Pensieve header: Finding the YB element for NOE-1.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
<< NOE-1.m
```

---

Solution to all-but-one equations, as found on 160807:

$$\begin{aligned}
&\texttt{ECAllButOne} = \Big\{ f_2[x_,y_] :\to y\,f_9[x,y] + y^2 f_{26}[x,y] - g_6[x],\\
&\quad f_3[x_,z_] :\to x^2 f_{18}[x,z],\ f_4[x_,y_] :\to y\,f_{13}[x,y] - g_1[x],\\
&\quad f_5[x_,y_] :\to y^2 f_{26}[x,y],\ f_8[x_,z_] :\to 0,\ f_{10}[x_,z_] :\to 2x\,f_{18}[x,z],\\
&\quad f_{11}[x_,y_] :\to 2x\,f_{18}[x,y] - 2e^x x\,f_{18}[x,y] - g_4[x],\ f_{12}[x_,z_] :\to 0,\\
&\quad f_{13}[x_,z_] :\to x\,f_{22}[x,z] - g_3[z],\ f_{14}[x_,y_] :\to y\,f_{22}[x,y] - g_2[x],\ f_{15}[x_,y_] :\to\\
&\qquad \frac{1}{(-1+e^y)\,x}\Big(-xy\,f_{23}[x,y] + e^y xy\,f_{23}[x,y] + x\,g_2[x] - e^x x\,g_2[x] - e^y x\,g_2[x] + e^{x+y} x\,g_2[x] -\\
&\qquad\quad e^y y\,g_2[y] + e^{x+y} y\,g_2[y] - y\,g_3[y] + e^x y\,g_3[y] - y\,g_4[y] + e^x y\,g_4[y] + 8y^2 g_5[y] -\\
&\qquad\quad 8e^x y^2 g_5[y] - 16e^y y^2 g_5[y] + 8e^{2y} y^2 g_5[y] + 16e^{x+y} y^2 g_5[y] - 8e^{x+2y} y^2 g_5[y] + 4y\,g_7[y] -\\
&\qquad\quad 4e^x y\,g_7[y] - 4e^y y\,g_7[y] + 4e^{x+y} y\,g_7[y]\Big),\ f_{16}[x_,z_] :\to 0,\ f_{17}[x_,y_] :\to 2y\,f_{26}[x,y],\\
&\quad f_{18}[x_,y_] :\to -\big((f_{19}[x,y] + 4g_5[x] - 8e^x g_5[x] + 4e^{2x} g_5[x]) \,/\, (2(-1+e^x))\big),\\
&\quad f_{19}[x_,y_] :\to \frac{1}{(-1+e^x)(-1+e^y)x^2 y}\\
&\qquad\Big(-1 + 2e^x - e^{2x} + e^y - 2e^{x+y} + e^{2x+y} + 2x^2 y\,f_{20}[x,y] - 2e^y x^2 y\,f_{20}[x,y] + xy\,g_4[x] -\\
&\qquad\quad e^x xy\,g_4[x] - e^y xy\,g_4[x] + e^{x+y} xy\,g_4[x] + y^2 g_4[y] - 2e^x y^2 g_4[y] + e^{2x} y^2 g_4[y] -\\
&\qquad\quad 4y^3 g_5[y] + 8e^x y^3 g_5[y] - 4e^{2x} y^3 g_5[y] + 8e^y y^3 g_5[y] - 4e^{2y} y^3 g_5[y] - 16e^{x+y} y^3 g_5[y] +\\
&\qquad\quad 8e^{2x+y} y^3 g_5[y] + 8e^{x+2y} y^3 g_5[y] - 4e^{2x+2y} y^3 g_5[y] - e^y y\,h_1[] + 2e^{x+y} y\,h_1[] - e^{2x+y} y\,h_1[]\Big),\\
&\quad f_{21}[x_,z_] :\to 0,\ f_{22}[x_,z_] :\to \frac{-x\,f_{23}[x,z] - 4g_7[z] + 4e^x g_7[z]}{(-1+e^x)\,x},\\
&\quad f_{24}[x_,z_] :\to 0,\ f_{25}[x_,z_] :\to 0,\ g_1[y_] :\to y\,g_2[y] + h_1[],\\
&\quad g_2[y_] :\to \frac{1}{2y^2} e^{-y}\Big(-2 + 2e^y - 2y^2 g_3[y] + 8y^3 g_5[y] - 16e^y y^3 g_5[y] +\\
&\qquad 8e^{2y} y^3 g_5[y] + 8y^2 g_7[y] - 8e^y y^2 g_7[y] - y\,h_1[] - e^y y\,h_1[]\Big),\ f_1[x_,z_] :\to\\
&\quad \frac{1}{2}\Big(-\big(\big(x\big(-2(-1+e^x)^2 f_6[x,z] - 2x\,f_{20}[x,z] + (-1+e^x)\big(g_4[x] + 4(-1+e^x)^2 x\,g_5[x]\big)\big)\big) \,/\\
&\qquad (-1+e^x)^2\big) + g_8[z]\Big)\Big\};
\end{aligned}$$

Solution to all equations, as found on 160809:

$$\begin{aligned}
&\texttt{ECAll} = \texttt{ECAllButOne} \bigcup \Big\{ f_6[x_,z_] \to \frac{-\frac{x\,f_7[x,z]}{-1+e^x} + g_9[x] + g_{10}[z]}{x},\ g_{10}[y_] \to \frac{1}{2(-1+e^y)^2 y}\\
&\qquad\Big(-1 + e^y - y^2 g_4[y] + 2e^y y^2 g_4[y] - 4y^3 g_5[y] + 8e^y y^3 g_5[y] - 4e^{2y} y^3 g_5[y] + 2e^y y\,g_6[y] -\\
&\qquad\quad 2e^{2y} y\,g_6[y] - y\,g_8[y] + e^y y\,g_8[y] - 2y\,g_9[y] + 4e^y y\,g_9[y] - 2e^{2y} y\,g_9[y] - e^y y\,h_1[]\Big)\Big\};
\end{aligned}$$

---

```
∈ U[cj] f1[bj, bk] + ∈ U[ck] f2[bj, bk] + ∈ U[cj, cj] f3[bj, bk] + ∈ U[cj, ck] f4[bj, bk] +
  ∈ U[ck, ck] f5[bj, bk] + ∈ U[uj, wj] f6[bj, bk] + ∈ U[uj, wk] f7[bj, bk] +
  ∈ U[uk, wj] f8[bj, bk] + ∈ U[uk, wk] f9[bj, bk] + ∈ U[cj, uj, wj] f10[bj, bk] +
  ∈ U[cj, uj, wk] f11[bj, bk] + ∈ U[cj, uk, wj] f12[bj, bk] + ∈ U[cj, uk, wk] f13[bj, bk] +
  ∈ U[ck, uj, wj] f14[bj, bk] + ∈ U[ck, uj, wk] f15[bj, bk] + ∈ U[ck, uk, wj] f16[bj, bk] +
  ∈ U[ck, uk, wk] f17[bj, bk] + ∈ U[uj, uj, wj, wj] f18[bj, bk] + ∈ U[uj, uj, wj, wk] f19[bj, bk] +
  ∈ U[uj, uj, wk, wk] f20[bj, bk] + ∈ U[uj, uk, wj, wj] f21[bj, bk] +
  ∈ U[uj, uk, wj, wk] f22[bj, bk] + ∈ U[uj, uk, wk, wk] f23[bj, bk] + ∈ U[uk, uk, wj, wj] f24[bj, bk] +
  ∈ U[uk, uk, wj, wk] f25[bj, bk] + ∈ U[uk, uk, wk, wk] f26[bj, bk] /. {U → Times, j → i, k → j}
```

$$\in c_i f_1[b_i, b_j] + \in c_j f_2[b_i, b_j] + \in c_i^2 f_3[b_i, b_j] + \in c_i c_j f_4[b_i, b_j] + \in c_j^2 f_5[b_i, b_j] + \in u_i w_i f_6[b_i, b_j] + \in u_i w_j f_7[b_i, b_j] + \in u_j w_i f_8[b_i, b_j] + \in u_j w_j f_9[b_i, b_j] + \in c_i u_i w_i f_{10}[b_i, b_j] + \in c_i u_i w_j f_{11}[b_i, b_j] + \in c_i u_j w_i f_{12}[b_i, b_j] + \in c_i u_j w_j f_{13}[b_i, b_j] + \in c_j u_i w_i f_{14}[b_i, b_j] + \in c_j u_i w_j f_{15}[b_i, b_j] + \in c_j u_j w_i f_{16}[b_i, b_j] + \in c_j u_j w_j f_{17}[b_i, b_j] + \in u_i^2 w_i^2 f_{18}[b_i, b_j] + \in u_i^2 w_i w_j f_{19}[b_i, b_j] + \in u_i^2 w_j^2 f_{20}[b_i, b_j] + \in u_i u_j w_i^2 f_{21}[b_i, b_j] + \in u_i u_j w_i w_j f_{22}[b_i, b_j] + \in u_i u_j w_j^2 f_{23}[b_i, b_j] + \in u_j^2 w_i^2 f_{24}[b_i, b_j] + \in u_j^2 w_i w_j f_{25}[b_i, b_j] + \in u_j^2 w_j^2 f_{26}[b_i, b_j]$$

```
Rp[i_, j_] :=
  𝔼[bi cj + (e^bi - 1)/bi ui wj] (1 + ∈ ci f1[bi, bj] + ∈ cj f2[bi, bj] + ∈ ci^2 f3[bi, bj] + ∈ ci cj f4[bi, bj] +
    ∈ cj^2 f5[bi, bj] + ∈ ui wi f6[bi, bj] + ∈ ui wj f7[bi, bj] + ∈ uj wi f8[bi, bj] +
    ∈ uj wj f9[bi, bj] + ∈ ci ui wi f10[bi, bj] + ∈ ci ui wj f11[bi, bj] + ∈ ci uj wi f12[bi, bj] +
    ∈ ci uj wj f13[bi, bj] + ∈ cj ui wi f14[bi, bj] + ∈ cj ui wj f15[bi, bj] +
    ∈ cj uj wi f16[bi, bj] + ∈ cj uj wj f17[bi, bj] + ∈ ui^2 wi^2 f18[bi, bj] + ∈ ui^2 wi wj f19[bi, bj] +
    ∈ ui^2 wj^2 f20[bi, bj] + ∈ ui uj wi^2 f21[bi, bj] + ∈ ui uj wi wj f22[bi, bj] +
    ∈ ui uj wj^2 f23[bi, bj] + ∈ uj^2 wi^2 f24[bi, bj] + ∈ uj^2 wi wj f25[bi, bj] + ∈ uj^2 wj^2 f26[bi, bj])
```

```
Short[t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, x] // m[1, 6, y] // m[2, 4, z]]
```

$$\frac{\mathbb{E}\left[\frac{\text{<<1>>} + \text{<<10>>} + \text{<<1>>}}{b_x\, b_y}\right] \left(\text{<<1>>}\right)}{b_x^2\, b_y^2}$$

```
t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, x] // m[2, 5, y] // m[4, 6, z]
```

$$\frac{1}{2\, b_x^2\, b_y^2}\, e^{-2 b_y}\, \mathbb{E}\left[\frac{\cdots 1 \cdots}{b_x\, b_y}\right] \left(2\, e^{2 b_y}\, b_x^2\, b_y^2 + \cdots 662 \cdots + 2\, e^{2 b_y} \in b_x^2\, b_y^2\, u_z^2\, w_z^2\, f_{26}[b_y, b_z]\right)$$

large output | show less | show more | show all | set size limit...

```
Short[t3 = (t1 ≡ t2)[[1]]]
```

$$\frac{e^{-2 b_y} \left(-4\ \text{<<6>>}\ w_z + \text{<<950>>} + 4\ \text{<<9>>}\right)}{2\, b_x^2\, b_y^2}$$

```
Short[Errors = CoefficientRules[Expand[t3], {cx, cy, cz, ux, uy, uz, wx, wy, wz}] /.
    {(_ → c_) :> c} /. {bx → x, by → y, bz → z}]
```

{ <<1>> }

```
EC = {}; E0 = Errors; gn = hn = 0;
```

```
EC = ECAllButOne; {gn = Cases[EC, g_n_[_] :> n, ∞] // Max, hn = Cases[EC, h_n_[] :> n, ∞] // Max}
```

```
{8, 1}
```

```
AddRule[ff_, rule_] := (
   Print["As ", e0, ", adding ", rule];
   done = False; EC = EC ⋃ {rule}
  );
MF[∅_, v_] := Module[{t = ∅, t1}, If[Simplify[t] === 0, 1,
    While[(t1 = Simplify[t /. v → 0]) === 0, t = D[t, v]]; t1
   ]];
```

```
done = False; While[! done, done = True;
 E0 = DeleteCases[Simplify[E0 //. EC], 0] // SortBy[LeafCount];
 Print["Length[E0]==", Length[E0],
  "; Length[EC]==", Length[EC], "; {gn,hn}==", {gn, hn}];
 For[k = 1, k ≤ Length@E0, ++k,
  e1 = Factor[e0 = E0[[k]]];
  If[Head[e1] =!= Times, e2 = e1,
   E0[[k]] = e2 = Select[e1, ! FreeQ[#, f_[_, _] | g_[_] | h_[]] &] ];
  If[e2 == 1, Print["Panic at ", e0, "! No solutions."]; Break[]];
  If[! FreeQ[e2, f_[_, _]] ∧ (FreeQ[e2, x] ⋁ FreeQ[e2, y] ⋁ FreeQ[e2, z]),
   {ff} = Cases[e2, f_[_, _], {0, ∞}, 1];
   {{sol}} = Solve[e2 == 0, ff];
   rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
   AddRule[ff, rule]; Break[]
  ];
  If[! FreeQ[e2, g_[_]] ∧ (FreeQ[e2, y | z] ⋁ FreeQ[e2, x | z] ⋁ FreeQ[e2, z | y]),
   {gg} = Cases[e2, g_[_], ∞, 1];
   {{sol}} = Solve[e2 == 0, gg];
   rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
   AddRule[gg, rule]; Break[]
  ];
  If[Head[e2] === Plus,
   s = List @@ Collect[e2, f_[_, _], Factor]; s1 = Select[s, FreeQ[f_[_, _]]];
   sxy = Cases[s, a_. * f_[x, y]];
   sxz = Cases[s, a_. * f_[x, z]]; syz = Cases[s, a_. * f_[y, z]];
   Which[
    sxy == {} ∧ sxz =!= {} ∧ syz =!= {}, (Print["here"];
     {ff} = Cases[sxz, a_. * f_k_[x, z] :> f_k[x, z], {1}, 1];
     mf = MF[First@sxz /. f_[x, z] → 1, x]; mf *= MF[First@syz /. f_[y, z] → 1, y];
     s1 = Plus @@ Simplify[s1 / mf];
     sxz = Plus @@ Simplify[sxz / mf]; syz = Plus @@ Simplify[syz / mf];
     Print[{mf, s1, sxz, syz}];
     If[
      FreeQ[sxz, y] ∧ FreeQ[syz, x] ∧ FreeQ[s1, x | y] ∧ Simplify[(sxz /. x → y) + syz == 0],
      {{sol}} = Solve[sxz == g_++gn[z], ff];
      rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
      AddRule[ff, rule]; Break[]
     ]
```

```
 ),
 syz == {} ⋀ sxy =!= {} ⋀ sxz =!= {}, (
  {ff} = Cases[sxy, a_. * f_k_[x, y] :> f_k[x, y], {1}, 1];
  mf = MF[First@sxy /. f_[x, y] → 1, y]; mf *= MF[First@sxz /. f_[x, z] → 1, z];
  s1 = Plus @@ Simplify[s1 / mf];
  sxy = Plus @@ Simplify[sxy / mf]; sxz = Plus @@ Simplify[sxz / mf];
  If[FreeQ[sxy, z] ⋀ FreeQ[sxz, y] ⋀ FreeQ[s1, y | z] ⋀
    Simplify[(sxz /. z → y) + sxy == 0],
   {{sol}} = Solve[sxy == g_{++gn}[x], ff];
   rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
   AddRule[ff, rule]; Break[]
  ]
 ),
 sxy =!= {} ⋀ sxz =!= {} ⋀ syz =!= {}, (
  kk = Union@Cases[e2, a_. * f_k_[x, y] :> k, ∞];
  If[Length[kk] == 1,
   {kk} = kk;
   {{sol}} = Solve[e2 == 0, f_kk[x, y]];
   sol = f_kk[x, y] /. sol;
   e3 = D[sol, z] // Factor;
   If[FreeQ[e3, f_[_, _]],
    If[Head[e3] === Times, e3 = Select[e3, ! FreeQ[#, f_^(0,1)[_, _] | g_[_] | h_[]] &]];
    s = Collect[e3, f_^(0,1)[_, _], Factor];
    s1 = Select[s, FreeQ[f_^(0,1)[_, _]]];
    pxz = Coefficient[s, f_kk^(0,1)[x, z]];
    pyz = Coefficient[s, f_kk^(0,1)[y, z]];
    mf = MF[pxz, x]; mf *= MF[pyz, y];
    {s1, pxz, pyz} = Simplify[{s1, pxz, pyz} / mf];
    If[FreeQ[pxz, y] ⋀ FreeQ[pyz, x] ⋀
      FreeQ[s1, x | y] ⋀ Simplify[(pyz /. y → x) + pxz == 0],
     rule = (f_kk[x_, z_] → g_{++gn}[z] / pxz + g_{++gn}[x]) /. Rule → RuleDelayed;
     AddRule[f_kk[x, z], rule]; Break[]
    ]
   ]
  ]
 )
];
If[FreeQ[e2, f_[_, _]] ⋀ ! FreeQ[e2, g_[_]],
 s = List @@ Collect[e2, g_[_], Factor]; s1 = Select[s, FreeQ[g_[_]]];
 sx = Cases[s, a_. * g_[x]]; sy = Cases[s, a_. * g_[y]]; sz = Cases[s, a_. * g_[z]];
 Which[
  FreeQ[e2, x] ⋀ sy =!= {} ⋀ sz =!= {}, (
   {gg} = Cases[sy, a_. * g_k_[y] :> g_k[y], {1}, 1];
   mf = MF[First@sy /. g_[y] → 1, y]; mf *= MF[First@sz /. g_[z] → 1, z];
   s1 = Plus @@ Simplify[s1 / mf];
   sy = Plus @@ Simplify[sy / mf]; sz = Plus @@ Simplify[sz / mf];
   If[FreeQ[sx, y] ⋀ FreeQ[sz, y] ⋀ FreeQ[s1, y | z] ⋀ Simplify[(sz /. z → y) + sy == 0],
    {{sol}} = Solve[sy == h_{++hn}[], gg];
```

```
        rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
        AddRule[gg, rule]; Break[]
       ]
      ),
      FreeQ[e2, z] ⋀ sy =!= {} ⋀ sx =!= {}, (
       {gg} = Cases[sy, a_. * g_k_[y] :> g_k[y], {1}, 1];
       mf = MF[First@sy /. g_[y] → 1, y]; mf *= MF[First@sx /. g_[x] → 1, x];
       s1 = Plus @@ Simplify[s1 / mf];
       sy = Plus @@ Simplify[sy / mf]; sx = Plus @@ Simplify[sx / mf];
       If[FreeQ[sz, y] ⋀ FreeQ[sx, y] ⋀ FreeQ[s1, y | x] ⋀ Simplify[(sx /. x → y) + sy == 0],
        {{sol}} = Solve[sy == h_{++hn}[], gg];
        rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
        AddRule[gg, rule]; Break[]
       ]
      )
     ] (* Which *)
    ] (* If *)
   ] (* If *)
  ] (* For *)
]; (* While *)
E0 = Union[DeleteCases[Simplify[E0 //. EC], 0]] // SortBy[LeafCount];
```

```
Length[E0]==1; Length[EC]==22; {gn,hn}=={8, 1}
```

**EC**

$$\Big\{f_2[x\_,y\_]\to y\,f_9[x,y]+y^2 f_{26}[x,y]-g_6[x],\ f_3[x\_,z\_]\to x^2 f_{18}[x,z],$$
$$f_4[x\_,y\_]\to y\,f_{13}[x,y]-g_1[x],\ f_5[x\_,y\_]\to y^2 f_{26}[x,y],\ f_8[x\_,z\_]\to 0,$$
$$f_{10}[x\_,z\_]\to 2x f_{18}[x,z],\ f_{11}[x\_,y\_]\to 2x f_{18}[x,y]-2e^x x f_{18}[x,y]-g_4[x],$$
$$f_{12}[x\_,z\_]\to 0,\ f_{13}[x\_,z\_]\to x f_{22}[x,z]-g_3[z],\ f_{14}[x\_,y\_]\to y f_{22}[x,y]-g_2[x],$$
$$f_{15}[x\_,y\_]\to \frac{1}{\left(-1+e^y\right)x}\Big(-x y f_{23}[x,y]+e^y x y f_{23}[x,y]+x g_2[x]-e^x x g_2[x]-$$
$$e^y x g_2[x]+e^{x+y} x g_2[x]-e^y y g_2[y]+e^{x+y} y g_2[y]-y g_3[y]+e^x y g_3[y]-y g_4[y]+$$
$$e^x y g_4[y]+8y^2 g_5[y]-8e^x y^2 g_5[y]-16e^y y^2 g_5[y]+8e^{2y} y^2 g_5[y]+16e^{x+y} y^2 g_5[y]-$$
$$8e^{x+2y} y^2 g_5[y]+4y g_7[y]-4e^x y g_7[y]-4e^y y g_7[y]+4e^{x+y} y g_7[y]\Big),\ f_{16}[x\_,z\_]\to 0,$$
$$f_{17}[x\_,y\_]\to 2y f_{26}[x,y],\ f_{18}[x\_,y\_]\to -\frac{f_{19}[x,y]+4g_5[x]-8e^x g_5[x]+4e^{2x} g_5[x]}{2\left(-1+e^x\right)},$$
$$f_{19}[x\_,y\_]\to$$
$$\frac{1}{\left(-1+e^x\right)\left(-1+e^y\right)x^2 y}\Big(-1+2e^x-e^{2x}+e^y-2e^{x+y}+e^{2x+y}+2x^2 y f_{20}[x,y]-2e^y x^2 y f_{20}[x,y]+$$
$$x y g_4[x]-e^x x y g_4[x]-e^y x y g_4[x]+e^{x+y} x y g_4[x]+y^2 g_4[y]-2e^x y^2 g_4[y]+e^{2x} y^2 g_4[y]-$$
$$4y^3 g_5[y]+8e^x y^3 g_5[y]-4e^{2x} y^3 g_5[y]+8e^y y^3 g_5[y]-4e^{2y} y^3 g_5[y]-16e^{x+y} y^3 g_5[y]+$$
$$8e^{2x+y} y^3 g_5[y]+8e^{x+2y} y^3 g_5[y]-4e^{2x+2y} y^3 g_5[y]-e^y y h_1[]+2e^{x+y} y h_1[]-e^{2x+y} y h_1[]\Big),$$
$$f_{21}[x\_,z\_]\to 0,\ f_{22}[x\_,z\_]\to \frac{-x f_{23}[x,z]-4g_7[z]+4e^x g_7[z]}{\left(-1+e^x\right)x},$$
$$f_{24}[x\_,z\_]\to 0,\ f_{25}[x\_,z\_]\to 0,$$
$$g_1[y\_]\to y g_2[y]+h_1[],$$
$$g_2[y\_]\to \frac{1}{2y^2}e^{-y}\Big(-2+2e^y-2y^2 g_3[y]+8y^3 g_5[y]-16e^y y^3 g_5[y]+8e^{2y} y^3 g_5[y]+$$
$$8y^2 g_7[y]-8e^y y^2 g_7[y]-y h_1[]-e^y y h_1[]\Big),\ f_1[x\_,z\_]\to \frac{1}{2}\Bigg(-\frac{1}{\left(-1+e^x\right)^2}$$
$$x\Big(-2\left(-1+e^x\right)^2 f_6[x,z]-2x f_{20}[x,z]+\left(-1+e^x\right)\left(g_4[x]+4\left(-1+e^x\right)^2 x g_5[x]\right)\Big)+g_8[z]\Bigg)\Big\}$$

**E0**

$$\Big\{-1+e^x+e^y-e^{x+y}+2\left(-1+e^x\right)\left(-1+e^y\right)^2 x y f_6[x,y]-2\left(-1+e^x\right)\left(-1+e^y\right)^2 x y f_6[x,z]-$$
$$2y^2 f_6[y,z]+2e^x y^2 f_6[y,z]+4e^y y^2 f_6[y,z]-2e^{2y} y^2 f_6[y,z]-4e^{x+y} y^2 f_6[y,z]+$$
$$2e^{x+2y} y^2 f_6[y,z]+2x y f_7[x,y]-4e^y x y f_7[x,y]+2e^{2y} x y f_7[x,y]-$$
$$2x y f_7[x,z]+4e^y x y f_7[x,z]-2e^{2y} x y f_7[x,z]+2y^2 f_7[y,z]-2e^x y^2 f_7[y,z]-$$
$$2e^y y^2 f_7[y,z]+2e^{x+y} y^2 f_7[y,z]-y^2 g_4[y]+e^x y^2 g_4[y]+2e^y y^2 g_4[y]-$$
$$2e^{x+y} y^2 g_4[y]-4y^3 g_5[y]+4e^x y^3 g_5[y]+8e^y y^3 g_5[y]-4e^{2y} y^3 g_5[y]-$$
$$8e^{x+y} y^3 g_5[y]+4e^{x+2y} y^3 g_5[y]+2e^y y g_6[y]-2e^{2y} y g_6[y]-2e^{x+y} y g_6[y]+$$
$$2e^{x+2y} y g_6[y]-y g_8[y]+e^x y g_8[y]+e^y y g_8[y]-e^{x+y} y g_8[y]-e^y y h_1[]+e^{x+y} y h_1[]\Big\}$$

A specific solution of the remaining equation:

```
E0[[1]] /. {f7[__] → 0, g_[_] → 0, h_[] → 0, f6[x_, y_] :→ 1/(2 x y (e^y - 1))} // Simplify
```

```
0
```

```
eq0 = Collect[E0[[1]]/((-1+e^x)(-1+e^y)), f6[__], FullSimplify]
```

$$2\left(-1+e^{y}\right)xy\,f_6[x,y]-2\left(-1+e^{y}\right)xy\,f_6[x,z]+2\left(-1+e^{y}\right)y^2 f_6[y,z]+\frac{1}{\left(-1+e^{x}\right)\left(-1+e^{y}\right)}\left(2\left(-1+e^{y}\right)^2 xy\,f_7[x,y]-2\left(-1+e^{y}\right)^2 xy\,f_7[x,z]+\left(-1+e^{x}\right)\left(1-e^{y}+y\left(2\left(-1+e^{y}\right)y\,f_7[y,z]+\left(y-2e^{y}y\right)g_4[y]+g_8[y]+e^{y}\left(8y^2\left(-1+\text{Cosh}[y]\right)g_5[y]+2\left(-1+e^{y}\right)g_6[y]-g_8[y]+h_1[\,]\right)\right)\right)\right)$$

```
eq1 = 1/(1/(-1+e^x)^2 2(-1+e^y)y) D[eq0, x] // Simplify
```

$$\left(-1+e^{x}\right)^2 f_6[x,y]-\left(-1+e^{x}\right)^2 f_6[x,z]-f_7[x,y]+e^{x}f_7[x,y]-e^{x}x\,f_7[x,y]+f_7[x,z]-e^{x}f_7[x,z]+e^{x}x\,f_7[x,z]+x\,f_6^{(1,0)}[x,y]-2e^{x}x\,f_6^{(1,0)}[x,y]+e^{2x}x\,f_6^{(1,0)}[x,y]-x\,f_6^{(1,0)}[x,z]+2e^{x}x\,f_6^{(1,0)}[x,z]-e^{2x}x\,f_6^{(1,0)}[x,z]-x\,f_7^{(1,0)}[x,y]+e^{x}x\,f_7^{(1,0)}[x,y]+x\,f_7^{(1,0)}[x,z]-e^{x}x\,f_7^{(1,0)}[x,z]$$

```
(eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0) + (eq1 /. f_[x, z] | f_^(1,0)[x, z] → 0) == eq1 // Simplify
```

True

```
FreeQ[(eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0), y]
```

True

```
FreeQ[(eq1 /. f_[x, z] | f_^(1,0)[x, z] → 0), z]
```

True

```
Collect[(-eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0), f6[x, z] | f6^(1,0)[x, z], Simplify]
```

$$\left(-1+e^{x}\right)^2 f_6[x,z]+\left(-1-e^{x}\left(-1+x\right)\right)f_7[x,z]+\left(-1+e^{x}\right)^2 x\,f_6^{(1,0)}[x,z]+\left(-1+e^{x}\right)x\,f_7^{(1,0)}[x,z]$$

```
DSolve[D[a[x] (-1+e^x)^2 x, x] == (-1+e^x)^2 a[x], a[x], x]
```

$$\left\{\left\{a[x]\to\frac{C[1]}{\left(-1+e^{x}\right)^2}\right\}\right\}$$

```
eq2 =
 Collect[1/(-1+e^x)^2 (-eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0), f6[x, z] | f6^(1,0)[x, z], Simplify]
```

$$f_6[x,z]+x\,f_6^{(1,0)}[x,z]+\frac{1}{\left(-1+e^{x}\right)^2}\left(\left(-1-e^{x}\left(-1+x\right)\right)f_7[x,z]+\left(-1+e^{x}\right)x\,f_7^{(1,0)}[x,z]\right)$$

```
Collect[1/(-1+e^x)^2 (-eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0), f7[x, z] | f7^(1,0)[x, z], Simplify]
```

$$f_6[x,z]-\frac{\left(1+e^{x}\left(-1+x\right)\right)f_7[x,z]}{\left(-1+e^{x}\right)^2}+x\,f_6^{(1,0)}[x,z]+\frac{x\,f_7^{(1,0)}[x,z]}{-1+e^{x}}$$

```
D[x f6[x, z] + (x f7[x, z])/(-1 + e^x), x] - eq2 // Simplify
```

```
0
```

```
Simplify[f6[x, z] /. First@Solve[x f6[x, z] + (x f7[x, z])/(-1 + e^x) == g9[x] + g10[z], f6[x, z]]]
```

$$\frac{-\frac{x\, f_7[x,z]}{-1+e^x} + g_9[x] + g_{10}[z]}{x}$$

```
eq3 = Simplify[eq0 /. f6[x_, z_] → (-(x f7[x,z])/(-1+e^x) + g9[x] + g10[z])/x]
```

$$\frac{1}{-1+e^y}\left(1 - e^y - \left(-1+2e^y\right) y^2 g_4[y] + 4\left(-1+e^y\right)^2 y^3 g_5[y] - 2e^y y\, g_6[y] + 2e^{2y} y\, g_6[y] + y\, g_8[y] - e^y y\, g_8[y] + 2y\, g_9[y] - 4e^y y\, g_9[y] + 2e^{2y} y\, g_9[y] + 2y\, g_{10}[y] - 4e^y y\, g_{10}[y] + 2e^{2y} y\, g_{10}[y] + e^y y\, h_1[\,]\right)$$

```
First@Solve[eq3 == 0, g10[y]]
```

$$\left\{g_{10}[y] \to \frac{1}{2\left(-1+e^y\right)^2 y}\left(-1+e^y - y^2 g_4[y] + 2e^y y^2 g_4[y] - 4y^3 g_5[y] + 8e^y y^3 g_5[y] - 4e^{2y} y^3 g_5[y] + 2e^y y\, g_6[y] - 2e^{2y} y\, g_6[y] - y\, g_8[y] + e^y y\, g_8[y] - 2y\, g_9[y] + 4e^y y\, g_9[y] - 2e^{2y} y\, g_9[y] - e^y y\, h_1[\,]\right)\right\}$$

```
Simplify[eq0 /. z → y]
```

$$2\left(-1+e^y\right) y^2 f_6[y,y] + \frac{1}{-1+e^y}\left(1 - e^y + y\left(2\left(-1+e^y\right) y\, f_7[y,y] + \left(y - 2e^y y\right) g_4[y] + g_8[y] + e^y\left(8y^2\left(-1+\text{Cosh}[y]\right) g_5[y] + 2\left(-1+e^y\right) g_6[y] - g_8[y] + h_1[\,]\right)\right)\right)$$

```
Simplify[eq0 /. x → y]
```

$$\frac{1}{-1+e^y}\left(1 - e^y + 2\left(-1+e^y\right)^2 y^2 f_6[y,y] + 2\left(-1+e^y\right) y^2 f_7[y,y] + y^2 g_4[y] - 2e^y y^2 g_4[y] - 8e^y y^3 g_5[y] + 8e^y y^3 \text{Cosh}[y]\, g_5[y] - 2e^y y\, g_6[y] + 2e^{2y} y\, g_6[y] + y\, g_8[y] - e^y y\, g_8[y] + e^y y\, h_1[\,]\right)$$

```
Simplify[eq0 /. z → x]
```

$$-2\left(-1+e^y\right) x y\, f_6[x,x] + 2\left(-1+e^y\right) x y\, f_6[x,y] + 2\left(-1+e^y\right) y^2 f_6[y,x] + \frac{1}{\left(-1+e^x\right)\left(-1+e^y\right)}\left(-2\left(-1+e^y\right)^2 x y\, f_7[x,x] + 2\left(-1+e^y\right)^2 x y\, f_7[x,y] + \left(-1+e^x\right)\left(1 - e^y + y\left(2\left(-1+e^y\right) y\, f_7[y,x] + \left(y - 2e^y y\right) g_4[y] + g_8[y] + e^y\left(8y^2\left(-1+\text{Cosh}[y]\right) g_5[y] + 2\left(-1+e^y\right) g_6[y] - g_8[y] + h_1[\,]\right)\right)\right)\right)$$

```
Simplify[E0 /. f6[x_, z_] → (-(x f7[x, z])/(-1+e^x) + g9[x] + g10[z])/x /. g10[y_] → 1/(2 (-1 + e^y)^2 y)]
```

$$\left(-1+e^{y}-y^{2} g_4[y]+2 e^{y} y^{2} g_4[y]-4 y^{3} g_5[y]+8 e^{y} y^{3} g_5[y]-4 e^{2 y} y^{3} g_5[y]+2 e^{y} y g_6[y]-2 e^{2 y} y g_6[y]-y g_8[y]+e^{y} y g_8[y]-2 y g_9[y]+4 e^{y} y g_9[y]-2 e^{2 y} y g_9[y]-e^{y} y h_1[]\right)$$

$$\{0\}$$