

Pensieve header: Implementing the scapegoated VdV algebra of <http://drorbn.net/bbs/show?shot=VanDerVeen-160731-121550.jpg>.

```

ε = 0;
PBWBasis = {c, u, w};
B[U@c, U@w] = - (B[U@w, U@c] = U@w);
B[U@u, U@c] = - (B[U@c, U@u] = U@u);
B[U@w, U@u] = - (B[U@u, U@w] = (t - 1) U[] (* -ε U[u,w] + ε t U[] + 2ε t U[c]*));

U_i[_ε_] := ε /. {t → t_i, u_U → Replace[u, x_ → x_i, 1]};

B[x_, x_] = 0;
B[U[(x_)i], U[(y_)i]] := U_i[B[U@x, U@y]];
B[U[(x_)i], U[(y_)j]] /. i != j := 0;
B[x_, y_] := x**y - y**x;

x_ ≤ y_ := OrderedQ[{x, y}]; x_ < y_ := !OrderedQ[{y, x}];
Simp[_ε_] := Collect[_ε_, _U, Expand];

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (#**y) & /@ x; x_ ** (y_Plus) := (x**#) & /@ y;
U[x_] ** U[y_] := (*U[x]**U[y] =*) If[x < y, U[x, y], U[y, x] + B[U@x, U@y]];
U[x_] ** U[y1_, yy_] := (*U[x] ** U[y1,yy] =*)
  If[x ≤ y1, U[x, y1, yy], (U@x**U@y1) ** U@yy];
U[xx_, xn_] ** U[yy_] := (*U[xx,xn]**U[yy] =*) U@xx ** (U@xn ** U@yy);

U[l___, x^-n_, r___] := U[l, Sequence@@Table[x, {n}], r];
U[l___, 1, r___] := U[l, r];

LBasis[n_Integer] := LBasis[Range[n]];
LBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[({ε → 2, c_ → 2, u_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ε U[]},
      Table[{U@c_i, U@u_i, U@w_i, ε U@c_i, ε U@u_i, ε U@w_i}, {i, S}],
      Table[{U@u_i, w_j}, ε U@u_i, w_j],
        ε U@@Sort@{c_i, c_j}, ε U@c_i, u_j}, ε U@c_i, w_j}}, {i, S}, {j, S}],
      Table[{ε U@c_i, u_j, w_k}, ε U@@Sort@{u_i, u_j, w_k}, ε U@@Sort@{u_i, w_j, w_k}},
        {i, S}, {j, S}, {k, S}],
      Table[ε U@@Sort@{u_i, u_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}]]]
  ]

```

```

BLBasis[n_Integer] := BLBasis[Range[n]];
BLBasis[S_] := DeleteCases[0]@
Module[{i, j, k, l}, SortBy[({# /. {e -> 2, c_ -> 2, u_ -> 2, w_ -> 2, U -> Times}) &][
  Union@Flatten[{{U[], e U[]},
    Table[{U@c_i, e U@c_i}, {i, S}],
    Table[{U[u_i, w_j], e U[u_i, w_j], e U@@Sort@{c_i, c_j}}, {i, S}, {j, S}],
    Table[{e U[c_i, u_j, w_k]}, {i, S}, {j, S}, {k, S}],
    Table[{e U@@Sort@{u_i, u_j, w_k, w_l}}, {i, S}, {j, S}, {k, S}, {l, S}]}]]
]

UExp[ε_, n_] := Module[{t = U[], k}, U[] + Sum[ $\frac{t = t ** \epsilon}{k!}$ , {k, n}]] // Simp

ToDegree[n_][ε_] :=
Simp[ε] /. {e -> ħ e, b_i -> ħ b_i, t_i -> eħ b_i, b -> ħ b, t -> eħ b, x_U -> ħCount[x, u|u_] x} /.
a_ . x_U -> Series[a, {ħ, 0, n}] * x

```

Testing AS and Jacobi

```

B[U@w, U@c]
U[w]

B[U@u, U@c]
-U[u]

B[U@w, U@u]
-(-1 + t) U[]

BLBasis[2]
{U[], U[c_1], U[c_2], U[u_1, w_1], U[u_1, w_2], U[u_2, w_1], U[u_2, w_2]}

bas = BLBasis[2]; Table[B[x, y] + B[y, x], {x, bas}, {y, bas}] // Flatten // Union
{0}

bas = BLBasis[2]; Timing[
Table[
  {x, y, z} = xyz;
  Simp[B[B[x, y], z] + B[B[y, z], x] + B[B[z, x], y]],
  {xyz, Subsets[bas, {3}]}
] // Flatten // Union
]
{0.046875, {0}}

```

Testing Yang-Baxter

```
R[i_, j_, d_] := Module[{nn, p},
  Sum[
    Sum[nn = p - m;
      
$$\frac{b_i^m}{(m!) (nn!)}$$
 U[u_i^nn, c_j^m, w_j^nn],
      {m, 0, p}],
    {p, 0, d}
  ]
]
```

```
R[i, j, 2]
```

$$U[] + b_i U[c_j] + \frac{1}{2} b_i^2 U[c_j, c_j] + U[u_i, w_j] + b_i U[u_i, c_j, w_j] + \frac{1}{2} U[u_i, u_i, w_j, w_j]$$

```
R[i, j, 3]
```

$$U[] + b_i U[c_j] + \frac{1}{2} b_i^2 U[c_j, c_j] + U[u_i, w_j] + \frac{1}{6} b_i^3 U[c_j, c_j, c_j] + b_i U[u_i, c_j, w_j] + \frac{1}{2} b_i^2 U[u_i, c_j, c_j, w_j] + \frac{1}{2} U[u_i, u_i, w_j, w_j] + \frac{1}{2} b_i U[u_i, u_i, c_j, w_j, w_j] + \frac{1}{6} U[u_i, u_i, u_i, w_j, w_j, w_j]$$

```
Series[ $\frac{1 - e^{-b_1}}{b_1}$ , {b_1, 0, 3}]
```

$$1 - \frac{b_1}{2} + \frac{b_1^2}{6} - \frac{b_1^3}{24} + O[b_1]^4$$

```
Simp[R[1, 2, 3]] // ToDegree[3]
```

$$U[] + (b_1 U[c_2] + U[u_1, w_2]) \hbar + \left(\frac{1}{2} b_1^2 U[c_2, c_2] + b_1 U[u_1, c_2, w_2] + \frac{1}{2} U[u_1, u_1, w_2, w_2] \right) \hbar^2 + \left(\frac{1}{6} b_1^3 U[c_2, c_2, c_2] + \frac{1}{2} b_1^2 U[u_1, c_2, c_2, w_2] + \frac{1}{2} b_1 U[u_1, u_1, c_2, w_2, w_2] + \frac{1}{6} U[u_1, u_1, u_1, w_2, w_2, w_2] \right) \hbar^3 + O[\hbar]^4$$

```
Simp[R[1, 2, 3] /. x_U ->  $\left(\frac{t_1 - 1}{b_1}\right)^{\text{Count}[x, u_1]} x$ ] // ToDegree[3]
```

$$U[] + (b_1 U[c_2] + U[u_1, w_2]) \hbar + \left(\frac{1}{2} b_1^2 U[c_2, c_2] + \frac{1}{2} b_1 U[u_1, w_2] + b_1 U[u_1, c_2, w_2] + \frac{1}{2} U[u_1, u_1, w_2, w_2] \right) \hbar^2 + \left(\frac{1}{6} b_1^2 U[u_1, w_2] + \frac{1}{6} b_1^3 U[c_2, c_2, c_2] + \frac{1}{2} b_1^2 U[u_1, c_2, w_2] + \frac{1}{2} b_1^2 U[u_1, c_2, c_2, w_2] + \frac{1}{2} b_1 U[u_1, u_1, w_2, w_2] + \frac{1}{2} b_1 U[u_1, u_1, c_2, w_2, w_2] + \frac{1}{6} U[u_1, u_1, u_1, w_2, w_2, w_2] \right) \hbar^3 + O[\hbar]^4$$

With[{**d** = 2}, **R**[1, 2, **d**] ** **R**[1, 3, **d**] ** **R**[2, 3, **d**] // **ToDegree**[**d**]

$$\begin{aligned}
 &U[] + (b_1 U[c_2] + (b_1 + b_2) U[c_3] + U[u_1, w_2] + U[u_1, w_3] + U[u_2, w_3]) \hbar + \\
 &\left(\frac{1}{2} b_1^2 U[c_2, c_2] + (b_1^2 + b_1 b_2) U[c_2, c_3] + \frac{1}{2} (b_1^2 + 2 b_1 b_2 + b_2^2) U[c_3, c_3] + \right. \\
 &\quad b_1 U[c_2, u_1, w_3] + b_1 U[c_2, u_2, w_3] + (b_1 + b_2) U[c_3, u_1, w_2] + b_2 U[c_3, u_1, w_3] + \\
 &\quad b_1 U[c_3, u_2, w_3] + b_1 U[u_1, c_2, w_2] + b_1 U[u_1, c_3, w_3] + b_2 U[u_2, c_3, w_3] + \\
 &\quad \frac{1}{2} U[u_1, u_1, w_2, w_2] + U[u_1, u_1, w_2, w_3] + \frac{1}{2} U[u_1, u_1, w_3, w_3] + \\
 &\quad \left. U[u_1, u_2, w_2, w_3] + U[u_1, u_2, w_3, w_3] + \frac{1}{2} U[u_2, u_2, w_3, w_3] \right) \hbar^2 + O[\hbar]^3
 \end{aligned}$$

R3[**d**_] := (**R**[1, 2, **d**] ** **R**[1, 3, **d**] ** **R**[2, 3, **d**]) - (**R**[2, 3, **d**] ** **R**[1, 3, **d**] ** **R**[1, 2, **d**])

R3[1] // **Simp** // **ToDegree**[1]

$$O[\hbar]^3$$

R3[2] // **ToDegree**[2]

$$O[\hbar]^3$$

R3[2] // **ToDegree**[3]

$$\begin{aligned}
 &((b_1^2 + b_1 b_2) U[c_2, c_3, u_1, w_2] - b_1^2 U[c_2, c_3, u_1, w_3] - \\
 &\quad b_1 b_2 U[c_2, c_3, u_2, w_3] + b_1^2 U[c_2, u_1, c_3, w_3] + b_1 b_2 U[c_2, u_2, c_3, w_3] + \\
 &\quad b_1 b_2 U[c_3, c_3, u_1, w_3] - b_1 b_2 U[c_3, c_3, u_2, w_3] + (-b_1^2 - b_1 b_2) U[c_3, u_1, c_2, w_2] - \\
 &\quad b_1 b_2 U[c_3, u_1, c_3, w_3] + b_1 b_2 U[c_3, u_2, c_3, w_3]) \hbar^3 + O[\hbar]^4
 \end{aligned}$$

With[{**p** = 3}, **Sum**[**n** = **p** - **m**;

$$\begin{aligned}
 &\frac{b_i^m}{m! n!} U[u_i^n, c_j^m, w_j^n], \\
 &\{\mathbf{m}, \mathbf{0}, \mathbf{p}\}]
 \end{aligned}$$

$$\begin{aligned}
 &\frac{1}{6} b_i^3 U[c_j, c_j, c_j] + \frac{1}{2} b_i^2 U[u_i, c_j, c_j, w_j] + \\
 &\frac{1}{2} b_i U[u_i, u_i, c_j, w_j, w_j] + \frac{1}{6} U[u_i, u_i, u_i, w_j, w_j, w_j]
 \end{aligned}$$

B[**U**@**u**, **U**@**w**]

$$(-1 + t) U[]$$

B[**U**@**u**, **U**@**c**]

$$-U[u]$$

With[{**p** = 1}, **Sum**[**n** = **p** - **m**;

$$\begin{aligned}
 &\frac{b_i^m}{m! n!} U[u_i^n, c_j^m, w_j^n], \\
 &\{\mathbf{m}, \mathbf{0}, \mathbf{p}\}]
 \end{aligned}$$

$$b_i U[c_j] + U[u_i, w_j]$$

a_{*i*,*j*} := **b**_{*i*} **U**[**c**_{*j*}] + **U**[**u**_{*i*}, **w**_{*j*}]

B[**a**_{1,2}, **a**_{1,2}] + **B**[**a**_{1,2}, **a**_{2,3}] + **B**[**a**_{1,3}, **a**_{2,3}] // **Simplify**

(1 + b₂ - t₂) U[u₁, w₃]

B[**a**_{1,2}, **a**_{1,2}] + **B**[**a**_{1,2}, **a**_{2,3}] + **B**[**a**_{1,3}, **a**_{2,3}] // **ToDegree**[3]

$-\frac{1}{2} (b_2^2 U[u_1, w_3]) \hbar^3 + O[\hbar]^4$

{**i**, **j**}

{**i**, **j**}

b

b

n

0

Simp[**Module**[{**nn**, **p**},

Sum[

Sum[**nn** = **p** - **m**;

$\frac{b_i^m}{(m!) (nn!)} U[u_i^{nn}, c_j^m, w_j^{nn}]$,

{**m**, 0, **p**},

{**p**, 0, 3}

]

] /. **x_U** => $\left(\frac{1 - t_i}{b_i}\right)^{\text{Count}[x, u_i]} x$ // **ToDegree**[3]

U[] + (b_i U[c_j] - U[u_i, w_j]) \hbar +

$\left(\frac{1}{2} b_i^2 U[c_j, c_j] - \frac{1}{2} b_i U[u_i, w_j] - b_i U[u_i, c_j, w_j] + \frac{1}{2} U[u_i, u_i, w_j, w_j]\right) \hbar^2 +$

$\left(-\frac{1}{6} b_i^2 U[u_i, w_j] + \frac{1}{6} b_i^3 U[c_j, c_j, c_j] - \frac{1}{2} b_i^2 U[u_i, c_j, w_j] - \frac{1}{2} b_i^2 U[u_i, c_j, c_j, w_j] +$

$\frac{1}{2} b_i U[u_i, u_i, w_j, w_j] + \frac{1}{2} b_i U[u_i, u_i, c_j, w_j, w_j] - \frac{1}{6} U[u_i, u_i, u_i, w_j, w_j, w_j]\right) \hbar^3 + O[\hbar]^4$

R[**i**_, **j**_, **d**_] := **Module**[{**nn**, **p**},

Sum[

Sum[**nn** = **p** - **m**;

$\frac{b_i^m}{(m!) (nn!)} U[u_i^{nn}, c_j^m, w_j^{nn}]$,

{**m**, 0, **p**},

{**p**, 0, **d**}

]

]

$$\text{Simp}[\mathbf{R}[1, 2, 3] /. \mathbf{x_U} \rightarrow \left(\frac{1 - \mathbf{t}_1}{\mathbf{b}_1}\right)^{\text{Count}[\mathbf{x}, \mathbf{u}_1]} \mathbf{x}] // \text{ToDegree}[3]$$

$$\begin{aligned} &U[] + (b_1 U[c_2] - U[u_1, w_2]) \hbar + \\ &\left(\frac{1}{2} b_1^2 U[c_2, c_2] - \frac{1}{2} b_1 U[u_1, w_2] - b_1 U[u_1, c_2, w_2] + \frac{1}{2} U[u_1, u_1, w_2, w_2]\right) \hbar^2 + \\ &\left(-\frac{1}{6} b_1^2 U[u_1, w_2] + \frac{1}{6} b_1^3 U[c_2, c_2, c_2] - \frac{1}{2} b_1^2 U[u_1, c_2, w_2] - \frac{1}{2} b_1^2 U[u_1, c_2, c_2, w_2] + \right. \\ &\quad \left. \frac{1}{2} b_1 U[u_1, u_1, w_2, w_2] + \frac{1}{2} b_1 U[u_1, u_1, c_2, w_2, w_2] - \frac{1}{6} U[u_1, u_1, u_1, w_2, w_2, w_2]\right) \hbar^3 + O[\hbar]^4 \end{aligned}$$

$$\mathbf{B}[\mathbf{R}[1, 2, 3], \mathbf{U@u}_1] // \text{ToDegree}[3]$$

$$(b_1 U[c_2, u_1, u_1, w_2] - b_1 U[u_1, u_1, c_2, w_2]) \hbar^3 + O[\hbar]^4$$

$$\mathbf{R}[1, 2, 3] ** \mathbf{U@u}_1$$

$$\begin{aligned} &U[u_1] + b_1 U[c_2, u_1] + \frac{1}{2} b_1^2 U[c_2, c_2, u_1] + U[u_1, u_1, w_2] + \frac{1}{6} b_1^3 U[c_2, c_2, c_2, u_1] + \\ &b_1 U[c_2, u_1, u_1, w_2] + \frac{1}{2} b_1^2 U[c_2, c_2, u_1, u_1, w_2] + \frac{1}{2} U[u_1, u_1, u_1, w_2, w_2] + \\ &\frac{1}{2} b_1 U[c_2, u_1, u_1, u_1, w_2, w_2] + \frac{1}{6} U[u_1, u_1, u_1, u_1, w_2, w_2, w_2] \end{aligned}$$