[1, 3] // R[2, 3]) - (ε // R[2, 3] // R[1, 3] // R[1, 2])]

R3[u1] /. ε → 0
0

R3[u2] /. ε → 0
0

R3[u3] /. ε → 0
0

Table[R3[yi] /. ε → 0, {y, {c, u, w}}, {i, 3}]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
    
```

R3 on u_1

$R3u1 = R3[u_1];$

The following were obtained by watching `Dynamic[R3u1 // Simp]` and repeatedly simplifying it:

Factor@Coefficient[R3u1, ε c1 u2]

$$- \frac{e^{-b_2} (-1 + e^{b_2}) b_1 b_3 f_{25}[b_1, b_3]}{b_2}$$

$f_{25}[x_, y_] := 0;$

Factor@Coefficient[R3u1, ε c3 u2]

$$- \frac{e^{-b_2} (-1 + e^{b_2}) b_1 b_3 f_{26}[b_1, b_3]}{b_2}$$

$f_{26}[x_, y_] := 0;$

Factor@Coefficient[R3u1, ε u2 u3 w1]

$$- \frac{4 e^{-2 b_2} (-1 + e^{b_2}) b_1 b_3 f_{44}[b_1, b_3]}{b_2}$$

$f_{44}[x_, y_] := 0;$

```

Factor@Coefficient[R3u1, e u1 u3 w2]
-2 (-1 + eb1) b1 f43[b1, b3]

f43[x_, y_] := 0;

Factor@Coefficient[R3u1, e u3 w2]
-2 (-1 + eb1) b1 f30[b1, b3]

f30[x_, y_] := 0;

Factor@Coefficient[R3u1, e u1 u3]
-e-b2 (-1 + eb2) (f12[b1, b3] - b1 f27[b1, b3])

f12[x_, y_] := x f27[x, y];

Factor@Coefficient[R3u1, e u2 u3 w3]
-  $\frac{2 e^{-b_2} (-1 + e^{b_2}) b_1 b_3 f_{47}[b_1, b_3]}{b_2}$ 

f47[x_, y_] := 0;

Factor@Coefficient[R3u1, e u22]
-  $\frac{e^{-2 b_2} (-1 + e^{b_2})^2 b_1 b_3^2 f_{28}[b_1, b_3]}{b_2^2}$ 

f28[x_, y_] := 0;

Factor@Coefficient[R3u1, e u3]
e-b2 (-1 + eb2) b1 f17[b1, b3]

f17[x_, y_] := 0;

Factor@Coefficient[R3u1, e u12 w2]
-  $\frac{2 (-1 + e^{b_1}) (f_7[b_1, b_3] - b_1 f_{22}[b_1, b_3] + b_1^2 f_{42}[b_1, b_3])}{b_1}$ 

f7[x_, y_] := x f22[x, y] - x2 f42[x, y]

Factor@Coefficient[R3u1, e u1 w2]
2 (-1 + eb1) (f14[b1, b3] - b1 f29[b1, b3])

f14[x_, y_] := x f29[x, y];

Factor@Coefficient[R3u1, e u12 w3]
- (-1 + eb2) (f31[b1, b2] - f31[b1, b3] - b1 f45[b1, b2] + b1 f45[b1, b3])

f31[x_, y_] := g1[x] + x f45[x, y]

```

Factor[Coefficient[R3u1, e u₂ w₃] /. u₁ → 0]

$$\frac{1}{b_2} (-1 + e^{b_2}) b_1 (-f_{15}[b_1, b_2] + f_{15}[b_1, b_3] + b_2 f_{39}[b_1, b_2] - b_3 f_{39}[b_1, b_3])$$

f₁₅[x_, y_] := g₂[x] + y f₃₉[x, y]

Factor[Coefficient[R3u1, e u₁ w₃] /. u₂ → 0]

$$-(-1 + e^{b_2}) (f_{18}[b_1, b_2] - f_{18}[b_1, b_3] - b_1 f_{38}[b_1, b_2] + b_1 f_{38}[b_1, b_3])$$

f₁₈[x_, y_] := g₃[x] + x f₃₈[x, y]

Factor[Coefficient[R3u1, e u₁ u₂ w₃]]

$$\frac{1}{b_2} (-1 + e^{b_2}) (f_8[b_1, b_2] - f_8[b_1, b_3] - b_1 f_{23}[b_1, b_2] + b_1 f_{23}[b_1, b_3] - b_2 f_{34}[b_1, b_2] + b_3 f_{34}[b_1, b_3] + b_1 b_2 f_{46}[b_1, b_2] - b_1 b_3 f_{46}[b_1, b_3])$$

f₈[x_, y_] := g₄[x] + x f₂₃[x, y] + y f₃₄[x, y] - x y f₄₆[x, y]

R3 on w₁

R3w1 = R3[w₁] /. {b₁ → x, b₂ → y, b₃ → z};

The following were obtained by watching Dynamic[R3w1 // Simp] and repeatedly simplifying it:

tt1 = Factor[Coefficient[R3w1, e u₁ w₂ w₃]]

$$-\frac{1}{x^2} (-1 + e^y) (-2 + 2 e^{2x} - x - 2 e^x x - e^{2x} x - 2 x^2 f_{32}[x, y] + 2 e^x x^2 f_{32}[x, y] + 4 x^3 f_{48}[x, y] + 2 x^2 y f_{49}[x, y] - 2 e^x x^2 y f_{49}[x, y] + 2 x^2 g_1[x] - 2 e^x x^2 g_1[x] - x g_4[x] + 2 e^x x g_4[x] - e^{2x} x g_4[x])$$

Solve[tt1 == 0, f₄₈[b₁, b₂]] /. {b₁ → x, b₂ → y, b₃ → z}

{{}}

f₄₈[x_, y_] :=

$$\frac{1}{4 x^3} (2 - 2 e^{2x} + x + 2 e^x x + e^{2x} x + 2 x^2 f_{32}[x, y] - 2 e^x x^2 f_{32}[x, y] - 2 x^2 y f_{49}[x, y] + 2 e^x x^2 y f_{49}[x, y] - 2 x^2 g_1[x] + 2 e^x x^2 g_1[x] + x g_4[x] - 2 e^x x g_4[x] + e^{2x} x g_4[x])$$

tt1 = Simplify[Coefficient[R3w1, e u₂ w₂] /. w₃ → 0]

$$(-1 + e^x) (f_{10}[y, z] - 2 y f_{24}[y, z] - z f_{27}[y, z])$$

f₁₀[x_, y_] := 2 x f₂₄[x, y] + y f₂₇[x, y]

tt1 = Simplify[Coefficient[R3w1, e u₂ w₂²] /. w₃ → 0]

$$(-1 + e^x) (f_{22}[y, z] - 2 y f_{42}[y, z])$$

f₂₂[x_, y_] := 2 x f₄₂[x, y]

```
tt1 = Simplify[Coefficient[R3w1, e c3] /. {w3 -> 0, w2 -> 0}]
- (-1 + e^x) (y f11[y, z] + z f13[y, z])
```

$$f_{11}[y_, z_] := \frac{-z}{y} f_{13}[y, z]$$

```
tt1 = Simplify[Coefficient[R3w1, e c3] /. {w3 -> 0, w2 -> 0}]
0
```

```
tt1 = Simplify[Coefficient[R3w1, e] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
- (-1 + e^x) (y f3[y, z] + z f4[y, z] + z f13[y, z] + 2 y^2 f24[y, z] + y z f27[y, z])
```

```
Solve[tt1 == 0, f3[y, z]]
```

$$\left\{ \left\{ f_3[y, z] \rightarrow \frac{-z f_4[y, z] - z f_{13}[y, z] - 2 y^2 f_{24}[y, z] - y z f_{27}[y, z]}{y} \right\} \right\}$$

$$f_3[y_, z_] := \frac{-z f_4[y, z] - z f_{13}[y, z] - 2 y^2 f_{24}[y, z] - y z f_{27}[y, z]}{y}$$

```
tt1 = Simplify[Coefficient[R3w1, e u3 w3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
(-1 + e^x) (f13[y, z] - y f36[y, z] - 2 z f37[y, z])
```

```
Solve[tt1 == 0, f13[y, z]]
```

$$\left\{ \left\{ f_{13}[y, z] \rightarrow y f_{36}[y, z] + 2 z f_{37}[y, z] \right\} \right\}$$

$$f_{13}[y_, z_] := y f_{36}[y, z] + 2 z f_{37}[y, z]$$

```
tt1 = Simplify[Coefficient[R3w1, e c2] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
-2 (-1 + e^x) y (y f24[y, z] + z f27[y, z])
```

```
Solve[tt1 == 0, f24[y, z]]
```

$$\left\{ \left\{ f_{24}[y, z] \rightarrow -\frac{z f_{27}[y, z]}{y} \right\} \right\}$$

$$f_{24}[y_, z_] := -\frac{z f_{27}[y, z]}{y}$$

```
tt1 = Simplify[Coefficient[R3w1, e c2] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
0
```

Not finished!

R3 on w3

```
R3w3 = R3[w3] /. {b1 -> x, b2 -> y, b3 -> z};
```

The following were obtained by watching `Dynamic[R3w3 // Simp]` and repeatedly simplifying it:

```
tt1 = Simplify[Coefficient[R3w3, e c1] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
```

```
-e^x (-1 + e^y) x z f27[x, z]
```

```
f27[x_, z_] := 0
```

```
tt1 = Simplify[Coefficient[R3w3, e w2 w3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
```

```
-e^{x+y} (-1 + e^x) (g2[x] - g2[y])
```

```
g2[x_] := 0
```

```
tt1 = Simplify[Coefficient[R3w3, e u2 w3^2] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
```

```

$$\frac{2 e^{x+y} (-1 + e^y) (f_9[x, z] + z (-f_{35}[x, z] + z f_{50}[x, z]))}{y}$$

```

```
Solve[tt1 == 0, f9[x, z]]
```

```
{{f9[x, z] -> -z (-f35[x, z] + z f50[x, z])}}
```

```
f9[x_, z_] := -z (-f35[x, z] + z f50[x, z])
```

```
tt1 = Simplify[Coefficient[R3w3, e u1 w2 w3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
```

```

$$\frac{e^{x+y} (-1 + e^x) (g4[x] - g4[y])}{x}$$

```

```
g4[x_] := 0
```

Not finished!

R3 on u_2

```
R3u2 = R3[u2];
```

The following were obtained by watching `Dynamic[R3u2 // Simp]` and repeatedly simplifying it:

```
Factor[Coefficient[R3u2, e u3 w3] /. u1 -> 0]
```

```
-e^{-b1} (-1 + e^{b1}) b2 (f39[b1, b3] - f39[b2, b3])
```

```
f39[x_, y_] := g5[y]
```

```
Factor[Coefficient[R3u2, e c1]]
```

```
-e^{-b1} (-1 + e^{b1}) b1 b2 f29[b1, b3]
```

```
f29[x_, y_] := 0
```

```
Factor[Coefficient[R3u2, e u1 u3 w3]]
```

```

$$-\frac{1}{b_1} e^{-b_1} (-1 + e^{b_1}) b_2 (-f_{34}[b_1, b_3] + f_{34}[b_2, b_3] + b_1 f_{46}[b_1, b_3] - b_2 f_{46}[b_2, b_3])$$

```

```
f34[x_, y_] := g6[y] + x f46[x, y]
```

Factor[Coefficient[R3u2, e u₁ w₃] /. u₂ → 0]

$$-\frac{1}{b_1} e^{-b_1} b_2 (2 b_1 f_{40}[b_1, b_2] - 2 e^{b_2} b_1 f_{40}[b_1, b_2] + g_3[b_1] - e^{b_1} g_3[b_1] - e^{b_2} g_3[b_1] + e^{b_1+b_2} g_3[b_1] - e^{b_1} g_3[b_2] + e^{2 b_1} g_3[b_2])$$

Solve[- $\frac{1}{b_1} e^{-b_1} b_2 (2 b_1 f_{40}[b_1, b_2] - 2 e^{b_2} b_1 f_{40}[b_1, b_2] + g_3[b_1] - e^{b_1} g_3[b_1] - e^{b_2} g_3[b_1] + e^{b_1+b_2} g_3[b_1] - e^{b_1} g_3[b_2] + e^{2 b_1} g_3[b_2]) = 0$, f₄₀[b₁, b₂]] /. {b₁ → x, b₂ → y}

$$\left\{ \left\{ f_{40}[x, y] \rightarrow \frac{(-1 + e^x) (-g_3[x] + e^y g_3[x] + e^x g_3[y])}{2 (-1 + e^y) x} \right\} \right\}$$

$$f_{40}[x_, y_] := \frac{(-1 + e^x) (-g_3[x] + e^y g_3[x] + e^x g_3[y])}{2 (-1 + e^y) x}$$

Factor[Coefficient[R3u2, e u₁² w₃]]

$$\frac{1}{2 b_1^2} e^{-b_1} (-1 + e^{b_1}) b_2 (-2 + 2 e^{b_1} + 2 e^{b_2} - 2 e^{b_1+b_2} - b_1 - e^{b_1} b_1 + e^{b_2} b_1 + e^{b_1+b_2} b_1 + 2 b_1^2 f_{32}[b_1, b_2] - 2 e^{b_2} b_1^2 f_{32}[b_1, b_2] - 2 b_1^2 b_2 f_{49}[b_1, b_2] + 2 e^{b_2} b_1^2 b_2 f_{49}[b_1, b_2] + 2 b_1 b_2 g_1[b_2] - 2 e^{b_1} b_1 b_2 g_1[b_2])$$

Solve[- $\frac{1}{2 b_1^2} e^{-b_1} (-1 + e^{b_1}) b_2 (-2 + 2 e^{b_1} + 2 e^{b_2} - 2 e^{b_1+b_2} - b_1 - e^{b_1} b_1 + e^{b_2} b_1 + e^{b_1+b_2} b_1 + 2 b_1^2 f_{32}[b_1, b_2] - 2 e^{b_2} b_1^2 f_{32}[b_1, b_2] - 2 b_1^2 b_2 f_{49}[b_1, b_2] + 2 e^{b_2} b_1^2 b_2 f_{49}[b_1, b_2] + 2 b_1 b_2 g_1[b_2] - 2 e^{b_1} b_1 b_2 g_1[b_2]) = 0$, f₄₉[b₁, b₂]] /. {b₁ → x, b₂ → y}

$$\left\{ \left\{ f_{49}[x, y] \rightarrow (2 - 2 e^x - 2 e^y + 2 e^{x+y} + x + e^x x - e^y x - e^{x+y} x - 2 x^2 f_{32}[x, y] + 2 e^y x^2 f_{32}[x, y] - 2 x y g_1[y] + 2 e^x x y g_1[y]) / (2 (-1 + e^y) x^2 y) \right\} \right\}$$

$$f_{49}[x_, y_] := (2 - 2 e^x - 2 e^y + 2 e^{x+y} + x + e^x x - e^y x - e^{x+y} x - 2 x^2 f_{32}[x, y] + 2 e^y x^2 f_{32}[x, y] - 2 x y g_1[y] + 2 e^x x y g_1[y]) / (2 (-1 + e^y) x^2 y)$$

Factor[Coefficient[R3u2, e u₂ w₃] /. u₁ → 0]

$$-e^{-b_1} (-2 f_{19}[b_1, b_2] + 2 e^{b_2} f_{19}[b_1, b_2] + 2 b_2 f_{41}[b_1, b_2] - 2 e^{b_2} b_2 f_{41}[b_1, b_2] - g_3[b_2] + e^{b_1} g_3[b_2])$$

Solve[- $e^{-b_1} (-2 f_{19}[b_1, b_2] + 2 e^{b_2} f_{19}[b_1, b_2] + 2 b_2 f_{41}[b_1, b_2] - 2 e^{b_2} b_2 f_{41}[b_1, b_2] - g_3[b_2] + e^{b_1} g_3[b_2]) = 0$, f₁₉[b₁, b₂]] /. {b₁ → x, b₂ → y}

$$\left\{ \left\{ f_{19}[x, y] \rightarrow \frac{-2 y f_{41}[x, y] + 2 e^y y f_{41}[x, y] + g_3[y] - e^x g_3[y]}{2 (-1 + e^y)} \right\} \right\}$$

$$f_{19}[x_, y_] := \frac{-2 y f_{41}[x, y] + 2 e^y y f_{41}[x, y] + g_3[y] - e^x g_3[y]}{2 (-1 + e^y)}$$

Factor[Coefficient[R3u2, e u₁]]

$$-\frac{1}{b_1} e^{-b_1} (-1 + e^{b_1}) b_2 (-f_1[b_1, b_3] + f_1[b_2, b_3] + b_1 f_{16}[b_1, b_3] - b_2 f_{16}[b_2, b_3] + b_1^2 f_{42}[b_1, b_3] - b_2^2 f_{42}[b_2, b_3])$$

$$\text{tt1} = \text{Factor}\left[\frac{b_1 \text{Coefficient}[\text{R3u2}, \epsilon u_1]}{e^{-b_1} (-1 + e^{b_1}) b_2}\right] /. \{b_1 \rightarrow x, b_2 \rightarrow y, b_3 \rightarrow z\} /. f_{-}[y, z] \rightarrow 0$$

$$f_1[x, z] - x f_{16}[x, z] - x^2 f_{42}[x, z]$$

$$f_1[x_, z_] := g_7[z] - (-x f_{16}[x, z] - x^2 f_{42}[x, z])$$

$$\text{Factor}[\text{Coefficient}[\text{R3u2}, \epsilon]]$$

$$-e^{-b_1} (-1 + e^{b_1}) b_2 (f_5[b_1, b_3] - f_5[b_2, b_3])$$

$$f_5[x_, y_] := g_8[y]$$

R3 on w₂

$$\text{R3w2} = \text{R3}[w_2] /. \{b_1 \rightarrow x, b_2 \rightarrow y, b_3 \rightarrow z\};$$

The following were obtained by watching `Dynamic[R3w2 // Simp]` and repeatedly simplifying it:

$$\text{tt1} = \text{Simplify}[\text{Coefficient}[\text{R3w2}, \epsilon u_3 w_3^2] /. \{u_{-} \rightarrow 0, w_{-} \rightarrow 0, c_{-} \rightarrow 0\}]$$

$$-e^x (-1 + e^y) (f_{35}[x, z] - 2 z f_{50}[x, z])$$

$$f_{35}[x_, z_] := 2 z f_{50}[x, z]$$

$$\text{tt1} = \text{Simplify}[\text{Coefficient}[\text{R3w2}, \epsilon c_1 w_3] /. \{u_{-} \rightarrow 0, w_{-} \rightarrow 0, c_{-} \rightarrow 0\}]$$

$$e^x (-1 + e^y) x (f_{23}[x, y] - f_{23}[x, z] - y f_{46}[x, y] + z f_{46}[x, z])$$

$$f_{23}[x_, y_] := g_9[x] + y f_{46}[x, y]$$

$$\text{tt1} = \text{Simplify}[\text{Coefficient}[\text{R3w2}, \epsilon w_3] /. \{u_{-} \rightarrow 0, w_{-} \rightarrow 0, c_{-} \rightarrow 0\}]$$

$$e^x (-1 + e^y) (f_2[x, y] - f_2[x, z] - y f_{21}[x, y] + z f_{21}[x, z] - y^2 f_{50}[x, y] + z^2 f_{50}[x, z])$$

$$f_2[x_, y_] := g_{10}[x] + y f_{21}[x, y] + y^2 f_{50}[x, y]$$

$$\text{tt1} = \text{Simplify}[\text{Coefficient}[\text{R3w2}, \epsilon c_3] /. \{u_{-} \rightarrow 0, w_{-} \rightarrow 0, c_{-} \rightarrow 0\}]$$

$$z (e^x (-1 + e^y) x f_{36}[x, z] - e^y (-1 + e^x) y f_{36}[y, z] + 2 z (e^x (-1 + e^y) f_{37}[x, z] - e^y (-1 + e^x) f_{37}[y, z]))$$

$$e^x (-1 + e^x)^{-1} x f_{36}[x, z] + 2 z e^x (-1 + e^x)^{-1} f_{37}[x, z] ==$$

$$e^y (-1 + e^y)^{-1} y f_{36}[y, z] + 2 z e^y (-1 + e^y)^{-1} f_{37}[y, z]$$

$$\frac{e^x x f_{36}[x, z]}{-1 + e^x} + \frac{2 e^x z f_{37}[x, z]}{-1 + e^x} == \frac{e^y y f_{36}[y, z]}{-1 + e^y} + \frac{2 e^y z f_{37}[y, z]}{-1 + e^y}$$

$$\text{Solve}[e^x (-1 + e^x)^{-1} x f_{36}[x, z] + 2 z e^x (-1 + e^x)^{-1} f_{37}[x, z] == g_{11}[z], f_{36}[x, z]]$$

$$\left\{ \left\{ f_{36}[x, z] \rightarrow -\frac{e^{-x} (2 e^x z f_{37}[x, z] + g_{11}[z] - e^x g_{11}[z])}{x} \right\} \right\}$$

$$f_{36}[x_, z_] := -\frac{e^{-x} (2 e^x z f_{37}[x, z] + g_{11}[z] - e^x g_{11}[z])}{x}$$

Assorted

`tt1 = Simplify[Coefficient[R3w3, e] /. {u_ -> 0, w_ -> 0, c_ -> 0}]`

$$-z (e^x (-1 + e^y) f_4[x, z] - e^y (-1 + e^x) f_4[y, z])$$

`Solve[e^x (-1 + e^x)^{-1} f_4[x, z] == g12[z], f_4[x, z]]`

$$\{\{f_4[x, z] \rightarrow e^{-x} (-1 + e^x) g_{12}[z]\}\}$$

$$f_4[x_, z_] := e^{-x} (-1 + e^x) g_{12}[z]$$

`tt1 = Simplify[Coefficient[R3w3, e u3 w3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]`

$$-2 z (e^x (-1 + e^y) f_{37}[x, z] - e^y (-1 + e^x) f_{37}[y, z])$$

$$f_{37}[x_, z_] := e^{-x} (-1 + e^x) g_{13}[z]$$

`tt1 = Simplify[Coefficient[R3w2, e u2 w3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]`

$$-\frac{1}{y} 2 e^{-y} (-1 + e^x) (-e^y y^2 f_{33}[y, z] + (-1 + e^y) (-e^y y g_{11}[y] + (-1 + e^y) z g_{11}[z] + e^y y^2 g_{13}[y] + z^2 g_{13}[z] - e^y z^2 g_{13}[z]))$$

`Solve[tt1 == 0, f33[y, z]]`

$$\{\{f_{33}[y, z] \rightarrow \frac{1}{y^2} e^{-y} (-1 + e^y) (-e^y y g_{11}[y] - z g_{11}[z] + e^y z g_{11}[z] + e^y y^2 g_{13}[y] + z^2 g_{13}[z] - e^y z^2 g_{13}[z])\}\}$$

$$f_{33}[y_, z_] := \frac{1}{y^2}$$

$$e^{-y} (-1 + e^y) (-e^y y g_{11}[y] - z g_{11}[z] + e^y z g_{11}[z] + e^y y^2 g_{13}[y] + z^2 g_{13}[z] - e^y z^2 g_{13}[z])$$

R3 on u3

`R3u3 = R3[u3] /. {b1 -> x, b2 -> y, b3 -> z};`

The following were obtained by watching `Dynamic[R3u3 // Simp]` and repeatedly simplifying it:

`Simplify[Coefficient[R3u3, e u3 w3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]`

$$-e^{-x-y} z ((-1 + e^y) f_{41}[x, z] - (-1 + e^x) f_{41}[y, z])$$

$$f_{41}[x_, y_] := g_{14}[y] (-1 + e^x)$$

`Simplify[Coefficient[R3u3, e c1 u1] /. {u_ -> 0, w_ -> 0, c_ -> 0}]`

$$e^{-x-y} (-1 + e^y) x z (f_{45}[x, y] - f_{45}[x, z])$$

$$f_{45}[x_, y_] := g_{15}[x]$$

`Simplify[Coefficient[R3u3, e w1 u1] /. {u_ -> 0, w_ -> 0, c_ -> 0}]`

$$e^{-x-y} (-1 + e^y) z (f_{38}[x, y] - f_{38}[x, z])$$

$f_{38}[x_, y_] := g_{16}[x]$

$tt1 = \text{Simplify}[\text{Coefficient}[\text{R3u3}, \epsilon c_2] /. \{u_ \rightarrow 0, w_ \rightarrow 0, c_ \rightarrow 0\}]$

$$\frac{1}{2} e^{-x-y} (-1 + e^x) z (g_3[y] + 2 y (e^y g_5[y] + (-1 + e^y) g_{14}[y] + g_{16}[y]))$$

$\text{Solve}[tt1 == 0, g_3[y]]$

$$\{\{g_3[y] \rightarrow -2 y (e^y g_5[y] - g_{14}[y] + e^y g_{14}[y] + g_{16}[y])\}\}$$

$g_3[y_] := -2 y (e^y g_5[y] - g_{14}[y] + e^y g_{14}[y] + g_{16}[y])$

$tt1 = \text{Simplify}[\text{Coefficient}[\text{R3u3}, \epsilon u_1 u_2 w_2] /. \{u_ \rightarrow 0, w_ \rightarrow 0, c_ \rightarrow 0\}]$

$$\frac{1}{x^2 y^2}$$

$$e^{-x-y} z \left(x - e^x x - e^y x + e^{x+y} x + y - e^x y - e^y y + e^{x+y} y + x y - e^{x+y} x y + (-1 + e^y) x^2 y f_{32}[x, y] + (-1 + e^x) x y^2 g_1[y] + e^y x y^2 g_6[y] - e^{x+y} x y^2 g_6[y] + x y^2 g_9[y] - e^x x y^2 g_9[y] - x y^3 g_{15}[y] + e^x x y^3 g_{15}[y] \right)$$

$\text{Solve}[tt1 == 0, f_{32}[x, y]]$

$$\left\{ \left\{ f_{32}[x, y] \rightarrow \frac{1}{(-1 + e^y) x^2 y} \left(-x + e^x x + e^y x - e^{x+y} x - y + e^x y + e^y y - e^{x+y} y - x y + e^{x+y} x y + x y^2 g_1[y] - e^x x y^2 g_1[y] - e^y x y^2 g_6[y] + e^{x+y} x y^2 g_6[y] - x y^2 g_9[y] + e^x x y^2 g_9[y] + x y^3 g_{15}[y] - e^x x y^3 g_{15}[y] \right) \right\} \right\}$$

$f_{32}[x_, y_] := \frac{1}{(-1 + e^y) x^2 y}$

$$\left(-x + e^x x + e^y x - e^{x+y} x - y + e^x y + e^y y - e^{x+y} y - x y + e^{x+y} x y + x y^2 g_1[y] - e^x x y^2 g_1[y] - e^y x y^2 g_6[y] + e^{x+y} x y^2 g_6[y] - x y^2 g_9[y] + e^x x y^2 g_9[y] + x y^3 g_{15}[y] - e^x x y^3 g_{15}[y] \right)$$

$tt1 = \text{Simplify}[\text{Coefficient}[\text{R3u3}, \epsilon] /. \{u_ \rightarrow 0, w_ \rightarrow 0, c_ \rightarrow 0\}]$

$$e^{-x-y} z \left((-1 + e^y) f_6[x, y] - (-1 + e^y) f_6[x, z] + (-1 + e^x) (f_6[y, z] + e^y g_8[y]) \right)$$

$tt2 = \text{Simplify}\left[\frac{tt1}{e^{-x-y} z}\right]$

$$(-1 + e^y) f_6[x, y] - (-1 + e^y) f_6[x, z] + (-1 + e^x) (f_6[y, z] + e^y g_8[y])$$

$D[tt2, z] // \text{Simplify}$

$$-(-1 + e^y) f_6^{(0,1)}[x, z] + (-1 + e^x) f_6^{(0,1)}[y, z]$$

$$f_6^{(0,1)}[x, z] == g[z] (-1 + e^x)$$

$$f_6^{(0,1)}[x, z] == (-1 + e^x) g[z]$$

$f_6[x_, z_] := (-1 + e^x) g_{17}[z] + g_{18}[x]$

$D[tt2, z] // \text{Simplify}$

0

```

tt1 = Simplify[Coefficient[R3u3, e] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
e^{-x-y} (-1 + e^x) z (e^y g_8[y] + (-1 + e^y) g_{17}[y] + g_{18}[y])

Solve[tt1 == 0, g_{18}[y]]
{{g_{18}[y] -> -e^y g_8[y] + g_{17}[y] - e^y g_{17}[y]}}

g_{18}[y_] := -e^y g_8[y] + g_{17}[y] - e^y g_{17}[y]

tt1 = Simplify[Coefficient[R3w1, e w_3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
-(-1 + e^y) x f_{20}[x, y] + (-1 + e^y) x f_{20}[x, z] -
(-1 + e^x) (y f_{20}[y, z] + y g_1[y] - e^y g_7[y] - g_{10}[y] + y^2 g_{15}[y])

tt2 = Simplify[D[tt1, z]]
(-1 + e^y) x f_{20}^{(0,1)}[x, z] - (-1 + e^x) y f_{20}^{(0,1)}[y, z]

f_{20}^{(0,1)}[x, z] == \frac{(-1 + e^x) g[z]}{x}
f_{20}^{(0,1)}[x, z] == \frac{(-1 + e^x) g[z]}{x}

f_{20}[x_, z_] := \frac{(-1 + e^x) g_{19}[z]}{x} + g_{20}[x]

tt2 = Simplify[D[tt1, z]]
0

tt1 = Simplify[Coefficient[R3w1, e w_3] /. {u_ -> 0, w_ -> 0, c_ -> 0}]
-(-1 + e^x) (y g_1[y] - e^y g_7[y] - g_{10}[y] + y^2 g_{15}[y] - g_{19}[y] + e^y g_{19}[y] + y g_{20}[y])

Solve[tt1 == 0, g_{10}[y]]
{{g_{10}[y] -> y g_1[y] - e^y g_7[y] + y^2 g_{15}[y] - g_{19}[y] + e^y g_{19}[y] + y g_{20}[y]}}

g_{10}[y_] := y g_1[y] - e^y g_7[y] + y^2 g_{15}[y] - g_{19}[y] + e^y g_{19}[y] + y g_{20}[y]

Table[R3[y_i] // Simp, {y, {c, u, w}}, {i, 3}]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```

Most General R

$\rho_0[1, 2]$ // **Simp**

$$\begin{aligned}
& \in \left(b_1^2 c_1^2 f_{42}[b_1, b_2] + b_2^2 c_2^2 f_{50}[b_1, b_2] + \right. \\
& \quad w_1 g_8[b_2] + u_2^2 \left(w_2^2 f_{50}[b_1, b_2] + e^{-b_1} (-1 + e^{b_1}) w_2 g_{13}[b_2] \right) + \\
& \quad u_2 \left(w_2 f_{21}[b_1, b_2] + w_1 w_2 g_5[b_2] + e^{-b_1} (-1 + e^{b_1}) g_{12}[b_2] + (-1 + e^{b_1}) w_2^2 g_{14}[b_2] \right) + \\
& \quad u_1^2 \left(w_1^2 f_{42}[b_1, b_2] + \left(w_2^2 \left(-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (-1 + e^{b_2}) b_1^2 g_1[b_1] + \right. \right. \right. \\
& \quad \quad \left. \left. \left. b_1 \left((1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1})^2 b_2 g_1[b_2] \right) \right) \right) / \left(2 (-1 + e^{b_2}) b_1^3 \right) + \frac{1}{b_1^2} \\
& \quad e^{-b_1} (-1 + e^{b_1}) w_2 \left(-e^{b_1} b_1 g_{11}[b_1] + e^{b_1} b_1^2 g_{13}[b_1] - (-1 + e^{b_1}) b_2 (-g_{11}[b_2] + b_2 g_{13}[b_2]) \right) + \\
& \quad \left. w_1 w_2 g_{15}[b_1] \right) + c_1 \left(b_1 f_{16}[b_1, b_2] + b_1^2 f_{42}[b_1, b_2] + u_2 w_2 (b_1 f_{46}[b_1, b_2] + g_6[b_2]) + \right. \\
& \quad g_7[b_2] + c_2 (b_2 g_6[b_2] + b_1 (b_2 f_{46}[b_1, b_2] + g_9[b_1])) + \\
& \quad u_1 (2 b_1 w_1 f_{42}[b_1, b_2] + w_2 (g_1[b_1] + b_1 g_{15}[b_1])) - \\
& \quad b_1 w_2 (2 e^{b_1} g_5[b_1] + 2 (-1 + e^{b_1}) g_{14}[b_1] + g_{16}[b_1]) \left. \right) + \\
& \quad w_2 \left(-e^{b_1} g_8[b_1] - (-1 + e^{b_1}) (g_{17}[b_1] - g_{17}[b_2]) \right) + \\
& \quad c_2 \left(b_2 f_{21}[b_1, b_2] + b_2^2 f_{50}[b_1, b_2] + b_2 w_1 g_5[b_2] - e^{b_1} g_7[b_1] + \right. \\
& \quad u_2 (2 b_2 w_2 f_{50}[b_1, b_2] + e^{-b_1} (-1 + e^{b_1}) g_{11}[b_2]) + b_1^2 g_{15}[b_1] + \\
& \quad u_1 \left(w_1 (b_2 f_{46}[b_1, b_2] + g_9[b_1]) - \frac{e^{-b_1} (-1 + e^{b_1}) b_2 g_{11}[b_2]}{b_1} + \frac{1}{(-1 + e^{b_2}) b_1^2 b_2} \right. \\
& \quad \quad \left. w_2 \left(-(-1 + e^{b_1}) (-1 + e^{b_2}) b_2 + b_1 \left(-(-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1+b_2}) b_2 - \right. \right. \right. \\
& \quad \quad \left. \left. \left. (-1 + e^{b_1}) b_2^2 (g_1[b_2] - e^{b_2} g_6[b_2] - g_9[b_2]) - (-1 + e^{b_1}) b_2^3 g_{15}[b_2] \right) \right) \right) + \\
& \quad \frac{1}{-1 + e^{b_2}} (-1 + e^{b_1}) b_2 w_2 (e^{b_2} g_5[b_2] + 2 (-1 + e^{b_2}) g_{14}[b_2] + g_{16}[b_2]) - \\
& \quad g_{19}[b_1] + e^{b_1} g_{19}[b_1] + b_1 (g_1[b_1] + g_{20}[b_1]) \left. \right) + \\
& \quad u_1 \left(-\frac{e^{-b_1} (-1 + e^{b_1}) b_2 (g_{11}[b_2] + g_{12}[b_2])}{b_1} + \right. \\
& \quad u_2 \left(w_1 w_2 f_{46}[b_1, b_2] + \frac{e^{-b_1} (-1 + e^{b_1}) w_2 (g_{11}[b_2] - 2 b_2 g_{13}[b_2])}{b_1} + \right. \\
& \quad \quad \left((-1 + e^{b_1}) w_2^2 (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2 + 2 b_2^2 (e^{b_2} g_6[b_2] + g_9[b_2]) - 2 b_2^3 g_{15}[b_2]) \right) / \\
& \quad \quad \left. \left(2 (-1 + e^{b_2}) b_1 b_2^2 \right) \right) + w_1 (f_{16}[b_1, b_2] + w_2 g_{16}[b_1]) - \\
& \quad \frac{1}{(-1 + e^{b_2}) b_1} (-1 + e^{b_1}) w_2^2 \left((-1 + e^{b_2}) b_1 (e^{b_1} g_5[b_1] + (-1 + e^{b_1}) g_{14}[b_1] + g_{16}[b_1]) + e^{b_1} \right. \\
& \quad \quad \left. b_2 (e^{b_2} g_5[b_2] + (-1 + e^{b_2}) g_{14}[b_2] + g_{16}[b_2]) \right) + w_2 \left(\frac{(-1 + e^{b_1}) g_{19}[b_2]}{b_1} + g_{20}[b_1] \right) \left. \right) \left. \right)
\end{aligned}$$

Special Rules I

u_1 // R[1, 2] // Simp

$$u_1 + \epsilon \left(\frac{u_1^2 w_2 (1 - e^{b_1} + b_1 + b_1^2 g_1[b_1])}{b_1^2} - b_1 u_2 w_2 g_5[b_2] + c_2 (-b_1 b_2 g_5[b_2] + u_1 (1 + b_2 g_6[b_2])) - \right. \\ \left. b_1 g_8[b_2] + u_1 (u_2 w_2 g_6[b_2] + g_7[b_2] - 2 b_1 w_2 (e^{b_1} g_5[b_1] + (-1 + e^{b_1}) g_{14}[b_1] + g_{16}[b_1])) \right)$$

u_2 // R[1, 2] // Simp

$$\frac{(1 - e^{-b_1}) b_2 u_1}{b_1} + e^{-b_1} u_2 + \epsilon \left(e^{-2 b_1} (-1 + e^{b_1}) u_2^2 (g_{11}[b_2] - b_2 g_{13}[b_2]) + \right. \\ \left. u_1^2 \left(\frac{e^{-b_1} (-1 + e^{b_1})^2 w_2 (-1 + e^{b_2} - b_2^2 g_1[b_2])}{(-1 + e^{b_2}) b_1^2} + \frac{1}{b_1^2} e^{-2 b_1} (-1 + e^{b_1}) b_2 (e^{b_1} b_1 g_{11}[b_1] - \right. \right. \\ \left. \left. e^{b_1} b_1^2 g_{13}[b_1] + (-1 + e^{b_1}) b_2 (-g_{11}[b_2] + b_2 g_{13}[b_2])) - e^{-b_1} b_2 w_1 g_{15}[b_1] \right) + \right. \\ \left. c_1 \left(e^{-b_1} u_2 (1 + b_1 g_9[b_1]) + \frac{e^{-b_1} b_2 u_1 (-1 + e^{b_1} - b_1 - b_1^2 g_1[b_1] - b_1^3 g_{15}[b_1])}{b_1^2} + \right. \right. \\ \left. \left. e^{-b_1} b_1 b_2 (2 e^{b_1} g_5[b_1] + 2 (-1 + e^{b_1}) g_{14}[b_1] + g_{16}[b_1]) \right) + c_2 \left(-\frac{1}{(-1 + e^{b_2}) b_1} \right. \right. \\ \left. \left. e^{-b_1} (-1 + e^{b_1}) u_1 (-1 + e^{b_2} + b_2 + b_2^2 (-g_1[b_2] + g_6[b_2] + g_9[b_2]) - b_2^3 g_{15}[b_2]) - \right. \right. \\ \left. \left. \frac{1}{-1 + e^{b_2}} e^{-b_1} (-1 + e^{b_1}) b_2^2 ((-1 + 2 e^{b_2}) g_5[b_2] + 2 (-1 + e^{b_2}) g_{14}[b_2] + g_{16}[b_2]) \right) + \right. \\ \left. e^{-b_1} b_2 (e^{b_1} g_8[b_1] - (-1 + e^{b_1}) (g_8[b_2] - g_{17}[b_1] + g_{17}[b_2])) + \right. \\ \left. u_1 \left(u_2 \left(e^{-b_1} w_1 g_9[b_1] + \frac{2 e^{-2 b_1} (-1 + e^{b_1}) b_2 (-g_{11}[b_2] + b_2 g_{13}[b_2])}{b_1} + \right. \right. \right. \\ \left. \left. (e^{-b_1} (-1 + e^{b_1}) w_2 (-1 + e^{b_2} - b_2 - b_2^2 (g_1[b_2] + g_6[b_2] + g_9[b_2]) + b_2^3 g_{15}[b_2])) \right) / \right. \\ \left. \left((-1 + e^{b_2}) b_1 b_2 \right) - e^{-b_1} b_2 w_1 g_{16}[b_1] + \frac{1}{(-1 + e^{b_2}) b_1} \right. \\ \left. 2 (-1 + e^{b_1}) b_2^2 w_2 (e^{b_2} g_5[b_2] + (-1 + e^{b_2}) g_{14}[b_2] + g_{16}[b_2]) + \frac{1}{b_1^2} \right. \\ \left. e^{-b_1} b_2 (1 - e^{b_1} + b_1 (1 + (-1 + e^{b_1}) g_7[b_2] - (-1 + e^{b_1}) g_{19}[b_2]) - b_1^2 g_{20}[b_1]) \right) + \right. \\ \left. u_2 \left(\frac{e^{-b_1} (-1 + e^{b_1}) b_2 w_2 (g_5[b_2] + g_{16}[b_2])}{-1 + e^{b_2}} + \right. \right. \\ \left. \left. e^{-b_1} (-e^{b_1} g_7[b_1] + b_1^2 g_{15}[b_1] + (-1 + e^{b_1}) g_{19}[b_1] + b_1 (g_1[b_1] + g_{20}[b_1])) \right) \right)$$

SR1 = ReplaceAll[

$$\{g_{1|5|6|7|8|11|12|13|14|15|17|19|20}[y_] \Rightarrow 0, f_{[_]} \Rightarrow 0, g_{16}[y_] \Rightarrow e^y - 1, g_9[y_] \Rightarrow -\frac{e^y}{y}\};$$

$\rho_0[1, 2]$ // SR1 // Simp

$$\in \left(\frac{(2 - 2 e^{b_1} + (1 + e^{b_1}) b_1) u_1^2 w_2^2}{2 b_1^3} + c_1 (-e^{b_1} c_2 - (-1 + e^{b_1}) b_1 w_2) + \right. \\ \left. u_1 \left((-1 + e^{b_1}) w_1 w_2 - \frac{(-1 + e^{b_1}) ((-1 + e^{b_1}) b_1 + e^{b_1} b_2) w_2^2}{b_1} - \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_1 b_2^2} \right) + \right. \\ \left. c_2 \left((-1 + e^{b_1}) b_2 w_2 + u_1 \left(-\frac{e^{b_1} w_1}{b_1} + \frac{(-(-1 + e^{b_1}) b_2 + b_1 (1 - e^{b_1} + b_2)) w_2}{b_1^2 b_2} \right) \right) \right)$$

 u_1 // R[1, 2] // SR1 // Simp

$$u_1 + \in \left(c_2 u_1 - 2 (-1 + e^{b_1}) b_1 u_1 w_2 + \frac{(1 - e^{b_1} + b_1) u_1^2 w_2}{b_1^2} \right)$$

 u_2 // R[1, 2] // SR1 // Simp

$$\frac{(1 - e^{-b_1}) b_2 u_1}{b_1} + e^{-b_1} u_2 + \in \left(c_2 \left((-1 + e^{-b_1}) b_2^2 + \frac{e^{-b_1} (-1 + e^{b_1}) (-1 + b_2) u_1}{b_1} \right) + \right. \\ \left. c_1 \left(e^{-b_1} (-1 + e^{b_1}) b_1 b_2 + \frac{e^{-b_1} (-1 + e^{b_1} - b_1) b_2 u_1}{b_1^2} + (-1 + e^{-b_1}) u_2 \right) + \right. \\ \left. \frac{e^{-b_1} (-1 + e^{b_1})^2 u_1^2 w_2}{b_1^2} + (1 - e^{-b_1}) b_2 u_2 w_2 + u_1 \left(-\frac{e^{-b_1} (-1 + e^{b_1} - b_1) b_2}{b_1^2} + \right. \right. \\ \left. \left. (-1 + e^{-b_1}) b_2 w_1 + \frac{2 (-1 + e^{b_1}) b_2^2 w_2}{b_1} + u_2 \left(-\frac{w_1}{b_1} + \frac{e^{-b_1} (-1 + e^{b_1}) (1 + b_2) w_2}{b_1 b_2} \right) \right) \right)$$

 w_1 // R[1, 2] // SR1 // Simp

$$w_1 + (1 - e^{b_1}) w_2 + \\ \in \left(\frac{(-1 + e^{b_1} - e^{b_1} (-2 + e^{b_1}) b_1) c_1 w_2}{b_1} + 2 (-1 + e^{b_1}) b_1 w_1 w_2 - (-1 + e^{b_1}) ((-1 + e^{b_1}) b_1 + b_2) w_2^2 - \right. \\ \left. \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_2^2} + c_2 \left(-w_1 + \frac{(1 - e^{b_1}) w_2}{b_2} \right) + \right. \\ \left. u_1 \left(\frac{(-1 + e^{b_1} + (-1 + e^{b_1} - e^{2 b_1}) b_1) w_1 w_2}{b_1^2} + \frac{(-1 + e^{2 b_1}) w_2^2}{2 b_1} \right) \right)$$

 w_2 // R[1, 2] // SR1 // Simp

$$e^{b_1} w_2 + \in \left(e^{b_1} (-1 + e^{b_1}) c_1 w_2 - e^{b_1} (-1 + e^{b_1}) b_2 w_2^2 + u_1 \left(\frac{e^{2 b_1} w_1 w_2}{b_1} - \frac{e^{b_1} (-1 + e^{b_1}) w_2^2}{2 b_1} \right) \right)$$

c_1 // R[1, 2] // SR1 // Simp

$$c_1 + \frac{(-1 + e^{b_1}) u_1 w_2}{b_1} + \epsilon \left(\frac{(-1 + e^{b_1} + e^{b_1} (-2 + e^{b_1}) b_1) c_1 u_1 w_2}{b_1^2} + \frac{(-1 + e^{b_1}) (1 + b_2) c_2 u_1 w_2}{b_1 b_2} + \right. \\ \left. u_1^2 \left(\frac{e^{b_1} (-1 + e^{b_1}) w_1 w_2}{b_1^2} - \frac{(-1 + e^{b_1})^2 w_2^2}{2 b_1^2} \right) + u_1 \left(- \frac{(-1 + e^{b_1} + e^{b_1} (-2 + e^{b_1}) b_1) w_2}{b_1^2} - \right. \right. \\ \left. \left. \frac{(-1 + e^{b_1}) ((-1 + e^{b_1}) b_1 - b_2) w_2^2}{b_1} + \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_1 b_2^2} \right) \right)$$

c_2 // R[1, 2] // SR1 // Simp

$$c_2 + \frac{(1 - e^{b_1}) u_1 w_2}{b_1} + \epsilon \left(c_1 \left(- (-1 + e^{b_1}) b_1 w_2 - \frac{(-1 + e^{b_1} + e^{b_1} (-2 + e^{b_1}) b_1) u_1 w_2}{b_1^2} \right) + \right. \\ \left. c_2 \left((-1 + e^{b_1}) b_2 w_2 - \frac{(-1 + e^{b_1}) (1 + b_2) u_1 w_2}{b_1 b_2} \right) + u_1^2 \left(- \frac{e^{b_1} (-1 + e^{b_1}) w_1 w_2}{b_1^2} + \frac{(-1 + e^{b_1})^2 w_2^2}{2 b_1^2} \right) \right) + \\ u_1 \left(\left(\frac{-1 + e^{b_1} + e^{b_1} (-2 + e^{b_1}) b_1}{b_1^2} + (-1 + e^{b_1}) w_1 \right) w_2 - \right. \\ \left. \frac{(-1 + e^{b_1}) b_2 w_2^2}{b_1} - \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_1 b_2^2} \right)$$

Special Rules 2

SR2[\mathcal{E}_-] := $\mathcal{E} / . \{u_i \rightarrow b_i u_i,$

$$g_1[y_-] \rightarrow \frac{(e^y - 1)}{y^2} gg_1[y], g_{16}[y_-] \rightarrow (e^y - 1) gg_2[y] - g_5[y], g_6[x_-] \rightarrow gg_6[x] / x,$$

$$g_9[x_-] \rightarrow gg_9[x] / x, g_{15}[x_-] \rightarrow gg_{15}[x] / x^2, g_{19}[x_-] \rightarrow gg_{19}[x] / x$$

$\} / . \{g_5[x_-] \rightarrow gg_5[x] / x, g_{11}[x_-] \rightarrow x g_{13}[x]\};$

Problem. The following cannot both be trigonometrized:

$$\{\text{Coefficient}\left[\frac{u_1}{b_1} // R[1, 2] // SR2 // Simp, \epsilon u_1^2 w_2\right],$$

$$\text{Coefficient}[w_2 // R[1, 2] // SR2 // Simp, \epsilon u_1 w_2^2]\}$$

$$\left\{ \frac{b_1 + (-1 + e^{b_1}) (-1 + gg_1[b_1])}{b_1}, - \frac{e^{b_1} (-1 + e^{b_1}) (b_2 - 2 gg_1[b_2])}{2 b_2} \right\}$$

$\frac{u_1}{b_1}$ // R[1, 2] // SR2 // Simp

$$u_1 + \epsilon \left(-g_8[b_2] + \frac{u_1^2 w_2 (b_1 + (-1 + e^{b_1}) (-1 + gg_1[b_1]))}{b_1} - u_2 w_2 gg_5[b_2] + \right. \\ \left. u_1 (g_7[b_2] - 2 (-1 + e^{b_1}) w_2 (b_1 (g_{14}[b_1] + gg_2[b_1]) + gg_5[b_1]) + u_2 w_2 gg_6[b_2]) \right) + \\ \left. c_2 (-gg_5[b_2] + u_1 (1 + gg_6[b_2])) \right)$$

$\frac{u_2}{b_2}$ // R[1, 2] // SR2 // Simp

$$\begin{aligned}
& (1 - e^{-b_1}) u_1 + e^{-b_1} u_2 + \epsilon \left(e^{-b_1} (e^{b_1} g_8[b_1] - (-1 + e^{b_1}) (g_8[b_2] - g_{17}[b_1] + g_{17}[b_2])) \right) + \\
& u_1^2 \left(-\frac{e^{-b_1} (-1 + e^{b_1})^2 w_2 (-1 + gg_1[b_2])}{b_2} - e^{-b_1} w_1 gg_{15}[b_1] \right) + \\
& c_1 \left(e^{-b_1} ((-1 + e^{b_1}) b_1 (2 g_{14}[b_1] + gg_2[b_1]) + (-1 + 2 e^{b_1}) gg_5[b_1]) + \right. \\
& \quad \left. e^{-b_1} u_2 (1 + gg_9[b_1]) + \frac{1}{b_1} e^{-b_1} u_1 (-(-1 + e^{b_1}) (-1 + gg_1[b_1]) - b_1 (1 + gg_{15}[b_1])) \right) + \\
& c_2 \left(-e^{-b_1} (-1 + e^{b_1}) (b_2 (2 g_{14}[b_2] + gg_2[b_2]) + 2 gg_5[b_2]) + \frac{1}{(-1 + e^{b_2}) b_2} \right. \\
& \quad \left. e^{-b_1} (-1 + e^{b_1}) u_1 ((-1 + e^{b_2}) (-1 + gg_1[b_2]) - b_2 (1 + gg_6[b_2] + gg_9[b_2] - gg_{15}[b_2])) \right) + \\
& u_2 \left(e^{-b_1} (-1 + e^{b_1}) b_2 w_2 gg_2[b_2] + \frac{1}{b_1} \right. \\
& \quad \left. e^{-b_1} (b_1^2 g_{20}[b_1] + b_1 (-e^{b_1} g_7[b_1] + gg_{15}[b_1]) + (-1 + e^{b_1}) (gg_1[b_1] + gg_{19}[b_1])) \right) + \\
& u_1 \left(e^{-b_1} w_1 (-(-1 + e^{b_1}) b_1 gg_2[b_1] + gg_5[b_1]) + 2 (-1 + e^{b_1}) w_2 \right. \\
& \quad (b_2 (g_{14}[b_2] + gg_2[b_2]) + gg_5[b_2]) + u_2 (e^{-b_1} w_1 gg_9[b_1] - (e^{-b_1} (-1 + e^{b_1}) w_2 ((-1 + e^{b_2}) \\
& \quad (-1 + gg_1[b_2]) + b_2 (1 + gg_6[b_2] + gg_9[b_2] - gg_{15}[b_2]))) / ((-1 + e^{b_2}) b_2)) + \\
& \quad \left. \frac{1}{b_1 b_2} e^{-b_1} (b_2 (1 - e^{b_1} + b_1 (1 + (-1 + e^{b_1}) g_7[b_2]) - b_1^2 g_{20}[b_1]) - (-1 + e^{b_1}) b_1 gg_{19}[b_2]) \right) \Big)
\end{aligned}$$

w₁ // R[1, 2] // SR2 // Simp

$$\begin{aligned}
& w_1 + (1 - e^{b_1}) w_2 + \epsilon \left(-(-1 + e^{b_1}) b_2 (g_{12}[b_2] + b_2 g_{13}[b_2]) + \right. \\
& w_1 (-g_7[b_2] + 2(-1 + e^{b_1}) w_2 (b_1 (g_{14}[b_1] + gg_2[b_1]) + gg_5[b_1])) - \\
& (-1 + e^{b_1}) w_2^2 ((-1 + e^{b_1}) b_1 (g_{14}[b_1] + gg_2[b_1]) + \\
& b_2 (g_{14}[b_2] + gg_2[b_2]) - gg_5[b_1] + e^{b_1} gg_5[b_1] + gg_5[b_2]) + \frac{1}{b_1} \\
& c_1 w_2 ((-1 + e^{b_1}) (1 + gg_1[b_1]) + b_1 (e^{b_1} + (-1 + e^{b_1}) gg_9[b_1] + gg_{15}[b_1])) + \\
& u_1 \left(\frac{(-1 + e^{b_1}) w_2^2 ((1 + e^{b_1}) b_1 + 2(-1 + e^{b_1}) gg_1[b_1])}{2 b_1} + \frac{1}{b_1} \right. \\
& \left. w_1 w_2 (-(-1 + e^{b_1}) (-1 + gg_1[b_1]) + b_1 (-1 + (-1 + e^{b_1}) gg_9[b_1] + gg_{15}[b_1])) \right) + \\
& u_2 \left(-(-1 + e^{b_1}) b_2^2 w_2 g_{13}[b_2] - w_1 w_2 gg_6[b_2] + ((-1 + e^{b_1}) w_2^2 \right. \\
& \left. (2 - 2 e^{b_2} + b_2 (1 + e^{b_2} + 2 e^{b_2} gg_6[b_2] + 2 gg_9[b_2] - 2 gg_{15}[b_2])) \right) / (2(-1 + e^{b_2}) b_2) + \\
& c_2 \left(-(-1 + e^{b_1}) b_2^2 g_{13}[b_2] + w_1 (-1 - gg_6[b_2]) + \frac{1}{(-1 + e^{b_2}) b_2} \right. \\
& \left. (-1 + e^{b_1}) w_2 (-(-1 + e^{b_2}) (1 + gg_1[b_2]) + b_2 (e^{b_2} + e^{b_2} gg_6[b_2] + gg_9[b_2] - gg_{15}[b_2])) \right) + \\
& \frac{1}{b_1 b_2} w_2 (e^{b_1} b_1^2 b_2 g_{20}[b_1] + (-1 + e^{b_1}) b_2 (e^{b_1} gg_1[b_1] + (-1 + e^{b_1}) gg_{19}[b_1])) + \\
& \left. b_1 (-e^{b_1} b_2 ((-1 + e^{b_1}) g_7[b_1] - gg_{15}[b_1]) + (-1 + e^{b_1}) gg_{19}[b_2]) \right)
\end{aligned}$$

w₂ // R[1, 2] // SR2 // Simp

$$\begin{aligned}
& e^{b_1} w_2 + \\
& \epsilon \left((-1 + e^{b_1}) b_2^2 c_2 g_{13}[b_2] + (-1 + e^{b_1}) b_2^2 u_2 w_2 g_{13}[b_2] + (-1 + e^{b_1}) b_2 (g_{12}[b_2] + b_2 g_{13}[b_2]) - \right. \\
& e^{b_1} (-1 + e^{b_1}) w_2^2 (b_2 (g_{14}[b_2] + gg_2[b_2]) + gg_5[b_2]) - e^{b_1} c_1 w_2 (1 + gg_9[b_1]) + \\
& u_1 \left(-\frac{e^{b_1} (-1 + e^{b_1}) w_2^2 (b_2 - 2 gg_1[b_2])}{2 b_2} - e^{b_1} w_1 w_2 gg_9[b_1] \right) + \frac{1}{b_1} \\
& \left. e^{b_1} w_2 (-b_1^2 g_{20}[b_1] + b_1 (e^{b_1} g_7[b_1] - gg_{15}[b_1]) - (-1 + e^{b_1}) (gg_1[b_1] + gg_{19}[b_1])) \right)
\end{aligned}$$

$\rho_0[1, 2]$ // SR2 // Simp

$$\begin{aligned}
& \in \left(b_1^2 c_1^2 f_{42}[b_1, b_2] + b_2^2 c_2^2 f_{50}[b_1, b_2] + \right. \\
& w_1 g_8[b_2] + u_2^2 \left(b_2^2 w_2^2 f_{50}[b_1, b_2] + e^{-b_1} (-1 + e^{b_1}) b_2^2 w_2 g_{13}[b_2] \right) + \\
& w_2 \left(-e^{b_1} g_8[b_1] - (-1 + e^{b_1}) (g_{17}[b_1] - g_{17}[b_2]) \right) + \\
& u_2 \left(b_2 w_2 f_{21}[b_1, b_2] + e^{-b_1} (-1 + e^{b_1}) b_2 g_{12}[b_2] + (-1 + e^{b_1}) b_2 w_2^2 g_{14}[b_2] + w_1 w_2 g g_5[b_2] \right) + \\
& u_1^2 \left(b_1^2 w_1^2 f_{42}[b_1, b_2] + \frac{1}{2 b_1 b_2} \right. \\
& w_2^2 \left(b_2 \left((1 + e^{b_1}) b_1 + (-1 + e^{b_1}) (-2 + (-1 + e^{b_1}) g g_1[b_1]) \right) + (-1 + e^{b_1})^2 b_1 g g_1[b_2] \right) + \\
& w_1 w_2 g g_{15}[b_1] \left. \right) + c_1 \left(b_1 f_{16}[b_1, b_2] + b_1^2 f_{42}[b_1, b_2] + g_7[b_2] + \right. \\
& w_2 \left(-(-1 + e^{b_1}) b_1 (2 g_{14}[b_1] + g g_2[b_1]) - (-1 + 2 e^{b_1}) g g_5[b_1] \right) + \\
& u_2 w_2 (b_1 b_2 f_{46}[b_1, b_2] + g g_6[b_2]) + c_2 (b_1 b_2 f_{46}[b_1, b_2] + g g_6[b_2] + g g_9[b_1]) + \\
& u_1 \left(2 b_1^2 w_1 f_{42}[b_1, b_2] + \frac{w_2 \left((-1 + e^{b_1}) g g_1[b_1] + b_1 g g_{15}[b_1] \right)}{b_1} \right) \left. \right) + \\
& c_2 \left(u_2 \left(2 b_2^2 w_2 f_{50}[b_1, b_2] + e^{-b_1} (-1 + e^{b_1}) b_2^2 g_{13}[b_2] \right) + w_1 g g_5[b_2] + \right. \\
& (-1 + e^{b_1}) w_2 (b_2 (2 g_{14}[b_2] + g g_2[b_2]) + g g_5[b_2]) + u_1 \left(-e^{-b_1} (-1 + e^{b_1}) b_2^2 g_{13}[b_2] + \right. \\
& w_1 (b_1 b_2 f_{46}[b_1, b_2] + g g_9[b_1]) + \frac{1}{(-1 + e^{b_2}) b_1 b_2} w_2 \left(-(-1 + e^{b_1}) (-1 + e^{b_2}) b_2 + \right. \\
& b_1 \left(-(-1 + e^{b_1}) (-1 + e^{b_2}) (1 + g g_1[b_2]) + b_2 (-1 + e^{b_1+b_2} + e^{b_2} (-1 + e^{b_1}) g g_6[b_2] + \right. \\
& \left. \left. (-1 + e^{b_1}) g g_9[b_2] + g g_{15}[b_2] - e^{b_1} g g_{15}[b_2] \right) \right) \left. \right) + \\
& \frac{1}{b_1} \left(b_1^2 g_{20}[b_1] + b_1 (b_2 f_{21}[b_1, b_2] + b_2^2 f_{50}[b_1, b_2] - e^{b_1} g_7[b_1] + g g_{15}[b_1]) \right) + \\
& (-1 + e^{b_1}) (g g_1[b_1] + g g_{19}[b_1]) \left. \right) + u_1 \left(-e^{-b_1} (-1 + e^{b_1}) b_2 (g_{12}[b_2] + b_2 g_{13}[b_2]) + \right. \\
& w_1 \left(b_1 f_{16}[b_1, b_2] + b_1 w_2 \left((-1 + e^{b_1}) g g_2[b_1] - \frac{g g_5[b_1]}{b_1} \right) \right) - \\
& (-1 + e^{b_1}) w_2^2 \left((-1 + e^{b_1}) b_1 (g_{14}[b_1] + g g_2[b_1]) + e^{b_1} b_2 (g_{14}[b_2] + g g_2[b_2]) - g g_5[b_1] + \right. \\
& e^{b_1} g g_5[b_1] + e^{b_1} g g_5[b_2]) + u_2 \left(b_1 b_2 w_1 w_2 f_{46}[b_1, b_2] - e^{-b_1} (-1 + e^{b_1}) b_2^2 w_2 g_{13}[b_2] + \right. \\
& \left((-1 + e^{b_1}) w_2^2 (2 - 2 e^{b_2} + b_2 (1 + e^{b_2} + 2 e^{b_2} g g_6[b_2] + 2 g g_9[b_2] - 2 g g_{15}[b_2])) \right) / \\
& \left. \left. (2 (-1 + e^{b_2}) b_2) \right) + \frac{w_2 (b_1 b_2 g_{20}[b_1] + (-1 + e^{b_1}) g g_{19}[b_2])}{b_2} \right) \left. \right)
\end{aligned}$$

Special Rules 3

First round of replacements below is the most general that makes the u-action trigonometric. The second round is arbitrary.

$$\begin{aligned}
 \text{SR3}[\mathcal{E}_-] &:= \mathcal{E} / . \{ \mathbf{u}_i \mapsto \mathbf{b}_i \mathbf{u}_i, \\
 \mathbf{g}_1[\mathbf{y}_-] &\mapsto \frac{(e^y - 1)}{y^2} + \frac{(e^y - 1) \mathbf{g}\mathbf{g}_1[\mathbf{y}]}{y}, \quad \mathbf{g}_5[\mathbf{x}_-] \mapsto \mathbf{g}\mathbf{g}_5[\mathbf{x}] / \mathbf{x}, \quad \mathbf{g}_6[\mathbf{x}_-] \mapsto \frac{(e^x - 1) \mathbf{g}\mathbf{g}_6[\mathbf{x}]}{\mathbf{x}}, \\
 \mathbf{g}_9[\mathbf{x}_-] &\mapsto \frac{(e^x - 1) \mathbf{g}\mathbf{g}_9[\mathbf{x}]}{\mathbf{x}}, \quad \mathbf{g}_{11}[\mathbf{y}_-] \mapsto \mathbf{y} \mathbf{g}_{13}[\mathbf{y}] - \mathbf{y}^{-1} \mathbf{g}\mathbf{g}_{11}[\mathbf{y}], \\
 \mathbf{g}_{14}[\mathbf{x}_-] &\mapsto \mathbf{g}\mathbf{g}_{14}[\mathbf{x}] / \mathbf{x}, \quad \mathbf{g}_{15}[\mathbf{x}_-] \mapsto \frac{(e^x - 1) \mathbf{g}\mathbf{g}_{15}[\mathbf{x}] + e^x}{\mathbf{x}^2}, \\
 \mathbf{g}_{16}[\mathbf{y}_-] &\mapsto \frac{1}{\mathbf{y}} (-\mathbf{g}\mathbf{g}_5[\mathbf{y}] + (e^y - 1) \mathbf{g}\mathbf{g}_{16}[\mathbf{y}]), \quad \mathbf{g}_{20}[\mathbf{x}_-] \mapsto \frac{1 - e^x}{\mathbf{x}^2} + \mathbf{g}\mathbf{g}_{20}[\mathbf{x}] / \mathbf{x} \\
 &\} / . \{ \mathbf{g}_{7|8|12|13|17|19}[_] \rightarrow 0, \quad \mathbf{g}\mathbf{g}_{1|5|6|9|11|14|15|16|20}[_] \rightarrow 0, \quad \mathbf{f}_{16|21|42|46|50}[_] \rightarrow 0 \};
 \end{aligned}$$

$$\frac{\mathbf{u}_1}{\mathbf{b}_1} // \mathbf{R}[1, 2] // \text{SR3} // \text{Simp}$$

$$\mathbf{u}_1 + \epsilon (c_2 \mathbf{u}_1 + \mathbf{u}_1^2 \mathbf{w}_2)$$

$$\text{Exponent}\left[\frac{\mathbf{u}_1}{\mathbf{b}_1} // \mathbf{R}[1, 2] // \text{SR3}, \#, \text{MinMax@*List}\right] \& /@ \{\mathbf{b}_1, \mathbf{b}_2\}$$

$$\{\{0, 0\}, \{0, 0\}\}$$

$$\frac{\mathbf{u}_2}{\mathbf{b}_2} // \mathbf{R}[1, 2] // \text{SR3} // \text{Simp}$$

$$\begin{aligned}
 &(1 - e^{-b_1}) \mathbf{u}_1 + e^{-b_1} \mathbf{u}_2 + \\
 &\epsilon \left((1 - e^{-b_1}) c_2 \mathbf{u}_1 + \mathbf{u}_2 + c_1 \left((-1 - e^{-b_1}) \mathbf{u}_1 + e^{-b_1} \mathbf{u}_2 \right) - \mathbf{u}_1^2 \mathbf{w}_1 + \mathbf{u}_1 \left(e^{-b_1} + (1 - e^{-b_1}) \mathbf{u}_2 \mathbf{w}_2 \right) \right)
 \end{aligned}$$

$$\text{Exponent}\left[\frac{\mathbf{u}_2}{\mathbf{b}_2} // \mathbf{R}[1, 2] // \text{SR3}, \#, \text{MinMax@*List}\right] \& /@ \{\mathbf{b}_1, \mathbf{b}_2\}$$

$$\{\{-1, 0\}, \{-1, 0\}\}$$

$$\mathbf{w}_1 // \mathbf{R}[1, 2] // \text{SR3} // \text{Simp}$$

$$\begin{aligned}
 &\mathbf{w}_1 + (1 - e^{b_1}) \mathbf{w}_2 + \\
 &\epsilon \left(e^{2b_1} \mathbf{w}_2 + \frac{2(-1 + e^{b_1} + e^{b_1} b_1) c_1 \mathbf{w}_2}{b_1} - \frac{(-1 + e^{b_1})(2 + b_2) \mathbf{u}_2 \mathbf{w}_2^2}{2 b_2} + c_2 \left(-\mathbf{w}_1 + \frac{(2 - 2 e^{b_1}) \mathbf{w}_2}{b_2} \right) + \right. \\
 &\quad \left. \mathbf{u}_1 \left((-1 + e^{b_1}) \mathbf{w}_1 \mathbf{w}_2 + \frac{(-1 + e^{b_1})(2(-1 + e^{b_1}) + (1 + e^{b_1}) b_1) \mathbf{w}_2^2}{2 b_1} \right) \right)
 \end{aligned}$$

$$\mathbf{w}_2 // \mathbf{R}[1, 2] // \text{SR3} // \text{Simp}$$

$$e^{b_1} \mathbf{w}_2 + \epsilon \left(-e^{2b_1} \mathbf{w}_2 - e^{b_1} c_1 \mathbf{w}_2 - \frac{e^{b_1}(-1 + e^{b_1})(-2 + b_2) \mathbf{u}_1 \mathbf{w}_2^2}{2 b_2} \right)$$

c₁ // R[1, 2] // SR3 // Simp

$$c_1 + (-1 + e^{b_1}) u_1 w_2 +$$

$$\in \left(-2 e^{b_1} c_1 u_1 w_2 + \frac{(-1 + e^{b_1}) (2 + b_2) c_2 u_1 w_2}{b_2} + u_1^2 \left(-e^{b_1} w_1 w_2 - \frac{1}{2} (-1 + e^{b_1})^2 w_2^2 \right) + \right.$$

$$\left. u_1 \left(-e^{b_1} (-2 + e^{b_1}) w_2 + \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_2} \right) \right)$$

c₂ // R[1, 2] // SR3 // Simp

$$c_2 + (1 - e^{b_1}) u_1 w_2 +$$

$$\in \left(2 e^{b_1} c_1 u_1 w_2 - \frac{(-1 + e^{b_1}) (2 + b_2) c_2 u_1 w_2}{b_2} + u_1^2 \left(e^{b_1} w_1 w_2 + \frac{1}{2} (-1 + e^{b_1})^2 w_2^2 \right) + \right.$$

$$\left. u_1 \left(e^{b_1} (-2 + e^{b_1}) w_2 - \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_2} \right) \right)$$

ρ0[1, 2] // SR3 // Simp

$$\in \left(\frac{(-1 + e^{b_1} + e^{b_1} b_1) c_1 u_1 w_2}{b_1} + c_2 \left(e^{b_1} + \left(\frac{1 - e^{b_1}}{b_1} + \frac{2 - 2 e^{b_1} + b_2}{b_2} \right) u_1 w_2 \right) + \right.$$

$$u_1^2 \left(e^{b_1} w_1 w_2 + \frac{1}{2 b_1 b_2} \left((3 - 4 e^{b_1} + e^{2 b_1}) b_2 + b_1 \left((-1 + e^{b_1})^2 + (1 + e^{b_1}) b_2 \right) \right) w_2^2 \right) +$$

$$\left. u_1 \left(\frac{(1 - e^{b_1}) w_2}{b_1} - \frac{(-1 + e^{b_1}) (2 + b_2) u_2 w_2^2}{2 b_2} \right) \right)$$