Pensieve header: Verifying the adjoint action.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606\\V1"];
<< OneCo.m
```

```
AutoAd[τ_ . a_i_,j_][x_] := Module[{bas, β, k, f, t, guess, InitCond, DiffCond, sol},
  bas = ReplacePart[LBasis[{i, j}], 1 → β];
  n = 0; guess = Sum[f_++n[t] y, {y, bas}];
  InitCond = Table[
    0 == Coefficient[((guess /. t → 0) - x, y]] /. {(c | u | w)_ → 0, ϵ → 0}, {y, bas}];
  DiffCond = Table[0 == Coefficient[D[guess, t] - b[Expand[τ a_i,j], guess], y] /.
     {(c | u | w)_ → 0, ϵ → 0}, {y, bas}];
  {sol} = DSolve[InitCond⋃DiffCond, Table[f_k[t], {k, n}], t];
  guess /. sol /. t → 1
 ]
```

```
x = c1
```

c_1

```
bas = ReplacePart[LBasis[{1, 2}], 1 → β]
```

$\{\beta, \epsilon, c_1, c_2, u_1, u_2, w_1, w_2, \epsilon c_1, \epsilon c_2, \epsilon u_1, \epsilon u_2, \epsilon w_1, u_1 w_1, u_2 w_1, \epsilon w_2, u_1 w_2,$
$u_2 w_2, \epsilon c_1^2, \epsilon c_1 c_2, \epsilon c_2^2, \epsilon c_1 u_1, \epsilon c_2 u_1, \epsilon c_1 u_2, \epsilon c_2 u_2, \epsilon c_1 w_1, \epsilon c_2 w_1, \epsilon u_1 w_1,$
$\epsilon u_2 w_1, \epsilon c_1 w_2, \epsilon c_2 w_2, \epsilon u_1 w_2, \epsilon u_2 w_2, \epsilon c_1 u_1 w_1, \epsilon c_2 u_1 w_1, \epsilon u_1^2 w_1, \epsilon c_1 u_2 w_1,$
$\epsilon c_2 u_2 w_1, \epsilon u_1 u_2 w_1, \epsilon u_2^2 w_1, \epsilon u_1 w_1^2, \epsilon u_2 w_1^2, \epsilon c_1 u_1 w_2, \epsilon c_2 u_1 w_2, \epsilon u_1^2 w_2,$
$\epsilon c_1 u_2 w_2, \epsilon c_2 u_2 w_2, \epsilon u_1 u_2 w_2, \epsilon u_2^2 w_2, \epsilon u_1 w_1 w_2, \epsilon u_2 w_1 w_2, \epsilon u_1 w_2^2, \epsilon u_2 w_2^2, \epsilon u_1^2 w_1^2,$
$\epsilon u_1 u_2 w_1^2, \epsilon u_2^2 w_1^2, \epsilon u_1^2 w_1 w_2, \epsilon u_1 u_2 w_1 w_2, \epsilon u_2^2 w_1 w_2, \epsilon u_1^2 w_2^2, \epsilon u_1 u_2 w_2^2, \epsilon u_2^2 w_2^2\}$

```
k = 0;
guess = Sum[f_++k[t] y, {y, bas}]
```

$\beta f_1[t] + \epsilon f_2[t] + c_1 f_3[t] + c_2 f_4[t] + u_1 f_5[t] + u_2 f_6[t] + w_1 f_7[t] + w_2 f_8[t] + \epsilon c_1 f_9[t] +$
$\epsilon c_2 f_{10}[t] + \epsilon u_1 f_{11}[t] + \epsilon u_2 f_{12}[t] + \epsilon w_1 f_{13}[t] + u_1 w_1 f_{14}[t] + u_2 w_1 f_{15}[t] + \epsilon w_2 f_{16}[t] +$
$u_1 w_2 f_{17}[t] + u_2 w_2 f_{18}[t] + \epsilon c_1^2 f_{19}[t] + \epsilon c_1 c_2 f_{20}[t] + \epsilon c_2^2 f_{21}[t] + \epsilon c_1 u_1 f_{22}[t] +$
$\epsilon c_2 u_1 f_{23}[t] + \epsilon c_1 u_2 f_{24}[t] + \epsilon c_2 u_2 f_{25}[t] + \epsilon c_1 w_1 f_{26}[t] + \epsilon c_2 w_1 f_{27}[t] +$
$\epsilon u_1 w_1 f_{28}[t] + \epsilon u_2 w_1 f_{29}[t] + \epsilon c_1 w_2 f_{30}[t] + \epsilon c_2 w_2 f_{31}[t] + \epsilon u_1 w_2 f_{32}[t] +$
$\epsilon u_2 w_2 f_{33}[t] + \epsilon c_1 u_1 w_1 f_{34}[t] + \epsilon c_2 u_1 w_1 f_{35}[t] + \epsilon u_1^2 w_1 f_{36}[t] + \epsilon c_1 u_2 w_1 f_{37}[t] +$
$\epsilon c_2 u_2 w_1 f_{38}[t] + \epsilon u_1 u_2 w_1 f_{39}[t] + \epsilon u_2^2 w_1 f_{40}[t] + \epsilon u_1 w_1^2 f_{41}[t] + \epsilon u_2 w_1^2 f_{42}[t] +$
$\epsilon c_1 u_1 w_2 f_{43}[t] + \epsilon c_2 u_1 w_2 f_{44}[t] + \epsilon u_1^2 w_2 f_{45}[t] + \epsilon c_1 u_2 w_2 f_{46}[t] + \epsilon c_2 u_2 w_2 f_{47}[t] +$
$\epsilon u_1 u_2 w_2 f_{48}[t] + \epsilon u_2^2 w_2 f_{49}[t] + \epsilon u_1 w_1 w_2 f_{50}[t] + \epsilon u_2 w_1 w_2 f_{51}[t] + \epsilon u_1 w_2^2 f_{52}[t] +$
$\epsilon u_2 w_2^2 f_{53}[t] + \epsilon u_1^2 w_1^2 f_{54}[t] + \epsilon u_1 u_2 w_1^2 f_{55}[t] + \epsilon u_2^2 w_1^2 f_{56}[t] + \epsilon u_1^2 w_1 w_2 f_{57}[t] +$
$\epsilon u_1 u_2 w_1 w_2 f_{58}[t] + \epsilon u_2^2 w_1 w_2 f_{59}[t] + \epsilon u_1^2 w_2^2 f_{60}[t] + \epsilon u_1 u_2 w_2^2 f_{61}[t] + \epsilon u_2^2 w_2^2 f_{62}[t]$

```
(guess /. t → 0) - x
```

$$-c_1+\beta\, f_1[0]+\epsilon\, f_2[0]+c_1\, f_3[0]+c_2\, f_4[0]+u_1\, f_5[0]+u_2\, f_6[0]+w_1\, f_7[0]+w_2\, f_8[0]+\epsilon\, c_1\, f_9[0]+\epsilon\, c_2\, f_{10}[0]+\epsilon\, u_1\, f_{11}[0]+\epsilon\, u_2\, f_{12}[0]+\epsilon\, w_1\, f_{13}[0]+u_1\, w_1\, f_{14}[0]+u_2\, w_1\, f_{15}[0]+\epsilon\, w_2\, f_{16}[0]+u_1\, w_2\, f_{17}[0]+u_2\, w_2\, f_{18}[0]+\epsilon\, c_1^2\, f_{19}[0]+\epsilon\, c_1\, c_2\, f_{20}[0]+\epsilon\, c_2^2\, f_{21}[0]+\epsilon\, c_1\, u_1\, f_{22}[0]+\epsilon\, c_2\, u_1\, f_{23}[0]+\epsilon\, c_1\, u_2\, f_{24}[0]+\epsilon\, c_2\, u_2\, f_{25}[0]+\epsilon\, c_1\, w_1\, f_{26}[0]+\epsilon\, c_2\, w_1\, f_{27}[0]+\epsilon\, u_1\, w_1\, f_{28}[0]+\epsilon\, u_2\, w_1\, f_{29}[0]+\epsilon\, c_1\, w_2\, f_{30}[0]+\epsilon\, c_2\, w_2\, f_{31}[0]+\epsilon\, u_1\, w_2\, f_{32}[0]+\epsilon\, u_2\, w_2\, f_{33}[0]+\epsilon\, c_1\, u_1\, w_1\, f_{34}[0]+\epsilon\, c_2\, u_1\, w_1\, f_{35}[0]+\epsilon\, u_1^2\, w_1\, f_{36}[0]+\epsilon\, c_1\, u_2\, w_1\, f_{37}[0]+\epsilon\, c_2\, u_2\, w_1\, f_{38}[0]+\epsilon\, u_1\, u_2\, w_1\, f_{39}[0]+\epsilon\, u_2^2\, w_1\, f_{40}[0]+\epsilon\, u_1\, w_1^2\, f_{41}[0]+\epsilon\, u_2\, w_1^2\, f_{42}[0]+\epsilon\, c_1\, u_1\, w_2\, f_{43}[0]+\epsilon\, c_2\, u_1\, w_2\, f_{44}[0]+\epsilon\, u_1^2\, w_2\, f_{45}[0]+\epsilon\, c_1\, u_2\, w_2\, f_{46}[0]+\epsilon\, c_2\, u_2\, w_2\, f_{47}[0]+\epsilon\, u_1\, u_2\, w_2\, f_{48}[0]+\epsilon\, u_2^2\, w_2\, f_{49}[0]+\epsilon\, u_1\, w_1\, w_2\, f_{50}[0]+\epsilon\, u_2\, w_1\, w_2\, f_{51}[0]+\epsilon\, u_1\, w_2^2\, f_{52}[0]+\epsilon\, u_2\, w_2^2\, f_{53}[0]+\epsilon\, u_1^2\, w_1^2\, f_{54}[0]+\epsilon\, u_1\, u_2\, w_1^2\, f_{55}[0]+\epsilon\, u_2^2\, w_1^2\, f_{56}[0]+\epsilon\, u_1^2\, w_1\, w_2\, f_{57}[0]+\epsilon\, u_1\, u_2\, w_1\, w_2\, f_{58}[0]+\epsilon\, u_2^2\, w_1\, w_2\, f_{59}[0]+\epsilon\, u_1^2\, w_2^2\, f_{60}[0]+\epsilon\, u_1\, u_2\, w_2^2\, f_{61}[0]+\epsilon\, u_2^2\, w_2^2\, f_{62}[0]$$

```
InitCond =
 Table[0 == Coefficient[(guess /. t → 0) - x, y] /. {(c | u | w)_ → 0, ϵ → 0}, {y, bas}]
```

$$\{0==f_1[0],\ 0==f_2[0],\ 0==-1+f_3[0],\ 0==f_4[0],\ 0==f_5[0],\ 0==f_6[0],\ 0==f_7[0],\ 0==f_8[0],\ 0==f_9[0],\ 0==f_{10}[0],\ 0==f_{11}[0],\ 0==f_{12}[0],\ 0==f_{13}[0],\ 0==f_{14}[0],\ 0==f_{15}[0],\ 0==f_{16}[0],\ 0==f_{17}[0],\ 0==f_{18}[0],\ 0==f_{19}[0],\ 0==f_{20}[0],\ 0==f_{21}[0],\ 0==f_{22}[0],\ 0==f_{23}[0],\ 0==f_{24}[0],\ 0==f_{25}[0],\ 0==f_{26}[0],\ 0==f_{27}[0],\ 0==f_{28}[0],\ 0==f_{29}[0],\ 0==f_{30}[0],\ 0==f_{31}[0],\ 0==f_{32}[0],\ 0==f_{33}[0],\ 0==f_{34}[0],\ 0==f_{35}[0],\ 0==f_{36}[0],\ 0==f_{37}[0],\ 0==f_{38}[0],\ 0==f_{39}[0],\ 0==f_{40}[0],\ 0==f_{41}[0],\ 0==f_{42}[0],\ 0==f_{43}[0],\ 0==f_{44}[0],\ 0==f_{45}[0],\ 0==f_{46}[0],\ 0==f_{47}[0],\ 0==f_{48}[0],\ 0==f_{49}[0],\ 0==f_{50}[0],\ 0==f_{51}[0],\ 0==f_{52}[0],\ 0==f_{53}[0],\ 0==f_{54}[0],\ 0==f_{55}[0],\ 0==f_{56}[0],\ 0==f_{57}[0],\ 0==f_{58}[0],\ 0==f_{59}[0],\ 0==f_{60}[0],\ 0==f_{61}[0],\ 0==f_{62}[0]\}$$

```
DiffCond = Table[
  0 == Coefficient[D[guess, t] - b[a_{1,2}, guess], y] /. {(c | u | w)_ → 0, ϵ → 0}, {y, bas}]
```

$$\{0 == f_1'[t],\ 0 == f_2'[t],\ 0 == f_3'[t],\ 0 == f_4'[t],\ 0 == b_2 f_6[t] + f_5'[t],\ 0 == -b_1 f_6[t] + f_6'[t],$$
$$0 == f_7'[t],\ 0 == -b_1 f_7[t] + b_1 f_8[t] + f_8'[t],\ 0 == f_9'[t],\ 0 == f_{10}'[t],$$
$$0 == b_2 f_{12}[t] - b_2 f_{24}[t] + b_2 f_{25}[t] + f_{11}'[t],\ 0 == -b_1 f_{12}[t] + f_{12}'[t],\ 0 == f_{13}'[t],$$
$$0 == b_2 f_{15}[t] + f_{14}'[t],\ 0 == -b_1 f_{15}[t] + f_{15}'[t],\ 0 == -b_1 f_{13}[t] + b_1 f_{16}[t] + f_{16}'[t],$$
$$0 == f_3[t] - f_4[t] - b_1 f_{14}[t] + b_1 f_{17}[t] + b_2 f_{18}[t] + f_{17}'[t],\ 0 == -b_1 f_{15}[t] + f_{18}'[t],$$
$$0 == f_{19}'[t],\ 0 == f_{20}'[t],\ 0 == f_{21}'[t],\ 0 == b_2 f_{24}[t] + f_{22}'[t],$$
$$0 == f_5[t] - 2 f_6[t] + b_2 f_{25}[t] + f_{23}'[t],\ 0 == f_6[t] - b_1 f_{24}[t] + f_{24}'[t],$$
$$0 == -b_1 f_{25}[t] + f_{25}'[t],\ 0 == f_{26}'[t],\ 0 == -f_7[t] + f_{27}'[t],$$
$$0 == b_2 f_{29}[t] - b_2 f_{37}[t] + b_2 f_{38}[t] + f_{28}'[t],\ 0 == f_{15}[t] - b_1 f_{29}[t] + f_{29}'[t],$$
$$0 == 2 f_7[t] - f_8[t] - b_1 f_{26}[t] + b_1 f_{30}[t] + f_{30}'[t],\ 0 == -b_1 f_{27}[t] + b_1 f_{31}[t] + f_{31}'[t],$$
$$0 == f_9[t] - f_{10}[t] - 2 f_{14}[t] + f_{17}[t] - f_{19}[t] + f_{20}[t] - f_{21}[t] - b_1 f_{28}[t] + b_1 f_{32}[t] +$$
$$b_2 f_{33}[t] - b_2 f_{46}[t] + b_2 f_{47}[t] + f_{32}'[t],\ 0 == -b_1 f_{29}[t] + f_{33}'[t],\ 0 == b_2 f_{37}[t] + f_{34}'[t],$$
$$0 == -2 f_{15}[t] + b_2 f_{38}[t] + f_{35}'[t],\ 0 == b_2 f_{39}[t] + f_{36}'[t],\ 0 == f_{15}[t] - b_1 f_{37}[t] + f_{37}'[t],$$
$$0 == -f_{15}[t] - b_1 f_{38}[t] + f_{38}'[t],\ 0 == -b_1 f_{39}[t] + 2 b_2 f_{40}[t] + f_{39}'[t],$$
$$0 == -2 b_1 f_{40}[t] + f_{40}'[t],\ 0 == b_2 f_{42}[t] + f_{41}'[t],\ 0 == -b_1 f_{42}[t] + f_{42}'[t],$$
$$0 == 2 f_{14}[t] - f_{17}[t] + 2 f_{19}[t] - f_{20}[t] - b_1 f_{34}[t] + b_1 f_{43}[t] + b_2 f_{46}[t] + f_{43}'[t],$$
$$0 == f_{17}[t] - 2 f_{18}[t] + f_{20}[t] - 2 f_{21}[t] - b_1 f_{35}[t] + b_1 f_{44}[t] + b_2 f_{47}[t] + f_{44}'[t],$$
$$0 == f_{22}[t] - f_{23}[t] - b_1 f_{36}[t] + b_1 f_{45}[t] + b_2 f_{48}[t] + f_{45}'[t],$$
$$0 == 2 f_{15}[t] - b_1 f_{37}[t] + f_{46}'[t],\ 0 == -b_1 f_{38}[t] + f_{47}'[t],$$
$$0 == f_{24}[t] - f_{25}[t] - b_1 f_{39}[t] + 2 b_2 f_{49}[t] + f_{48}'[t],\ 0 == -b_1 f_{40}[t] - b_1 f_{49}[t] + f_{49}'[t],$$
$$0 == f_{26}[t] - f_{27}[t] - 2 b_1 f_{41}[t] + b_1 f_{50}[t] + b_2 f_{51}[t] + f_{50}'[t],$$
$$0 == -2 b_1 f_{42}[t] + f_{51}'[t],\ 0 == f_{30}[t] - f_{31}[t] - b_1 f_{50}[t] + 2 b_1 f_{52}[t] + b_2 f_{53}[t] + f_{52}'[t],$$
$$0 == -b_1 f_{51}[t] + b_1 f_{53}[t] + f_{53}'[t],\ 0 == b_2 f_{55}[t] + f_{54}'[t],$$
$$0 == -b_1 f_{55}[t] + 2 b_2 f_{56}[t] + f_{55}'[t],\ 0 == -2 b_1 f_{56}[t] + f_{56}'[t],$$
$$0 == f_{34}[t] - f_{35}[t] - 2 b_1 f_{54}[t] + b_1 f_{57}[t] + b_2 f_{58}[t] + f_{57}'[t],$$
$$0 == f_{37}[t] - f_{38}[t] - 2 b_1 f_{55}[t] + 2 b_2 f_{59}[t] + f_{58}'[t],\ 0 == -2 b_1 f_{56}[t] - b_1 f_{59}[t] + f_{59}'[t],$$
$$0 == f_{43}[t] - f_{44}[t] - b_1 f_{57}[t] + 2 b_1 f_{60}[t] + b_2 f_{61}[t] + f_{60}'[t],$$
$$0 == f_{46}[t] - f_{47}[t] - b_1 f_{58}[t] + b_1 f_{61}[t] + 2 b_2 f_{62}[t] + f_{61}'[t],\ 0 == -b_1 f_{59}[t] + f_{62}'[t]\}$$

```
{sol} = DSolve[InitCond⋃DiffCond, Table[f_i[t], {i, k}], t]
```

$$\Big\{\Big\{f_1[t]\to 0,\ f_2[t]\to 1,\ f_3[t]\to 0,\ f_8[t]\to 0,\ f_9[t]\to 0,\ f_{13}[t]\to 0,\ f_{14}[t]\to 0,$$
$$f_{16}[t]\to -\frac{e^{-t\,b_1}\left(-1+e^{t\,b_1}\right)}{b_1},\ f_{17}[t]\to 0,\ f_{18}[t]\to 0,\ f_{19}[t]\to 0,\ f_{20}[t]\to 0,\ f_{27}[t]\to 0,$$
$$f_{28}[t]\to 0,\ f_{31}[t]\to \frac{e^{-t\,b_1}\left(-1+e^{t\,b_1}-t\,b_1\right)}{b_1^2},\ f_{32}[t]\to 0,\ f_{33}[t]\to 0,\ f_{34}[t]\to 0,$$
$$f_{36}[t]\to 0,\ f_{37}[t]\to 0,\ f_{42}[t]\to -\frac{e^{-t\,b_1}\left(-1+e^{t\,b_1}-t\,b_1\right)}{b_1^2},\ f_{43}[t]\to \frac{e^{-t\,b_1}\left(-1+e^{t\,b_1}-t\,b_1\right)}{b_1^2},$$
$$f_{45}[t]\to 0,\ f_{46}[t]\to 0,\ f_{53}[t]\to 0,\ f_{54}[t]\to 0,\ f_{55}[t]\to 0,\ f_{56}[t]\to 0,\ f_{57}[t]\to 0,$$
$$f_{58}[t]\to 0,\ f_{59}[t]\to \frac{e^{-2\,t\,b_1}\left(-1+e^{2\,t\,b_1}-2\,e^{t\,b_1}\,t\,b_1\right)}{b_1^3},\ f_{60}[t]\to 0,\ f_{61}[t]\to 0,$$
$$f_4[t]\to 0,\ f_5[t]\to 0,\ f_{10}[t]\to 0,\ f_{11}[t]\to 0,\ f_{21}[t]\to 0,\ f_{22}[t]\to 0,\ f_{23}[t]\to 0,$$
$$f_{24}[t]\to 0,\ f_{35}[t]\to 0,\ f_{38}[t]\to 0,\ f_{39}[t]\to 0,\ f_{44}[t]\to 0,\ f_{47}[t]\to 0,\ f_{48}[t]\to 0,$$
$$f_6[t]\to 0,\ f_7[t]\to 0,\ f_{25}[t]\to 0,\ f_{26}[t]\to 0,\ f_{29}[t]\to 0,\ f_{30}[t]\to 0,\ f_{40}[t]\to 0,$$
$$f_{41}[t]\to 0,\ f_{49}[t]\to 0,\ f_{50}[t]\to 0,\ f_{51}[t]\to 0,\ f_{52}[t]\to 0,\ f_{12}[t]\to 0,\ f_{15}[t]\to 0\Big\}\Big\}$$

```
guess /. sol
```

$$c_1-\frac{e^{-t\,b_1}\left(-1+e^{t\,b_1}\right)u_1\,w_2}{b_1}+\frac{e^{-t\,b_1}\in\left(-1+e^{t\,b_1}-t\,b_1\right)u_1\,w_2}{b_1^2}-\frac{e^{-t\,b_1}\in\left(-1+e^{t\,b_1}-t\,b_1\right)c_1\,u_1\,w_2}{b_1^2}+$$
$$\frac{e^{-t\,b_1}\in\left(-1+e^{t\,b_1}-t\,b_1\right)c_2\,u_1\,w_2}{b_1^2}+\frac{e^{-2\,t\,b_1}\in\left(-1+e^{2\,t\,b_1}-2\,e^{t\,b_1}\,t\,b_1\right)u_1^2\,w_2^2}{b_1^3}$$

```
AutoAd[τ a_{i,j}][c_i]
```

$$c_i-\frac{e^{-\tau\,b_i}\left(-1+e^{\tau\,b_i}\right)u_i\,w_j}{b_i}+\frac{e^{-\tau\,b_i}\in\left(-1+e^{\tau\,b_i}-\tau\,b_i\right)u_i\,w_j}{b_i^2}-\frac{e^{-\tau\,b_i}\in\left(-1+e^{\tau\,b_i}-\tau\,b_i\right)c_i\,u_i\,w_j}{b_i^2}+$$
$$\frac{e^{-\tau\,b_i}\in\left(-1+e^{\tau\,b_i}-\tau\,b_i\right)c_j\,u_i\,w_j}{b_i^2}+\frac{e^{-2\,\tau\,b_i}\in\left(-1+e^{2\,\tau\,b_i}-2\,e^{\tau\,b_i}\,\tau\,b_i\right)u_i^2\,w_j^2}{b_i^3}$$

```
AutoAd[τ a_{i,j}][c_j]
```

$$c_j+\frac{e^{-\tau\,b_i}\left(-1+e^{\tau\,b_i}\right)u_i\,w_j}{b_i}-\frac{e^{-\tau\,b_i}\in\left(-1+e^{\tau\,b_i}-\tau\,b_i\right)u_i\,w_j}{b_i^2}+\frac{e^{-\tau\,b_i}\in\left(-1+e^{\tau\,b_i}-\tau\,b_i\right)c_i\,u_i\,w_j}{b_i^2}-$$
$$\frac{e^{-\tau\,b_i}\in\left(-1+e^{\tau\,b_i}-\tau\,b_i\right)c_j\,u_i\,w_j}{b_i^2}-\frac{e^{-2\,\tau\,b_i}\in\left(-1+e^{2\,\tau\,b_i}-2\,e^{\tau\,b_i}\,\tau\,b_i\right)u_i^2\,w_j^2}{b_i^3}$$

```
AutoAd[τ a_{i,j}][u_i]
```

$$u_i-\in\tau\,c_j\,u_i-\frac{e^{-\tau\,b_i}\in\left(1-e^{\tau\,b_i}+e^{\tau\,b_i}\,\tau\,b_i\right)u_i^2\,w_j}{b_i^2}$$

AutoAd[τ a_{i,j}][u_j]

$$-\frac{\left(-1+e^{\tau b_i}\right) b_j u_i}{b_i}-\frac{\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) b_j u_i}{b_i^2}+\frac{\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) b_j c_i u_i}{b_i^2}+$$
$$\frac{\epsilon\left(-2 b_i+2 e^{\tau b_i} b_i-b_j+e^{\tau b_i} b_j-\tau b_i b_j\right) c_j u_i}{b_i^2}+e^{\tau b_i} u_j-e^{\tau b_i}\epsilon\tau c_i u_j-\frac{1}{b_i^3}$$
$$e^{-\tau b_i}\epsilon\left(-b_i+2 e^{\tau b_i} b_i-e^{2\tau b_i} b_i+2 e^{\tau b_i} b_j-2 e^{2\tau b_i} b_j+e^{\tau b_i}\tau b_i b_j+e^{2\tau b_i}\tau b_i b_j\right) u_i^2 w_j+$$
$$\frac{\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) u_i u_j w_j}{b_i^2}$$

AutoAd[τ a_{i,j}][w_i]

$$w_i+\epsilon\tau c_j w_i+e^{-\tau b_i}\left(-1+e^{\tau b_i}\right) w_j-$$
$$\frac{e^{-\tau b_i}\epsilon\left(-1+e^{\tau b_i}+\tau b_i\right) c_i w_j}{b_i}+\frac{e^{-\tau b_i}\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) c_j w_j}{b_i}+$$
$$\frac{e^{-\tau b_i}\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) u_i w_i w_j}{b_i^2}+\frac{e^{-2\tau b_i}\left(1+e^{\tau b_i}\right)\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) u_i w_j^2}{b_i^2}$$

AutoAd[τ a_{i,j}][w_j]

$$e^{-\tau b_i} w_j+e^{-\tau b_i}\epsilon\tau c_i w_j-\frac{e^{-2\tau b_i}\epsilon\left(1-e^{\tau b_i}+e^{\tau b_i}\tau b_i\right) u_i w_j^2}{b_i^2}$$

AutoAd[β]

0

b[AutoAd[u_i], AutoAd[w_i]] + 2 ϵ AutoAd[c_i] // Expand

b_i

b[AutoAd[u_j], AutoAd[w_j]] + 2 ϵ AutoAd[c_j] // Expand

b_j

Table[Expand[R_0[1, 1, 2][y_k] - AutoAd[a_{1,2}][y_k]], {y, {c, u, w}}, {k, 3}]

{{0, 0, c_3}, {0, 0, u_3}, {0, 0, w_3}}

b[c_1, w_1]

$-w_1$

Ad[τ a_{1,2}][b[c_1, w_1]]

$$-w_1-\epsilon\tau c_2 w_1-e^{-\tau b_1}\left(-1+e^{\tau b_1}\right) w_2+$$
$$\frac{e^{-\tau b_1}\epsilon\left(-1+e^{\tau b_1}+\tau b_1\right) c_1 w_2}{b_1}-\frac{e^{-\tau b_1}\epsilon\left(1-e^{\tau b_1}+e^{\tau b_1}\tau b_1\right) c_2 w_2}{b_1}-$$
$$\frac{e^{-\tau b_1}\epsilon\left(1-e^{\tau b_1}+e^{\tau b_1}\tau b_1\right) u_1 w_1 w_2}{b_1^2}-\frac{e^{-2\tau b_1}\left(1+e^{\tau b_1}\right)\epsilon\left(1-e^{\tau b_1}+e^{\tau b_1}\tau b_1\right) u_1 w_2^2}{b_1^2}$$

```
Ad[τ (b1 c2 - ϵ c1 c2 + u1 w2)][w1]
```

$$w_1 + \epsilon\,\tau\,c_2\,w_1 + e^{-\tau\,b_1}\left(-1 + e^{\tau\,b_1}\right)w_2 - \frac{e^{-\tau\,b_1}\,\epsilon\left(-1 + e^{\tau\,b_1} + \tau\,b_1\right)c_1\,w_2}{b_1} + \frac{e^{-\tau\,b_1}\,\epsilon\left(1 - e^{\tau\,b_1} + e^{\tau\,b_1}\,\tau\,b_1\right)c_2\,w_2}{b_1} + \frac{e^{-\tau\,b_1}\,\epsilon\left(1 - e^{\tau\,b_1} + e^{\tau\,b_1}\,\tau\,b_1\right)u_1\,w_1\,w_2}{b_1^2} + \frac{e^{-2\,\tau\,b_1}\left(1 + e^{\tau\,b_1}\right)\epsilon\left(1 - e^{\tau\,b_1} + e^{\tau\,b_1}\,\tau\,b_1\right)u_1\,w_2^2}{b_1^2}$$