Pensieve header: Normally Ordered Exponentials at 1-Co in the t variables.

For pragmatic reasons, $\mathbb{E}[\omega, L, Q, P]$ means $\omega^{-1}(1+\epsilon\,\omega^{-4}\,P)\,\mathrm{Exp}[L+\omega^{-1}\,Q]$, where ω is an ϵ-free scalar, L is linear and contains only c's and b's, Q is a balanced quadratic in the u's and the w's and contains no c's and b's, and P is a balanced quartic polynomial in the c's, u's, and w's, . $\mathbb{E}$ is also a casting operator: $\mathbb{E}[\omega^{-1}(1+\epsilon\,\omega^{-4}\,P)\,\mathrm{Exp}[L+\omega^{-1}\,Q]]$ returns $\mathbb{E}[\omega, L, Q, P]$, meaning $\mathbb{E}[G\,\mathrm{Exp}[L+Q]]$ computes $\omega=(G\mid_{\epsilon=0})^{-1}$ and returns $\mathbb{E}[\omega, L, \omega\,Q, \omega^5\,\partial_\epsilon G,]$.

```
DP[P_, x_ → Dα_, y_ → Dβ_][f_] :=
 Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :> c D[f, {α, m}, {β, n}]]
```

```
𝔼 /: Simplify[𝔼[ω_, L_, Q_, P_]] := 𝔼[Expand@Together[ω /. bL_ :> Log[tL]],
   Expand[L], Expand@Together[Q /. bL_ :> Log[tL]], Expand@Together[P /. bL_ :> Log[tL]]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] ≡ 𝔼[ω2_, L2_, Q2_, P2_] := (ω1 == ω2 ⋀ L1 == L2 ⋀ Q1 == Q2 ⋀ P1 == P2);
𝔼[G_. ℯ^F_] /; FreeQ[G, ℯ_] :=
  With[{ω = (G /. ϵ → 0)^-1}, Simplify@𝔼[ω, F /. u_ | w_ | α | β → 0, ω F /. c_ → 0, ω^5 ∂ϵ G]];
𝔼[G_] /; FreeQ[G, ℯ_] := With[{ω = (G /. ϵ → 0)^-1}, Simplify@𝔼[ω, 0, 0, ω^5 ∂ϵ G]];
𝔼[ℰ_] := 𝔼[Factor[ℰ]];
```

```
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  Simplify@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

```
NO[ui_, cj_, k_][𝔼[ω_, L_, Q_, P_]] := 𝔼[1, L /. cj → 0, ω^-1 Q /. ui → 0, 0] (
    𝔼[ω^-5 DP[ω^4 + ϵ P, cj → Dγ, ui → Dβ][ℯ^(ℯ^-γ β uk + γ ck)] /. {γ → ∂cj L, β → ω^-1 ∂ui Q}]);
```

```
NO[wi_, cj_, k_][𝔼[ω_, L_, Q_, P_]] := 𝔼[1, L /. cj → 0, ω^-1 Q /. wi → 0, 0] (
    𝔼[ω^-5 DP[ω^4 + ϵ P, cj → Dγ, wi → Dβ][ℯ^(ℯ^γ β wk + γ ck)] /. {γ → ∂cj L, β → ω^-1 ∂wi Q}]);
```

```
NO[wi_, uj_, k_][𝔼[ω_, L_, Q_, P_]] := With[{q = (1 - tk) ν α β + ν β uk + ν δ uk wk + ν α wk},
  𝔼[1, L, ω^-1 Q /. wi | uj → 0, 0] (
    𝔼[ν ω^-5 DP[ω^4 + ϵ P, wi → Dα, uj → Dβ][ℯ^q] + ϵ ν^5 ω^-1 Λ ℯ^q] /. {
      ν → (1 + (tk - 1) δ)^-1,
      Λ → -1/2 (-1 + tk) (α^2 β^2 + 4 α β δ (1 + (-1 + tk) δ) + 2 δ^2 (1 + (-1 + tk) δ)^2) +
        2 (1 + (-1 + tk) δ)^2 (α β + δ + (-1 + tk) δ^2) ck - β (1 + 2 (-1 + tk) δ) (α β + 2 δ (1 + (-1 + tk) δ)) uk +
        2 β δ (1 + (-1 + tk) δ)^2 ck uk - 1/2 β^2 δ (2 + 3 (-1 + tk) δ) uk^2 + α (α β + 2 δ (1 + (-1 + tk) δ)) wk +
        2 α δ (1 + (-1 + tk) δ)^2 ck wk - 2 (-1 + tk) δ^2 (α β + δ (1 + (-1 + tk) δ)) uk wk + 2 δ^2 (1 + (-1 + tk) δ)^2
         ck uk wk - β δ^2 (1 + 2 (-1 + tk) δ) uk^2 wk + 1/2 α^2 δ (2 + (-1 + tk) δ) wk^2 + α δ^2 uk wk^2 - 1/2 (-1 + tk) δ^4 uk^2 wk^2
    } /. {α → ω^-1 (∂wi Q /. uj → 0), β → ω^-1 (∂uj Q /. wi → 0), δ → ω^-1 ∂wi,uj Q}]];
```

```
m[i_, j_, kk_][Z_] := Module[{x, y},
  Z // ReplaceAll[{bi|j → bkk, ti|j → tkk}] // NO[wi, cj, x] // NO[wx, uj, y] //
    ReplaceAll[{cx|y → cx, wj → wy}] // NO[ui, cx, x] //
   ReplaceAll[{ci|x → ckk, ux|y → ukk, wy → wkk, bx|y → bkk, tx|y → tkk}] // Simplify]
```

Done line

```
Q0 = 𝔼[ⅇ^(u1 w1 + u2 w3)];
t1 = Q0 // m[1, 2, 4]
```

$$\mathbb{E}\left[1,\ 0,\ 2\,u_4\,w_3 - t_4\,u_4\,w_3 + u_4\,w_4,\ -2\,u_4\,w_3 + 2\,c_4\,u_4\,w_3 - \frac{5}{2}\,u_4^2\,w_3^2 + \frac{3}{2}\,t_4\,u_4^2\,w_3^2 - u_4^2\,w_3\,w_4\right]$$

```
Q0 = 𝔼[ⅇ^(u1 w1 + u3 w2)];

Q0 = 𝔼[Exp[Sum[a_{i,j} u_i w_j + l_{i,j} b_i c_j, {i, 3}, {j, 3}]]]
```

$$\mathbb{E}[1,\ b_1\,c_1\,l_{1,1} + b_1\,c_2\,l_{1,2} + b_1\,c_3\,l_{1,3} + b_2\,c_1\,l_{2,1} + b_2\,c_2\,l_{2,2} + b_2\,c_3\,l_{2,3} + b_3\,c_1\,l_{3,1} + b_3\,c_2\,l_{3,2} + b_3\,c_3\,l_{3,3},$$
$$u_1\,w_1\,a_{1,1} + u_1\,w_2\,a_{1,2} + u_1\,w_3\,a_{1,3} + u_2\,w_1\,a_{2,1} + u_2\,w_2\,a_{2,2} + u_2\,w_3\,a_{2,3} + u_3\,w_1\,a_{3,1} + u_3\,w_2\,a_{3,2} + u_3\,w_3\,a_{3,3},\ 0]$$

```
SeedRandom[6];
Q0 = 𝔼[Exp[Sum[RandomInteger[{-2, 2}] u_i w_j + RandomInteger[{-2, 2}] b_i c_j, {i, 3}, {j, 3}]]]
t1 = Q0 // m[1, 2, 4] // m[4, 3, 5]
t2 = Q0 // m[2, 3, 4] // m[1, 4, 5]
t3 = (t1 ≡ t2)
```

$$\mathbb{E}[1,\ -2\,b_1\,c_1 - 2\,b_2\,c_1 + 2\,b_3\,c_1 + b_1\,c_2 - 2\,b_3\,c_2 - b_1\,c_3 - b_2\,c_3,$$
$$-2\,u_1\,w_1 + u_2\,w_1 + 2\,u_3\,w_1 - 2\,u_1\,w_2 + 2\,u_2\,w_2 - u_1\,w_3 + 2\,u_2\,w_3 - u_3\,w_3,\ 0]$$

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$$\mathrm{NO}[u_4,\ c_5,\ x\$333]\left[\mathrm{NO}[w_{x\$333},\ u_3,\ y\$333]\left[\mathrm{NO}[w_4,\ c_3,\ x\$333]\left[\right.\right.\right.$$
$$\mathrm{Hold}\left[\mathbb{E}\left[\mathrm{Factor}\left[\frac{1}{2\,t_5^2}\,ⅇ^{-b_5+(-b_5+b_5)\,c_5+ⅇ^{b_5-b_5}\,u_4\,w_5}\left(2\,ⅇ^{b_5}\,t_5^2 + 4\,ⅇ^{b_5} \in c_5\,t_5\,t_5\,u_3\,w_5 + ⅇ^{b_5} \in t_5^2\,u_3^2\,w_5^2 - ⅇ^{b_5} \in t_5^2\,t_5\,u_3^2\,w_5^2 - \right.\right.\right.\right.$$
$$4\,ⅇ^{b_5} \in t_5\,t_5\,u_3\,u_4\,w_5^2 + 2\,ⅇ^{b_5} \in t_5\,t_5\,u_3\,u_4\,w_5^2 - 8\,ⅇ^{b_5} \in c_5\,t_5\,t_5\,u_3\,w_4 - 6\,ⅇ^{b_5} \in t_5^2\,u_3^2\,w_5\,w_4 + 4\,ⅇ^{b_5} \in t_5^2\,t_5\,u_3^2\,w_5\,w_4 +$$
$$\left.\left.\left.\left.\left.8\,ⅇ^{b_5} \in t_5\,t_5\,u_3\,u_4\,w_5\,w_4 - 8\,ⅇ^{b_5} \in t_5\,t_5\,u_3\,u_4\,w_5\,w_4 + 8\,ⅇ^{b_5} \in t_5^2\,u_3^2\,w_4^2 - 4\,ⅇ^{b_5} \in t_5^2\,t_5\,u_3^2\,w_4^2 + 8\,ⅇ^{b_5} \in t_5\,t_5\,u_3\,u_4\,w_4^2\right)\right]\right]\right]$$
$$\left.\left.\left.\mathbb{E}\left[1,\ b_5\,(2\,c_3 - c_5),\ \left(\left(-1 + t_5\right)\,u_3 + u_4\right)\,\left(2\,w_4 - w_5\right),\ 0\right]\right]\right]\right] \equiv t2$$

```
Simplify[t3 /. b5 → Log[t5]]
```

$$\frac{\left(-1 + t_5\right)\,\left(1 + 2\,t_5^2 - 4\,t_5^3 + 2\,t_5^4\right)}{t_5} == 0\ \&\&\ \mathrm{Log}[t_5] == 0\ \&\&$$
$$\frac{1}{t_5}\left(-6\,\mathrm{Log}[t_5] + 10\,\mathrm{Log}[t_5]\,t_5^5 + 5\,u_5\,w_5 + 5\,t_5\,\left(\mathrm{Log}[t_5] - 2\,u_5\,w_5\right) - 2\,t_5^4\,\left(16\,\mathrm{Log}[t_5] + u_5\,w_5\right) + \right.$$
$$\left.2\,t_5^3\,\left(17\,\mathrm{Log}[t_5] + u_5\,w_5\right) + t_5^2\,\left(-12\,\mathrm{Log}[t_5] + 5\,u_5\,w_5\right)\right) == 0\ \&\&$$
$$\frac{1}{t_5}\left(-1 + t_5^4\right)\,\left(-64\,t_5^{17} - 6\,u_5^2\,w_5^2 + 4\,t_5\,u_5^2\,w_5^2 + t_5^{16}\,\left(480 - 64\,u_5\,w_5\right) - 4\,t_5^4\,u_5\,w_5\,\left(3 + 38\,u_5\,w_5\right) - t_5^2\,u_5\,w_5\,\left(8 + 51\,u_5\,w_5\right) + 4\,t_5^5\,u_5\right.$$
$$w_5\,\left(21 + 95\,u_5\,w_5\right) + t_5^3\,u_5\,w_5\,\left(8 + 127\,u_5\,w_5\right) + t_5^{12}\,\left(2176 + 6384\,u_5\,w_5 - 992\,u_5^2\,w_5^2\right) + t_5^{10}\,\left(928 + 7456\,u_5\,w_5 - 388\,u_5^2\,w_5^2\right) +$$
$$t_5^6\,\left(24 + 216\,u_5\,w_5 - 326\,u_5^2\,w_5^2\right) - 16\,t_5^{15}\,\left(96 - 16\,u_5\,w_5 + u_5^2\,w_5^2\right) + 16\,t_5^{14}\,\left(170 + 2\,u_5\,w_5 + u_5^2\,w_5^2\right) +$$
$$16\,t_5^{13}\,\left(-184 - 141\,u_5\,w_5 + 16\,u_5^2\,w_5^2\right) - 16\,t_5^9\,\left(29 + 239\,u_5\,w_5 + 100\,u_5^2\,w_5^2\right) - 4\,t_5^7\,\left(16 + 236\,u_5\,w_5 + 223\,u_5^2\,w_5^2\right) +$$
$$4\,t_5^{11}\,\left(-352 - 2264\,u_5\,w_5 + 361\,u_5^2\,w_5^2\right) + 4\,t_5^8\,\left(38 + 432\,u_5\,w_5 + 551\,u_5^2\,w_5^2\right) + 4\,c_5\,\left(t_5 + 2\,t_5^3 - 4\,t_5^4 + 2\,t_5^5\right)^2\,\left(8\,t_5^7 + 5\,u_5\,w_5 - \right.$$
$$\left.\left.4\,t_5\,u_5\,w_5 + 4\,t_5^6\,\left(-6 + u_5\,w_5\right) + 2\,t_5^4\,\left(-4 + u_5\,w_5\right) - 8\,t_5^5\,\left(-3 + u_5\,w_5\right) - 4\,t_5^2\,\left(1 + 3\,u_5\,w_5\right) + 4\,t_5^3\,\left(1 + 3\,u_5\,w_5\right)\right)\right) == 0$$

```
Rp[1, 2] Rm[4, 3]
```

$$\mathbb{E}\left[b_1\,c_2 - b_4\,c_3 + \frac{\left(-1 + ⅇ^{b_1}\right)\,u_1\,w_2}{b_1} + \frac{\left(-1 + ⅇ^{-b_4}\right)\,u_4\,w_3}{b_4}\right]$$

```
Rp[1, 2] Rm[4, 3] // m[1, 4, 1] // m[2, 3, 2]
```

$$\mathbb{E}[0]$$

```
Rm[1, 2] Rp[4, 3] // m[4, 1, 4] // m[2, 3, 2]
```

$$\mathbb{E}[0]$$

```
t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, a] // m[1, 6, b] // m[2, 4, c]
```

$$\mathbb{E}\left[\frac{b_a^2\, b_b\, (c_b + c_c) + \left(-1 + e^{b_a}\right) b_b\, u_a\, (w_b + w_c) + b_a \left(b_b^2\, c_c + \left(-1 + e^{b_b}\right) u_b\, w_c\right)}{b_a\, b_b}\right]$$

```
t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, a] // m[2, 5, b] // m[4, 6, c]
```

$$\mathbb{E}\left[\frac{b_a^2\, b_b\, (c_b + c_c) + \left(-1 + e^{b_a}\right) b_b\, u_a\, (w_b + w_c) + b_a \left(b_b^2\, c_c + \left(-1 + e^{b_b}\right) u_b\, w_c\right)}{b_a\, b_b}\right]$$

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t1 ≡ t2
```

True

```
t3 = Rm[12, 1] Rm[2, 7] Rm[8, 3] Rm[4, 11] Rp[16, 5] Rp[6, 13] Rp[14, 9] Rp[10, 15]
```

$$\mathbb{E}\left[-b_{12}\, c_1 - b_8\, c_3 + b_{16}\, c_5 - b_2\, c_7 + b_{14}\, c_9 - b_4\, c_{11} + b_6\, c_{13} + b_{10}\, c_{15} + \frac{\left(-1 + e^{-b_{12}}\right) u_{12}\, w_1}{b_{12}} + \frac{\left(-1 + e^{-b_8}\right) u_8\, w_3}{b_8} + \frac{\left(-1 + e^{b_{16}}\right) u_{16}\, w_5}{b_{16}} + \frac{\left(-1 + e^{-b_2}\right) u_2\, w_7}{b_2} + \frac{\left(-1 + e^{b_{14}}\right) u_{14}\, w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right) u_4\, w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right) u_6\, w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right) u_{10}\, w_{15}}{b_{10}}\right]$$

```
t3 // m[1, 2, 1]
```

$$\mathbb{E}\left[-b_{12}\, c_1 - b_8\, c_3 + b_{16}\, c_5 - b_1\, c_7 + b_{14}\, c_9 - b_4\, c_{11} + b_6\, c_{13} + b_{10}\, c_{15} + \frac{\left(-1 + e^{-b_{12}}\right) u_{12}\, w_1}{b_{12}} + \frac{\left(-1 + e^{-b_8}\right) u_8\, w_3}{b_8} + \frac{\left(-1 + e^{b_{16}}\right) u_{16}\, w_5}{b_{16}} + \frac{\left(-1 + e^{-b_1}\right) u_1\, w_7}{b_1} - \frac{\left(-1 + e^{-b_1}\right)\left(-1 + e^{-b_{12}}\right) u_{12}\, w_7}{b_{12}} + \frac{\left(-1 + e^{b_{14}}\right) u_{14}\, w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right) u_4\, w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right) u_6\, w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right) u_{10}\, w_{15}}{b_{10}}\right]$$

```
t3 // m[1, 2, 1] // m[1, 3, 1]
```

$$\mathbb{E}\left[-b_8\, c_1 - b_{12}\, c_1 + b_{16}\, c_5 - b_1\, c_7 + b_{14}\, c_9 - b_4\, c_{11} + b_6\, c_{13} + b_{10}\, c_{15} + \frac{\left(-1 + e^{-b_8}\right) u_8\, w_1}{b_8} + \frac{e^{-b_8}\left(-1 + e^{-b_{12}}\right) u_{12}\, w_1}{b_{12}} + \frac{\left(-1 + e^{b_{16}}\right) u_{16}\, w_5}{b_{16}} + \frac{e^{b_8}\left(-1 + e^{-b_1}\right) u_1\, w_7}{b_1} - \frac{\left(-1 + e^{-b_1}\right)\left(-1 + e^{-b_{12}}\right) u_{12}\, w_7}{b_{12}} + \frac{\left(-1 + e^{b_{14}}\right) u_{14}\, w_9}{b_{14}} + \frac{\left(-1 + e^{-b_4}\right) u_4\, w_{11}}{b_4} + \frac{\left(-1 + e^{b_6}\right) u_6\, w_{13}}{b_6} + \frac{\left(-1 + e^{b_{10}}\right) u_{10}\, w_{15}}{b_{10}}\right]$$

```
Do[t3 = t3 // m[1, kk, 1], {kk, 2, 16}]; t3
```

$$-\frac{e^{3\, b_1}\, \mathbb{E}[0]}{1 - 4\, e^{b_1} + 8\, e^{2\, b_1} - 11\, e^{3\, b_1} + 8\, e^{4\, b_1} - 4\, e^{5\, b_1} + e^{6\, b_1}}$$

```
Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 4, 4] // m[4, 5, 5] // m[5, 2, 2] // m[2, 3, 3] // m[3, 6, 6] // m[6, 1, 1]
```

$$\frac{\mathbb{E}\left[3\, b_1\, c_1 + \frac{\left(1 - e^{-3\, b_1}\right) u_1\, w_1}{b_1}\right]}{1 + e^{b_1}\left(-1 + e^{b_1}\right)}$$